JOURNAL OF APPROXIMATION THEORY **49**, 99–109 (1987)

Approximation by Compact Operators between $C(X)$ Spaces

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Communicated by Allan Pinkus

Received July 23, 1984; revised February 25, 1985

A closed subspace M of a Banach space E is said to be proximinal if every $a \in E$ admits a closest point in M, i.e., a point $x \in M$ for which $\|a - x\| = d(a, M)$, the distance of a from M. Many authors have considered the problem of determining whether $K(E, F)$, the space of compact operators from E to F, is proximinal in $B(E, F)$, the corresponding space of bounded linear operators. We attempt to solve this problem for the case when $E = C(X)$ and $F = C(Y)$ are the usual function spaces over compact Hausdorff spaces X and Y. If Y is extremally disconnected, we can completely characterize those X for which $K(C(X), C(Y))$ is proximinal. Except where stated otherwise, our results are valid for both real and complex scalars.

In each case, we will establish proximinality of the compact operators by establishing the $1\frac{1}{2}$-ball property. Recall that a subspace M has the $1\frac{1}{2}$-ball property in E if, whenever $a \in E$, $r \geqslant 0$, $\|a\| < r + 1$ and the closed ball $B(a, r)$ meets M, then $M \cap B(0, 1) \cap B(a, r)$ is non-empty. Every subspace with the $1\frac{1}{2}$-ball property is proximinal, and even more is true.

PROPOSITION 1 [16, Theorem 1.2]. *Suppose M has the $1\frac{1}{2}$-ball property in E. Then there exists a continuous, homogeneous map $\Pi: E \to M$ satisfying $\|x - \Pi(x)\| = d(x, M)$ and also $\Pi(x + m) = \Pi(x) + m$ whenever $m \in M$.*

Proposition 1 generalizes the corresponding result for M-ideals [5]. A number of authors, including [1, 4, 10, 11, 12], have established proximinality of $K(E, F)$, for suitable E and F, by showing that $K(E, F)$ is an M-ideal in $B(E, F)$. Rather than repeat the definition of M-ideals, we simply recall that every M-ideal has the $1\frac{1}{2}$-ball property [17].

Before starting our work, we need the following two observations. They are well known and easy to prove.

* Research supported by a Queen Elizabeth Fellowship.

0021-9045/87 $3.00

PROPOSITION 2. *The map $T \mapsto T^*|F$ is a linear isometry from $B(E, F^*)$ onto $B(F, E^*)$ which sends $K(E, F^*)$ to $K(F, E^*)$.*

PROPOSITION 3. *Let M and N be the ranges of contractive projections on E and F, respectively. If $K(E, F)$ is proximinal (or has the $1\frac{1}{2}$-ball property) in $B(E, F)$, then the same is true of $K(M, N)$ in $B(M, N)$.*

Our first result actually concerns certain spaces of measurable functions. Case (iv) improves a result proved for real scalars by Lau [7, Theorem 6.4]. Case (i) is obviously a special case of (iv), and is stated separately only to streamline the proof.

THEOREM 4. *In each of the following cases, $K(E, F)$ has the $1\frac{1}{2}$-ball property in $B(E, F)$:*

(i) *$E = l_1(\Delta)$ and $F = l_1(\Gamma)$ for discrete sets Γ and Δ.*

(ii) *$E^* = l_1(\Gamma)$ and $F = C(Y)$, where Γ is discrete and Y is extremally disconnected.*

(iii) *$E^* = l_1(\Gamma)$ and $F = L_\infty(S, \mu)$, where Γ is discrete and (S, μ) is any measure space.*

(iv) *$E = L_1(S, \mu)$ and $F = l_1(\Gamma)$, where (S, μ) is any measure space and Γ is discrete.*

Proof. (i) This is a trivial generalization of [16, Proposition 2.8].

(ii) If Y is the Stone–Čech compactification of some discrete set Γ, then $C(Y) = l_\infty(\Gamma)$, and the result follows from case (i) and Proposition 2. In general, the result follows from Proposition 3 and the fact that $C(Y)$ is the range of a contractive projection on some $l_\infty(\Gamma)$ [6, Corollary 11.2].

(iii) This is a special case of (ii). It is worth recalling that a Banach space is isometric to the range of a contractive projection on every superspace if and only if it is isometric to $C(Y)$, for some extremally disconnected Y. Every space $L_\infty(S, \mu)$ has this property. See [3; 6, Sect. 11].

(iv) This follows from Proposition 2 and case (iii). ∎

Although the proof of [16, Proposition 2.8] was constructive, the proof of Theorem 4 is not.

Now we can give the promised results about spaces of continuous functions.

THEOREM 5. *If Y is extremally disconnected, then the following are equivalent:*

(i) X is dispersed (i.e., every subset contains an isolated point)

(ii) $K(C(X), C(Y))$ has the $1\frac{1}{2}$-ball property in $B(C(X), C(Y))$

(iii) $K(C(X), C(Y))$ is proximinal in $B(C(X), C(Y))$.

Proof. (i)$\Rightarrow$(ii). This follows from Theorem 4 and [6, Theorems 8.9 and 8.10].

(ii)$\Rightarrow$(iii). This is Proposition 1.

(iii)$\Rightarrow$(i). Feder [2, Theorem 3] proved that $K(l_1, L_1(0, 1))$ is not proximinal in $B(l_1, L_1(0, 1))$. If X is not dispersed, then $L_1(0, 1)$ is isometric to the range of a contractive projection on $C(X)^*$ [6, Theorems 14.11 and 18.5]. Propositions 2 and 3 then show that $K(C(X), l_\infty)$ is not proximinal in $B(C(X), l_\infty)$. Since the Stone–Čech compactification of the integers is a continuous image of Y, l_∞ is the range of a contractive projection on $C(Y)$. Another application of Proposition 3 completes the proof. ∎

It is natural to ask if these results hold without the assumption that Y is extremally disconnected. For the 1-point compactification of the integers, they do not.

EXAMPLE 6. If the scalars are complex, then $K(\mathscr{C})$ does not have the $1\frac{1}{2}$-ball property in $B(\mathscr{C})$.

Proof. We follow the notation of Taylor [15, Sect. 4.51]. If $(\xi_1, \xi_2, \ldots)$ is any sequence in $\mathscr{C}$, we let ξ_0 denote its limit. Each $A \in B(\mathscr{C})$ corresponds to an infinite matrix (a_{jk}), where $j = 1, 2, 3, \ldots$ and $k = 0, 1, 2, \ldots$, for which $\sum_{k=0}^{\infty} a_{jk}$ converges as $j \to \infty$, as does $(a_{jk})_{j \to \infty}$ for $k = 1, 2, 3, \ldots$. If $(\eta_n) = A(\xi_n)$ then, of course,

$$\eta_j = \sum_{k=0}^{\infty} a_{jk} \xi_k \qquad \text{for} \quad j = 1, 2, 3, \ldots.$$

The norm of A is given by

$$\|A\| = \sup_{j=1}^{\infty} \sum_{k=0}^{\infty} |a_{jk}|,$$

but there is no simple formula for $d(A, K(\mathscr{C}))$.

Let $\{e, e_1, e_2, \ldots\}$ be the usual basis for $\mathscr{C}$, where $e = (1, 1, 1, \ldots)$. Define $A: \mathscr{C} \to \mathscr{C}$ by

$$Ae_1 = \tfrac{1}{2}e - e_1, \qquad Ae_n = (-1)^n e_n$$

for $n \geq 2$ and $Ae = (i + \frac{1}{2})e$. It is routine to verify that $\|A\| < 3$ and that $K(\mathscr{C}) \cap B(A, 2)$ is non-empty. However,

$$K(\mathscr{C}) \cap B(A, 2) \cap B(0, 1) = \varnothing.$$

To see this, suppose $T \in K(\mathscr{C}) \cap B(A, 2)$. Then

$$\sum_{k=0}^{\infty} |a_{jk} - t_{jk}| \leqslant 2$$

for all j, so

$$|i - 1 - t_{j0}| + |\tfrac{1}{2} - t_{j1}| + |1 - t_{jj}| \leqslant 2 \qquad \text{for } j \text{ even}$$

and

$$|i + 1 - t_{j0}| + |\tfrac{1}{2} - t_{j1}| + |-1 - t_{jj}| \leqslant 2 \qquad \text{for } j \text{ odd}, \quad j \neq 1.$$

Let

$$t_1 = \lim_{j \to \infty} t_{j1}.$$

Since T is compact, $t_0 = \lim_{j \to \infty} t_{j0}$ exists, and also $\lim_{j \to \infty} t_{jj} = 0$. Thus

$$|i \pm 1 - t_0| + |\tfrac{1}{2} - t_1| \leqslant 1.$$

This forces $t_0 = i$ and $t_1 = \frac{1}{2}$, so $T \notin B(0, 1)$. ∎

The classical sequence space $\mathscr{C}$ seems to have received no attention in the literature. Curiously, we have a positive result (with a constructive proof) if the scalars are real.

THEOREM 7. *For real scalars, $K(\mathscr{C})$ does have the $1\frac{1}{2}$-ball property in $B(\mathscr{C})$.*

Proof. If $S = (s_{jk})$ has the property that, for some N, $s_{jk} = 0$ for all $k > N$, then S is a compact operator. Conversely, the set of operators with this property is dense in $K(\mathscr{C})$.

Now suppose we are given $A \in B(\mathscr{C})$ with $\|A\| < r + 1$ and $K(\mathscr{C}) \cap B(A, r) \neq \varnothing$, and choose ε so that $0 < \varepsilon < r + 1 - \|A\|$. Then

$$\sum_{k=0}^{\infty} |a_{jk}| \leqslant r + 1 - \varepsilon$$

for all j, and also $\|A - S\| < r + \varepsilon$ for some S of the above form. We may also suppose that $s_{j0} = s$ for all but finitely many j.

Let

$$a_k = \lim_{j \to \infty} a_{jk},$$

for $1 \leqslant k \leqslant N$. Then choose M so that, if $j > M$, then $|a_k - a_{jk}| < \varepsilon/N$ and $s_{j0} = s$. Next, put

$$\sigma_j = \sum_{k > N} |a_{jk}| \qquad \text{for all } j,$$

and

$$\sigma = \sum_{k=1}^{N} |a_k|.$$

We now have, for all $j > M$,

$$|a_{j0} - s| + \sigma_j \leqslant \|A - S\| < r + \varepsilon \tag{1}$$

and

$$|a_{j0}| + \sigma + \sigma_j < r + 1. \tag{2}$$

There are two cases to consider, depending on the value of σ.

Case I. Suppose $\sigma \geqslant 1$. Then we find $n \leqslant N$ and $\lambda \in [0, 1]$ so that

$$\sum_{k=1}^{n-1} |a_k| + \lambda |a_n| = 1.$$

Put

$$s_k = a_k \quad \text{for} \quad 1 \leqslant k < n, \qquad s_n = \lambda a_n$$

and

$$s_k = 0 \quad \text{for} \quad k > n \quad \text{or} \quad k = 0.$$

Then, for $j > M$,

$$|a_{j0} - s_0| + \sum_{k=1}^{N} |a_k - s_k| + \sigma_j = |a_{j0}| + \sum_{k=1}^{N} |a_k| - 1 + \sigma_j < r + \varepsilon.$$

Clearly

$$\sum_{k=0}^{N} |s_k| \leqslant 1.$$

Case II. Suppose $\sigma < 1$. This time, we put $s_k = a_k$ for $1 \leqslant k \leqslant N$. Choosing s_0 is a little more difficult. First note that, for all $j > M$, $b_j = r + \varepsilon - \sigma_j > 0$. From (2) it follows that $-a_{j0} \leqslant r + 1 - \sigma - \sigma_j$ and so

$$-(1 - \sigma) \leqslant a_{j0} + b_j.$$

Similarly $a_{j0} - b_j \leqslant 1 - \sigma$ and so

$$\sup(\{a_{j0} - b_j : j > M\} \cup \{-(1 - \sigma)\}) \leqslant \inf(\{a_{j0} + b_j : j > M\} \cup \{1 - \sigma\}).$$

Hence we can find a real number s_0 satisfying

$$-(1 - \sigma) \leqslant s_0 \leqslant 1 - \sigma$$

and

$$a_{j0} - (r + \varepsilon - \sigma_j) \leqslant s_0 \leqslant a_{j0} + (r + \varepsilon - \sigma_j),$$

for all $j > M$. Then, as in the previous case, we have

$$|a_{j0} - s_0| + \sum_{k=1}^{N} |a_k - s_k| + \sigma_j = |a_{j0} - s_0| + \sigma_j \leqslant r + \varepsilon$$

and

$$\sum_{k=0}^{N} |s_k| = |s_0| + \sigma \leqslant 1.$$

Now define $T = (t_{jk})$ by

$$t_{jk} = a_{jk}/(r+1) \qquad \text{for} \quad j \leqslant M,$$

$$t_{jk} = s_k \qquad \text{for} \quad j > M \quad \text{and} \quad k \leqslant N,$$

and

$$t_{jk} = 0 \qquad \text{for} \quad j > M \quad \text{and} \quad k > N.$$

Then the image of T lies in the linear span of $\{e, e_1, e_2, \ldots, e_M\}$, so T is compact. Clearly $\|T\| \leqslant 1$. Furthermore, for $j \leqslant M$,

$$\sum_{k=0}^{\infty} |a_{jk} - t_{jk}| = \frac{r}{r+1} \sum_{k=0}^{\infty} |a_{jk}| \leqslant r,$$

and for $j > M$,

$$\sum_{k=0}^{\infty} |a_{jk} - t_{jk}| < |a_{j0} - s_0| + \sum_{k=1}^{N} |a_k - s_k| + \varepsilon + \sigma_j$$

$$\leqslant r + 2\varepsilon.$$

Thus $\|T - A\| \leqslant r + 2\varepsilon$.

We have now shown that

$$K(\mathscr{C}) \cap B(A, r + 2\varepsilon) \cap B(0, 1)$$

is non-empty. By [17, Theorem 3] this establishes the $1\frac{1}{2}$-ball property. ∎

By severely restricting the domain space, we can completely dispense with the extremally disconnected assumption on the range space. To be precise, we can show that $K(\mathscr{C}_0, C(X))$ has the $1\frac{1}{2}$-ball property in $B(\mathscr{C}_0, C(X))$, at least if the scalars are real. Before proving this, we discuss the difficulties that arise in the complex case.

If S is any metric space, let 2^S denote the collection of closed, bounded, non-empty subsets of S. It is standard to make 2^S into a metric space by giving it the Hausdorff metric, defined for $A, B \in 2^S$ by

$$d(A, B) = \sup(\{d(x, A) : x \in B\} \cup \{d(x, B) : x \in A\}).$$

If E is a Banach space and $f \in E$, let us define

$$\Psi = \Psi_f\colon [(\|f\| - 1)^+, \infty) \to 2^E$$

by

$$\Psi(r) = B(0, 1) \cap B(f, r).$$

With the usual lack of imagination, we will say that E has property (P) if the family of maps $\{\Psi_f : f \in E\}$ is uniformly equicontinuous. Recall that E is said to have the 3.2 intersection property if, whenever B_1, B_2, B_3 are closed balls in E which meet pairwise, then

$$B_1 \cap B_2 \cap B_3 \neq \emptyset.$$

If E has the 3.2 intersection property, it is easy to verify that

$$d(\Psi_f(r), \Psi_f(r + \varepsilon)) \leqslant \varepsilon.$$

Thus, the 3.2 intersection property implies (P). It follows [9, Theorem 4.6(c)] that the real Banach space l_1 has (P).

Conjecture 8. The complex Banach space l_1 has property (P).

This ideal is crucial in the proof of the next theorem. We have been unable to determine whether Conjecture 8 is true or false.

Assuming property (P) for l_1, we will show that $K(\mathscr{C}_0, C(X))$ has the $1\frac{1}{2}$-ball property in $B(\mathscr{C}_0, C(X))$ for any compact Hausdorff space X. Since l_1 is the dual of $\mathscr{C}_0$, we may identify $B(\mathscr{C}_0, C(X))$ with the sup-normed space $CW^*(X, l_1)$ of weak* continuous maps $f\colon X \to l_1$, and $K(\mathscr{C}_0, C(X))$ with the subspace $C(X, l_1)$ of norm continuous maps. The identification is the obvious one, given by

$$(Ta)(x) = f(x)(a) \qquad \text{for all } a \in \mathscr{C}_0,\ x \in X$$

and $T\colon \mathscr{C}_0 \to C(X)$.

Now fix $f \in CW^*(X, l_1)$ and put

$$d(x) = \limsup_{y \to x} \|f(y) - f(x)\|.$$

Replacing f with $f - g$, where $g \in C(X, l_1)$, leaves the value of $d(x)$ unaltered. The idea of introducing $d(\cdot)$ is due to Mach [11], who used similar techniques to prove the proximinality of $K(\mathscr{C}_0, C(X))$, for either scalar field.

LEMMA 9. *If $x, y \in l_1 = \mathscr{C}_0^*$ and $x_\alpha \to 0$ weak*, then*

$$\|x_\alpha + y\| - \|x_\alpha\| \to \|y\|.$$

Proof. For any $A \subset \mathbb{N}$ we have

$$| \, \|x_\alpha + y\| - \|x_\alpha\| - \|y\| \, | \leqslant \sum_{n \in A} 2|x_\alpha(n)| + \sum_{n \notin A} 2|y(n)|.$$

A routine truncation argument completes the proof. ▮

If we regard l_1 as the dual of some other Banach space, such as $\mathscr{C}$, then Lemma 9 does not hold.

LEMMA 10. *Let f, d be as above and fix $x \in X$. Then*

(i) *for any $y \in X$,*

$$\limsup_{z \to y} \|f(z) - f(x)\| = \|f(y) - f(x)\| + d(y).$$

(ii) for any $y \in X$,

$$\limsup_{z \to y} \|f(z)\| = \|f(y)\| + d(y).$$

(iii)
$$d(x) = \limsup_{y \to x} (\|f(x) - f(y)\| + d(y)).$$

(iv) for any $g \in C(X, l_1)$,

$$\|f(x) - g(x)\| + d(x) \leqslant \|f - g\|.$$

Proof. (i) Since f is weak*-continuous, the previous lemma gives

$$\begin{aligned}\limsup_{z \to y} \|f(z) - f(x)\| &= \lim_{z \to y} (\|f(z) - f(x)\| - \|f(z) - f(y)\|) \\ &\quad + \limsup_{z \to y} \|f(z) - f(y)\| \\ &= \|f(y) - f(x)\| + d(y).\end{aligned}$$

(ii) The constant function $g = f(x)$ certainly lies in $C(X, l_1)$. Replace f by $f - g$ in (i).

(iii) From the definition of $d(\cdot)$, and (i), we have

$$\begin{aligned}d(x) &\leqslant \limsup_{y \to x} (\|f(x) - f(y)\| + d(y)) \\ &= \limsup_{y \to x} \limsup_{z \to y} \|f(z) - f(x)\| \\ &\leqslant \limsup_{z \to x} \|f(z) - f(x)\| = d(x).\end{aligned}$$

(iv) Assume without loss of generality that $g = 0$. Then, by (ii),

$$\|f(x)\| + d(x) = \limsup_{y \to x} \|f(y)\| \leqslant \|f\|.$$

THEOREM 11. *Let X be any compact Hausdorff space. Then $K(\mathscr{C}_0, C(X))$ has the $1\frac{1}{2}$-ball property in $B(\mathscr{C}_0, C(X))$ if the scalars are real, or if Conjecture 8 is true.*

Proof. Suppose that $C(X, l_1) \cap B(f, r) \neq \varnothing$ and $\|f\| \leqslant r + 1$. We must show that

$$C(X, l_1) \cap B(0, 1) \cap B(f, r) \neq \varnothing.$$

The last part of Lemma 10, with $g \in C(X, l_1) \cap B(f, r)$, shows that $r \geqslant d(x)$ for all $x \in X$. With $g = 0$ it shows that

$$\|f(x)\| \leqslant r + 1 - d(x),$$

for each x. Thus we may define $\Psi: X \to 2^{l_1}$ by

$$\Psi(x) = B(0, 1) \cap B(f(x), r - d(x)).$$

Clearly each $\Psi(x)$ is closed, convex, and non-empty; we claim that Ψ is lower semicontinuous. This means that if K is any closed subset of l_1, we have to show that $\{x: \Psi(x) \subseteq K\}$ is closed.

Suppose then that $x_\alpha \to x$ in X, and that each $\Psi(x_\alpha) \subseteq K$. Choose $a \in \Psi(x)$ and put

$$\lambda_\alpha = \|a - f(x_\alpha)\| + d(x_\alpha) - r.$$

By Lemma 10(iii)

$$\limsup \lambda_\alpha \leqslant \|a - f(x)\| + d(x) - r \leqslant 0.$$

Hence $\varepsilon_\alpha = \max\{\lambda_\alpha, 0\} \to 0$ and also

$$\|a - f(x_\alpha)\| = r - d(x_\alpha) + \lambda_\alpha \leqslant r - d(x_\alpha) + \varepsilon_\alpha$$

and $\|a\| \leqslant 1$. Let

$$\delta(\varepsilon) = \sup\{d(B(0, 1) \cap B(g, s), B(0, 1) \cap B(g, s + \varepsilon)): g \in l_1,$$
$$s > 0, \|g\| \leqslant s + 1\}.$$

Assuming l_1 has property (P), we have $\delta(\varepsilon) \to 0$ as $\varepsilon \to 0$. Thus, from the definition of $\delta(\varepsilon)$, we can find a_α with $\|a_\alpha\| \leqslant 1$,

$$\|a_\alpha - f(x_\alpha)\| \leqslant r - d(x_\alpha)$$

and $\|a - a_\alpha\| \leqslant \delta(\varepsilon_\alpha)$. Then

$$a_\alpha \in \Psi(x_\alpha) \subseteq K$$

and $a_\alpha \to a$. This proves that $\Psi(x) \subseteq K$. Michael's theorem [14] now gives us a continuous selection for Ψ, which clearly belongs to

$$C(X, l_1) \cap B(0, 1) \cap B(f, r). \quad \blacksquare$$

To show that the preceding examples are not M-ideals, first note that there is a function $f \in CW^*(X, l_1)$ whose range contains the standard basis $\{e_1, e_2, ...\}$. This is easy to see if X contains a convergent sequence of distinct points. For the general case, recall that every compact space can be mapped onto a Hausdorff space which contains a convergent sequence.

Choose $x_n \in X$ so that $f(x_n) = e_n$, and let $\mathrm{LIM} \in l_\infty^*$ be any Banach limit. We define two functionals $\Psi, \phi \in CW^*(X, l_1)^*$ by

$$\Psi(g) = \mathop{\mathrm{LIM}}_n g_1(x_n) \qquad \text{and} \qquad \phi(g) = \mathop{\mathrm{LIM}}_n \{g_1(x_n) + g_n(x_n)\}.$$

It is clear that $\|\Psi\| \leqslant 1$, $\|\phi\| \leqslant 1$ and that $\Psi(f) = 0 \neq 1 = \phi(f)$. If $g \in C(X, l_1)$ then $g(X)$ is norm compact in l_1, and so $g_n(x) \to 0$ (as $n \to \infty$) uniformly with respect to $x \in X$. It follows that

$$\Psi \mid C(X, l_1) = \phi \mid C(X, l_1) = \eta,$$

say. If $g \in C(X, l_1)$ is the constant function $g(x) = e_1$ then $\|g\| = n(g) = 1$. Thus ϕ and Ψ are two distinct norm-preserving extensions of η.

So $K(\mathscr{C}_0, C(X))$ does not have the unique extension property in $B(\mathscr{C}_0, C(X))$. It follows [17, Theorem 4] that $K(\mathscr{C}_0, C(X))$ is not an M-ideal in $B(\mathscr{C}_0, C(X))$.

The following result provides some evidence that Theorem 11 may be true for both scalar fields.

PROPOSITION 12. *For either scalar field, $K(\mathscr{C}_0, \mathscr{C})$ has the $1\frac{1}{2}$-ball property in $B(\mathscr{C}_0, \mathscr{C})$.*

Proof. Again following [15], any $A \in B(\mathscr{C}_0, \mathscr{C})$ corresponds to an infinite matrix (a_{jk}), where $j = 1, 2, 3,...$ and $k = 1, 2, 3....$ Imitating the proof of Theorem 7, we find that some simplifications are caused by the absence of zeroth columns in elements of $B(\mathscr{C}_0, \mathscr{C})$. In particular, it is not necessary to define s_0. Doing so, in Case II of the previous proof, was the only point at which the scalars were required to be real. $\blacksquare$

We recall that for any Banach space E, $K(E, \mathscr{C}_0)$ is actually an M-ideal in $B(E, \mathscr{C}_0)$. This was observed independently by several authors [8, 12, 16]. To see how special the role of $\mathscr{C}_0$ is in this result, we note that $K(L_p(S, \mu), C(X))$ fails the $1\frac{1}{2}$-ball property in $B(L_p(S, \mu), C(X))$, whenever $L_p(S, \mu)$ and $C(X)$ are infinite dimensional, and $1 < p < \infty$. By Proposition 3, and the remarks preceding Lemma 9, it suffices to show that

$C(X, l_p)$ fails the $1\frac{1}{2}$-ball property in $CW^*(X, l_p)$. This follows from a generalization of the argument of [16, p. 296].

We finish with another negative result.

PROPOSITION 13. *Suppose X and Y both contain uncountable, metrizable, closed subsets. Then $K(C(X), C(Y))$ is not proximinal in $B(C(X), C(Y))$.*

Proof. Benyamini [2, Appendix] proved this in the case $X = Y = [0, 1]$. If $[0, 1]$ is replaced by the Cantor set, Z, throughout the proof, it works just as well. By the Borsuk–Dugundji extension theorem [13, Sect. 7], the result holds whenever X and Y contain homeomorphic copies of Z. But every uncountable compact metric space contains a copy of Z (this follows from the Cantor–Bendixson theorem and a standard argument). ∎

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