The Laurent Expansion for a Nearly Singular Matrix

C. E. LANGENHOP*
University of Kentucky
Lexington, Kentucky

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ABSTRACT

Let \( A_0, A_1 \) be \( n \times n \) matrices of complex numbers and let \( E^n \) be the vector space of \( n \times 1 \) matrices of complex numbers. Let \( N_1 = \{ x \in E^n | A_1 x = 0 \} \), \( N_{0,-1} = \{ 0 \} \subseteq E^n \), and for \( k \geq -1 \) define \( R_{1k} = A_1 N_{0k} \) and \( N_{0k+1} = \{ x \in E^n | A_0 x \in R_{1k} \} \). In any case \( \mu = \min \{ k \geq -1 | N_{0,k+1} = N_{0k} \} \) exists and \( \mu \geq 0 \) or \( \mu = -1 \) according as \( A_0 \) is singular or not. The main result presented is the following: There exists \( \delta > 0 \) such that the matrix \( A_0 + zA_1 \) is invertible for all complex numbers \( z \) such that \( 0 < |z| < \delta \) if and only if \( N_1 \cap X_{0k} = \{ 0 \} \) for all \( k > 0 \). Moreover, if this condition holds, then there exist \( n \times n \) matrices \( Q_k \) such that \( (A_0 + zA_1)^{-1} = \sum_{k=-\mu-1}^{\infty} z^k Q_k \), the series converging for \( 0 < |z| < \delta \) for some \( \delta > 0 \), and \( Q_{-\mu-1} \neq 0 \).

1. INTRODUCTION

Let \( A_0 \) and \( A_1 \) be \( n \times n \) matrices of complex numbers. We are interested here in the invertibility of the matrix \( A_0 + zA_1 \) for complex numbers \( z \neq 0 \). This requires that \( \det(A_0 + zA_1) \), as a polynomial in \( z \), be not identically zero and thus when \( (A_0 + zA_1)^{-1} \) exists its elements are rational functions of \( z \). Hence, for some \( \delta > 0 \), if \( 0 < |z| < \delta \), then

\[
(A_0 + zA_1)^{-1} = z^{-m}Q_{-m} + \cdots + z^{-1}Q_{-1} + Q_0 + zQ_1 + \cdots, \tag{1.1}
\]

where the \( n \times n \) matrices \( Q_k \) are independent of \( z \) and \( Q_{-m} \neq 0 \) for some \( m \geq 0 \). If \( A_0 \) is nonsingular, we may write \( (A_0 + zA_1)^{-1} = A_0^{-1}(I + zA_1A_0^{-1})^{-1} \) and \( m = 0 \) since \( (I + zK)^{-1} = I - zK + z^2K^2 - \cdots \) in some

* Currently with Southern Illinois University, Carbondale, Illinois.

neighborhood of \( z = 0 \). The interesting case then is when \( A_0 \) is singular so that \( m > 0 \), and it is for this case that \( A_0 + zA_1 \) is "nearly singular" when \( z \) is near zero.

If \( A_0 \) is nonsingular, we may also write \((A_0 + zA_1)^{-1} = z^{-1}A_0^{-1}(z^{-1}I + A_1A_0^{-1})^{-1}\), so in this case the expansion of \((A_0 + zA_1)^{-1}\) in a deleted neighborhood of \( z = 0 \) is related to the expansion of the resolvent operator \((\zeta I - K)^{-1}\) of \( K = -A_1A_0^{-1}\) about \( \zeta = \infty \). Thus many of the lemmas and definitions which appear below are analogous to corresponding ones in the spectral analysis theory of a linear operator \( K \) on a linear space. (See, for example, [1, Chapter 5].) Since we are primarily concerned here with the case when \( A_0 \) is singular, we have had to establish appropriate modifications of such standard results.

In this paper we present what might be called an intrinsic condition which is necessary and sufficient that \((A_0 + zA_1)^{-1}\) have a valid Laurent expansion of the form (1.1). The parameter \( m \) and the matrices \( Q_k \) are characterized in terms of certain subspaces of \( E^n \) exhibiting the interaction between \( A_0 \) and \( A_1 \). Here \( E^n \) denotes the vector space of \( n \times 1 \) matrices \( x \) over the complex numbers, and we interpret a \( n \times n \) matrix \( B \) as describing or effecting a linear transformation \( x \rightarrow Bx \) of \( E^n \) into \( E^n \).

In the problem at hand the ranges \( R_i = \{A_ix | x \in E^n\} \) and null spaces \( N_i = \{x \in E^n | A_ix = 0\} \), \( i = 0, 1, \) are important of course. They do not tell the whole story, however, and we must introduce two sequences of related subspaces which play the fundamental role.

2. THE BASIC SUBSPACES AND A NECESSARY CONDITION

In the following definition the fundamental sequences of related subspaces are presented.

**Definition 2.1.** Let \( N_{0,-1} = \{0\} \), the trivial subspace of \( E^n \), and for \( k \geq -1 \) define

\[
R_{ik} = A_1N_{0,k}, \quad N_{0,k+1} = \{x \in E^n | A_0x \in R_{ik}\}. \tag{2.1}
\]

Let \( R_{0,-1} = E^n \) and for \( k \geq -1 \) define

\[
N_{1k} = \{x \in E^n | A_1x \in R_{0,k}\}, \quad R_{0,k+1} = A_0N_{1k}. \tag{2.2}
\]

Of course we mean, for example, \( A_0N_{1k} = \{A_0x | x \in N_{1k}\} \), and it is evident inductively that all the sets \( R_{ik}, N_{ik} \) defined by (2.1) and (2.2) are subspaces of \( E^n \). We note that \( R_{1,-1} = \{0\} \) so \( N_{00} = N_0 \), and \( N_{1,-1} = \ldots \)

\[ E^n \text{ so } R_{00} = R_0. \] The statements in the next lemma are easy consequences of Definition 2.1, and we prove only (2.4) to illustrate.

**Lemma 2.1.** For all integers \( k \geq 0, \)

\[
\begin{align*}
N_0 &\subseteq N_{0k} \subseteq N_{0,k+1}; & R_{1k} &\subseteq R_{1,k+1} \subseteq R_1; \\
N_1 &\subseteq N_{1,k+1} \subseteq N_{1k}; & R_{0,k+1} &\subseteq R_{0k} \subseteq R_0;
\end{align*}
\]

(2.3)

if \( x \in N_{0,k}, \) then

\[ A_0 x = A_1 \bar{x} \text{ for some } \bar{x} \in N_{0,k-1}. \]

(2.5)

**Proof of (2.4).** Since \( A_1 N_1 = \{0\} \subseteq R_{0k} \) for any \( k \geq 0, \) then \( N_1 \subseteq N_{1k} \) for \( k \geq 0. \) Now obviously \( N_{10} \subseteq N_{1,-1} = E^n, \) so \( R_{01} = A_0 N_{00} \subseteq A_0 N_{1,-1} = R_{00} = R_0 \) whence also \( N_{11} \subseteq N_{10}. \) Thus \( R_{0,k+1} \subseteq R_{0k} \) holds for \( k = 0. \) Assuming it for some \( k \geq 0, \) we then get \( N_{1,k+1} \subseteq N_{1k} \) from the first of (2.2), and thus \( R_{0,k+2} = A_0 N_{1,k+1} \subseteq A_0 N_{1k} = R_{0,k+1}. \) Thus the monotonicity of \( N_{1k} \) and \( R_{0k} \) is true by induction.

Assertion (2.5) enables us to obtain a necessary condition for invertibility of \( A_0 + zA_1 \) as formulated in the following theorem. An obvious analog of (2.5) is also valid, but we have no need for it in our proofs.

**Theorem 2.1.** If \( A_0 + zA_1 \) is invertible for some \( z \neq 0, \) then \( N_1 \cap N_{0k} = \{0\} \) for all \( k \geq 0. \) (\( N_1 \cap N_{0,-1} = \{0\} \) trivially.)

**Proof.** Let \( z \neq 0 \) be such that \( A_0 + zA_1 \) is invertible and suppose \( k \geq 0 \) is such that \( N_1 \cap N_{0k} \neq \{0\}. \) Moreover, we assume \( k \) is the least such index so that \( N_1 \cap N_{0,k-1} = \{0\}. \) Now let \( x \in N_1 \cap N_{0k}, \) \( x \neq 0. \) We note first that \( k > 0 \) for, if \( k = 0, \) then \( A_0 x = A_1 x = 0 \) and \( (A_0 + zA_1)x = 0 \) with \( x \neq 0, \) thus contradicting the invertibility of \( A_0 + zA_1. \) We denote \( x_k = x \) and then by (2.5) there exist \( x_j \in N_{0,j}, j = 0, 1, \ldots, k - 1, \) such that \( A_0 x_{j+1} = A_1 x_j. \) If we define \( \xi = \sum_{j=0}^{k} (-z)^j x_j, \) then

\[
(A_0 + zA_1)\xi = \sum_{j=0}^{k} (-z)^j A_0 x_j - \sum_{j=0}^{k} (-z)^{j+1} A_1 x_j
\]

\[
= A_0 x_0 - (-z)^{k+1} A_1 x_k = 0
\]

since \( x_0 \in N_0 \) and \( x_k \in N_1. \) But \( A_0 + zA_1 \) is invertible, so \( \xi = 0 \) and \( x = x_k = -(-z)^k (x_0 - z x_1 + \cdots + (-z)^{k-1} x_{k-1}) \in N_{0,k-1} \) by (2.3).
Thus $x \in N_1 \cap N_{0,k-1}$, so $N_1 \cap N_{0,k-1} \neq \{0\}$, which contradicts the choice of $k$; therefore the theorem is proved.

For convenience in later references we shall denote the condition of interest by $D$; thus

$$\text{(D)} \quad N_1 \cap N_{0,k} = \{0\}, \quad k = 0, 1, 2, \ldots$$

In the next section we establish some further properties of the spaces $N_{ik}$ and $R_{ik}$ and some consequences of condition $D$. In Section 4 we prove, in effect, the converse of Theorem 2.1.

3. PROPERTIES OF THE BASIC SUBSPACES

Because of the monotonicity properties, (2.3) and (2.4), there are in fact only a finite number of distinct subspaces $N_{ik}$ and $R_{ik}$. Their dimensions are integer-valued, bounded monotone functions of $k$ so they must be constant from some value of $k$ on, and the sequences of subspaces then also are constant from some value of $k$ on. In fact, each is strictly monotone up to the beginning of the constant tail of the sequence.

**Lemma 3.1.** If $N_{0,k+1} = N_{0,k}$, then $N_{0,k+j} = N_{0,k}$ and $R_{1,k+j} = R_{1,k}$ for $j = 1, 2, \ldots$. If $R_{0,k+1} = R_{0,k}$, then $R_{0,k+j} = R_{0,k}$ and $N_{1,k+j} = N_{1,k}$ for $j = 1, 2, \ldots$.

**Proof.** Suppose $N_{0,k+1} = N_{0,k}$ so that $N_{0,k+j} = N_{0,k}$ for some $j \geq 1$. Then also $R_{1,k+j} = A_1 N_{0,k+j} = A_1 N_{0,k} = R_{1,k}$ by (2.1) and, moreover, $N_{0,k+j+1} = N_{0,k+1} = N_{0,k}$, whence $R_{1,k+j+1} = R_{1,k}$. Thus the first statement in the lemma is true by induction, and the second can be shown similarly.

The remarks immediately preceding this lemma justify the following definitions

$$\mu = \min\{k \geq -1 | N_{0,k+1} = N_{0,k}\}, \quad (3.1)$$

$$\lambda = \min\{k \geq -1 | R_{0,k+1} = R_{0,k}\}, \quad (3.2)$$

since the sets involved here are not empty. Lemma 3.1 implies that these sets consist of all $k \geq \mu$ and all $k \geq \lambda$, respectively. We note here that the nonsingularity of $A_0$ is equivalent to $\mu = -1$ and also to $\lambda = -1$.

Thus we assume in what follows that $\mu \geq 0$ and $\lambda \geq 0$, although the results are usually valid also in case $\mu = \lambda = -1$. It is convenient, though, to be able to characterize $\mu$ by $N_{0,\mu+1} = N_{0,\mu} \neq N_{0,\mu-1}$ and to characterize $\lambda$ by $R_{0,\lambda+1} = R_{0,\lambda} \neq R_{0,\lambda-1}$.

**Lemma 3.2.** For all integers $k \geq 0, j \geq 0$ the following hold:

\[
A_0(N_{0k} \cap N_{1,j-1}) = R_{1,k-1} \cap R_{0j}, 
\]

\[
A_1(N_{0k} \cap N_{1,j-1}) = R_{1k} \cap R_{0,j-1}, 
\]

\[
A_1 N_{1k} = R_{1} \cap R_{0k}. 
\]

Hence also the following hold:

\[
dim N_{0k} \cap N_{1,j-1} = \dim N_{0} \cap N_{1,j-1} + \dim R_{1,k-1} \cap R_{0j}, 
\]

\[
dim N_{0k} \cap N_{1,j-1} = \dim N_{1} \cap N_{0k} + \dim R_{1k} \cap R_{0,j-1}, 
\]

\[
dim N_{1k} = \dim N_{1} \cap N_{0k} + \dim R_{1k}. 
\]

**Proof.** Because the proofs of these are similar, we merely illustrate. Thus, for example, if $x \in N_{0k} \cap N_{1,j-1}$, it is clear from Definition 2.1 that $A_0x \in R_{1,k-1} \cap R_{0j}$. On the other hand, if $y \in R_{1,k-1} \cap R_{0j}$, then $y = A_0x$ for some $x \in N_{1,j-1}$ and, since also $A_0x - y \in R_{1,k-1}$, then $x \in N_{0k}$ as well. Thus (3.3) holds and (3.6) follows from this since $N_{0} \subseteq N_{0k}$, so the null space of $A_0$ on $N_{0k} \cap N_{1,j-1}$ is $N_{0} \cap N_{1,j-1}$.

Besides the general results (3.6) and (3.7) we will need especially the case $j = 0$ in which $N_{1,j-1} = E^n$. Thus

\[
dim N_{0k} = \dim N_{0} + \dim R_{1,k-1} \cap R_{0j}, \quad k \geq 0, 
\]

\[
dim N_{0k} = \dim N_{1} \cap N_{0k} + \dim R_{1k}, \quad k \geq 0. 
\]

One additional relation of an analogous nature will be needed. It arises from $R_{0,k+1} = A_0N_{1k}$ in Definition 2.1. Thus

\[
dim N_{1k} = \dim N_{0} \cap N_{1k} + \dim R_{0,k+1}. 
\]

**Lemma 3.3.** For all integers $k \geq 0$,

\[
dim(R_{1k} + R_0) = \dim R_{1k} - \dim N_{0,k+1} + n, 
\]

\[
dim(R_{0k} + R_1) = \dim R_{0k} - \dim N_{1k} + n. 
\]
Proof. These relations follow easily from (3.9) and (3.8), respectively.

**Lemma 3.4.** If condition D holds, then

\[ \dim R_{1k} = \dim N_{0k}, \quad k = 0, 1, \ldots, \]  

\[ R_{1k} + R_0 = E^n, \quad \text{if and only if} \quad k \geq \mu. \]  

**Proof.** Under condition D \((N_1 \cap N_{0k} = \{0\}) \) for \(k \geq 0\) we see that (3.10) reduces to (3.14), so in (3.12) we have \( \dim(R_{1k} + R_0) = \dim N_{0k} - \dim N_{0,k+1} + n \). By (3.1) and (2.3) we see that this is less than \(n\) if \(k \leq \mu - 1\) and equal to \(n\) if \(k \geq \mu\) so (3.15) must hold.

**Lemma 3.5.** Let \( k \geq 0 \), let \( S_0 \) be a subspace of \( E^n \) such that \( E^n = R_{1k} \oplus S_0 \), and define \( M_0 = \{ x | A_1x \in S_0 \} \). If condition D holds, then \( E^n = N_{0k} \oplus M_0 \).

**Proof.** If \( x \in N_{0k} \cap M_0 \), then \( A_1x \in R_{1k} \cap S_0 \). Then \( A_1x = 0 \), so \( x \in N_1 \cap N_{0k} \) and \( x = 0 \) if D holds. Hence \( N_{0k} + M_0 = N_{0k} \oplus M_0 \).

Note that \( N_1 \subset M_0 \) and, since \( A_1M_0 = R_1 \cap S_0 \), then \( \dim R_1 \cap S_0 = \dim M_0 - \dim N_1 \). Now \( R_{1k} \subset R_1 \), so \( R_1 = R_1 \cap (R_{1k} \oplus S_0) = R_{1k} \oplus (R_1 \cap S_0) \). Hence, by (3.14),

\[ n - \dim N_1 = \dim R_1 = \dim R_{1k} + \dim R_1 \cap S_0 \]

\[ - \dim N_{0k} + \dim M_0 - \dim N_1. \]

Thus \( \dim(N_{0k} \oplus M_0) = n \) and the lemma is proved.

**Lemma 3.6.** Let \( k \geq 0 \), let \( M_0 \) be a subspace of \( E^n \) such that \( E^n = N_{0k} \oplus M_0 \), and define \( S_1 = A_0 M_0 \). If D holds, then

\[ \dim(R_{1k} + S_1) = n - \dim M_0 \cap N_{0,k+1} \]  

(3.16)

and

\[ E^n = R_{1k} \oplus S_1 \quad \text{if and only if} \quad k \geq \mu. \]  

(3.17)

**Proof.** Since \( N_0 \subset N_{0k} \) for \(k \geq 0\), then \( N_0 \cap M_0 = \{0\} \) so \( \dim S_1 = \dim M_0 \). We easily see that \( A_0(M_0 \cap N_{0,k+1}) = S_1 \cap R_{1k} \), so we have, similarly, \( \dim S_1 \cap R_{1k} = \dim M_0 \cap N_{0,k+1} \). Thus, by (3.14), \( \dim(R_{1k} + \)}
$S_1 = \dim N_{0k} + \dim M_0 - \dim M_0 \cap N_{0,k+1}$, and (3.16) follows from $E^n = N_{0k} \oplus M_0$. Now since $N_{0k} \subset N_{0,k+1}$ we see that $\dim M_0 \cap N_{0,k+1} = 0$ if and only if $N_{0,k+1} = N_{0k}$, which is equivalent to $k \geq \mu$. Thus $E^n - R_{1k} + S_1$ if and only if $k \geq \mu$ and, when this holds, $S_1 \cap R_{1k} = A_0\{0\} = \{0\}$ so the sum is direct and (3.17) is proved.

**Lemma 3.7.** If $D$ holds, then for all integers $k \geq 0$,

\begin{align}
R_{1k} + R_{0k} &= E^n, \\
N_{0a} + N_{1k} &= E^n.
\end{align}

**Proof.** From (3.15) we see that (3.18) holds for $k = 0$, so we let $k$ be any nonnegative integer for which it holds. Then there is a subspace $S_0 \subset R_{0k}$ such that $E^n = R_{1k} \oplus S_0$ and we define $M_0 = \{x | A_1 x \in S_0\}$. Clearly $M_0 \subset N_{1k}$, so $S_1 = A_0 M_0 \subset R_{0,k+1}$. By Lemma 3.5 we have $E^n = N_{0a} \oplus M_0$, whence by Lemma 3.6 we have $E^n = R_{1k} \oplus S_1 \subset R_{1k} + R_{0,k+1}$. Thus (3.18) also holds with $k$ replaced by $k + 1$, so it holds for all integers $k \geq 0$. We may get (3.19) from parts of the proof of (3.18). Thus $E^n = N_{0a} \oplus M_0 \subset N_{0a} + N_{1k}$.

**Lemma 3.8.** If $D$ holds, then $N_0 \cap N_{1k} = \{0\}$ if and only if $k \geq \lambda$.

**Proof.** From (3.18) it follows that $R_1 + R_{0k} = E^n$ for all $k \geq 0$. Thus by (3.13) we have

\[ \dim R_{0k} = \dim N_{1k}, \quad k = 0, 1, \ldots \]

(3.20)

if $D$ holds. Using this in (3.11), we conclude that

\[ \dim R_{0,k+1} = \dim R_{0k} - \dim N_0 \cap N_{1k}. \]

(3.21)

By (2.4), Lemma 3.1, and the definition of $\lambda$ in (3.2), it is clear from (3.21) that $N_0 \cap N_{1k} = \{0\}$ if and only if $k \geq \lambda$.

**Lemma 3.9.** If $D$ holds, then, for all integers $k \geq 0$, $j \geq 0$,

\[ \dim N_{0,k+1} \cap N_{1,j+1} = \sum_{l=j-1}^{j+k} \dim N_0 \cap N_{1l}. \]

(3.22)

**Proof.** By (3.6) and (3.7) we have

\[ \dim N_{0, k+1} \cap N_{1,j-1} = \dim N_0 \cap N_{1,j-1} + \dim N_{0k} \cap N_{1j} \]

when D holds. Using this to substitute in the last term, we get

\[ \dim N_{0, k+1} \cap N_{1,j-1} = \dim N_0 \cap N_{1,j-1} + \dim N_0 \cap N_{1j} + \dim N_{0, k-1} \cap N_{1,j+1}. \]

Repeating this substitution, we finally arrive at (3.22).

**Lemma 3.10.** If D holds, then \( \mu = \lambda \).

**Proof.** We first specialize (3.22) to the case \( j = 0 \). Thus, since \( N_{1,-1} = \mathbb{E}^n \),

\[ \dim N_{0, k+1} = \dim N_0 + \sum_{i=0}^{k} \dim N_0 \cap N_{1i}, \]

from which it is clear that

\[ \dim N_{0, k+1} = \dim N_{0k} + \dim N_0 \cap N_{1k}. \]

(Note that this is also valid for \( k = -1 \).) Since \( N_{0k} \subset N_{0, k+1} \), we see that \( N_{0, k+1} = N_{0k} \) if and only if \( N_0 \cap N_{1k} = \{ 0 \} \). By Lemma 3.8 we then see that \( N_{0, k+1} = N_{0k} \) if and only if \( k \geq \lambda \). But by (3.1) this is equivalent to \( \mu = \lambda \).

**Lemma 3.11.** If D holds, then \( E^n = N_{0a} \oplus N_{1a} \) and \( E^n = R_{1a} \oplus R_{0a} \). Moreover, \( A_1 \) maps \( N_{0a} \) one-to-one onto \( R_{1a} \), \( A_0 \) maps \( N_{1a} \) one-to-one onto \( R_{0a} \), and the inverses of these maps are linear transformations.

**Proof.** This is trivially true if \( \mu = -1 \), so we suppose \( \mu \geq 0 \). Then in (3.22) we take \( k = \mu - 1 \) (note that (3.22) holds trivially for \( k = -1 \)) and \( j = \mu + 1 \) and we get \( \dim N_{0a} \cap N_{1a} = \sum_{i=\mu}^{2\mu} \dim N_0 \cap N_{1i} = 0 \) by Lemmas 3.8 and 3.10. Thus with \( k = \mu \) in (3.19) we have \( E^n = N_{0a} \oplus N_{1a} \). From (3.7) we have \( \dim R_{1a} \cap R_{0a} = \dim N_{0a} \cap N_{1a} = 0 \) since \( N \cap N_{0a} = \{ 0 \} \) by condition D. Thus with \( k = \mu \) in (3.18) we have \( E^n = R_{1a} \oplus R_{0a} \). Now \( R_{1a} = A_1 N_{0a} \) by (2.1), and since \( N_1 \cap N_{0a} = \{ 0 \} \) this mapping from \( N_{0a} \) to \( R_{1a} \) is one-to-one. By (2.2) we have \( A_0 N_{1a} = R_{0a} \) and by Lemmas 3.8 and 3.10 we have \( N_0 \cap N_{1a} = \{ 0 \} \), so this mapping from \( N_{1a} \) to \( R_{0a} \) is also one-to-one. The linearity of the inverses follows, of course, from that of the direct mappings.
4. SUFFICIENCY OF CONDITION D AND CONSTRUCTION OF THE LAURENT SERIES

Throughout this section we shall assume that condition D holds even when this is not mentioned explicitly. The inverses of the maps mentioned in Lemma 3.11 may be extended in various ways to the complementary spaces $R_{0,n}$ and $R_{1,n}$, respectively, so as to define linear transformations on $E^n$ into $E^n$. In particular, by virtue of Lemma 3.11, two $n \times n$ matrices $Q_0$ and $Q_{-1}$ are uniquely defined by the following rules.

**Definition 4.1.**

\[
Q_0y = \begin{cases} 
0, & \text{if } y \in R_{1,n} \\
X, & \text{if } y \in R_{0,n} \text{ and } y = A_0x \text{ with } x \in N_{1,n}.
\end{cases} \tag{4.1}
\]

\[
Q_{-1}y = \begin{cases} 
0, & \text{if } y \in R_{0,n} \\
X, & \text{if } y \in R_{1,n} \text{ and } y = A_1x \text{ with } x \in N_{0,n}.
\end{cases} \tag{4.2}
\]

We now develop some basic properties of the matrices $Q_0$ and $Q_{-1}$ in relation to $A_0$ and $A_1$.

**Lemma 4.1.** The matrix $Q_0A_0$ effects the projection onto $N_{1,n}$ along $N_{0,n}$, and $Q_{-1}A_1$ the projection onto $N_{0,n}$ along $N_{1,n}$. Consequently,

\[
Q_0A_0 + Q_{-1}A_1 = I. \tag{4.3}
\]

**Proof.** If $x \in N_{0,n}$, then $A_0x \in R_{1,n}$ since $N_{0,n} = N_{0,n}^{a+1}$, so by (4.1) we see that $Q_0A_0x = 0$. If $x \in N_{1,n}$, then $A_0x \in R_{0,n}^{a+1} = R_{0,n}$ so $Q_0A_0x = x$ by (4.1). Thus $Q_0A_0$ projects onto $N_{1,n}$ along $N_{0,n}$ since $E^n = N_{0,n} \oplus N_{1,n}$. A similar argument shows that $Q_{-1}A_1$ projects onto $N_{0,n}$ along $N_{1,n}$ and (4.3) then follows.

**Lemma 4.2.** For any integers $k \geq 0$ and $j \geq 0$,

\[
Q_{-1}A_0(N_{0,n} \cap N_{1,n-j}) = N_{0,n} \cap N_{1,n-j}. \tag{4.4}
\]

**Proof.** By (3.3) we see that (4.4) is equivalent to

\[
Q_{-1}(R_{1,n} \cap R_{0,j}) = N_{0,n} \cap N_{1,n-j}. \tag{4.5}
\]

Now suppose $y \in R_{1,n} \cap R_{0,j}$. Then $y = A_1x$ for some $x \in N_{0,n} \subset N_{0,n}$. 

and, since $R_{1,k-1} \subseteq R_{1,0}$, then $x = Q_{-1}y$ by (4.2). Since also $A_1x = y \in R_{0,j}$, then $x \in N_{1,j}$ so $Q_{-1}y \in N_{0,k-1} \cap N_{1,j}$. On the other hand, if $x \in N_{0,k-1} \cap N_{1,j}$, then $y = A_1x \in R_{1,k-1} \cap R_{0,j}$ and $x = Q_{-1}y$ by (4.2).

**Definition 4.2.** For $k = 1, 2, 3, \ldots$ define

$$Q_{-(k+1)} = (-Q_{-1}A_0)^k Q_{-1}. \quad (4.6)$$

**Lemma 4.3.** For all integers $k \geq 1$,

$$Q_{-k}A_0 + Q_{-(k+1)}A_1 = 0. \quad (4.7)$$

**Proof.** By (4.6) and (4.3) we have

$$Q_{-2}A_1 = Q_{-1}A_0 Q_{-1}A_1 = Q_{-1}A_0 + Q_{-1}A_0 A_0 A_0.$$  

Lemma 4.1 and the definitions of $\mu$ and $Q_{-1}$ imply that $Q_{-1}A_0 Q_0 A_0 = 0$, and we see that (4.7) holds for $k = 1$. Completion of the induction proof is easy.

**Lemma 4.4.** $Q_{-(\mu+1)} \neq 0$, but $Q_{-k} = 0$ for $k \geq \mu + 2$.

**Proof.** We first show that

$$Q_{-(k+1)}E^n = N_{0,\mu-k} \cap N_{1,k-1}, \quad k = 0, 1, \ldots, \mu. \quad (4.8)$$

Indeed this is true for $k = 0$ by (4.2) since $N_{1,-1} = E^n$, $A_1$ maps $N_{0\mu}$ onto $R_{1,0}$, and $E^n = R_{1,0} \oplus R_{0,\mu}$. Furthermore, if (4.8) holds for some $k$, such that $0 \leq k \leq \mu - 1$, then using (4.6) and (4.4) we find that it also holds with $k$ replaced by $k + 1$, so (4.8) is established. Now, if we set $k = \mu$ in (4.8), we get $Q_{-(\mu+1)}E^n = N_0 \cap N_{1,\mu-1}$. By Lemma 3.8 and the fact that $\mu = \lambda$, we see that $Q_{-(\mu+1)} \neq 0$. However, $A_0 Q_{-(\mu+1)}E^n = \{0\}$, so $Q_{-(\mu+2)} = -Q_{-1}A_0 Q_{-(\mu+1)} = 0$, and from (4.6) we see that $Q_{-k} = 0$ for all $k \geq \mu + 2$.

**Definition 4.3.** For $k = 1, 2, 3, \ldots$ define

$$Q_k = (-Q_0 A_1)^k Q_0. \quad (4.9)$$

**Lemma 4.5.** For all integers $k \geq 0$,

$$Q_{k+1}A_0 + Q_k A_1 = 0. \quad (4.10)$$
Proof. By (4.9) and (4.3) we have

\[ Q_1 A_0 = -Q_0 A_1 Q_0 A_0 = -Q_0 A_1 + Q_0 A_1 Q_1 A_1. \]

For any \( x \in \mathbb{R}^n \) we have \( Q_{-1} A_1 x \in N_{0_k} \) by Lemma 4.1. Hence \( A_1 Q_{-1} A_1 x \in R_{1_1} \), so \( Q_0 A_1 Q_{-1} A_1 x = 0 \) by (4.1). Thus \( Q_0 A_1 Q_{-1} A_1 = 0 \), so (4.10) holds for \( k = 0 \). Now, if it holds for some \( k \geq 0 \), then by (4.9)

\[ Q_{k+1} A_0 = (-Q_0 A_1)Q_{k+1} A_0 = (-Q_0 A_1)(-Q_0 A_1) = -Q_{k+1} A_1, \]

so (4.10) holds for all \( k \geq 0 \) by induction.

We can now prove the sufficiency of condition D for the invertibility of \( A_0 + zA_1 \) in some deleted neighborhood of \( z = 0 \).

**Theorem 4.1.** If \( N_1 \cap N_{0_k} = \{0\} \) for all integers \( k \geq 0 \), then

\[ Q(z) = \sum_{k=-(p+1)}^\infty z^k Q_k \]

(4.11)

converges for \( 0 < |z| < \delta \) for some \( \delta > 0 \). Moreover, for \( z \) in this range \( Q(z) = (A_0 + zA_1)^{-1} \).

Proof. From (4.9) we see that, for \( k \geq 0 \) and for any matrix norm (in the operator sense),

\[ ||z^k Q_k|| = |z|^k ||Q_k|| \leq |z|^k \cdot ||Q_0 A_1||^k \cdot ||Q_0||. \]

(4.12)

Thus, if \( ||Q_0 A_1|| \neq 0 \), then the series (4.11) converges for \( 0 < |z| < \delta = ||Q_0 A_1||^{-1} \) while, if \( ||Q_0 A_1|| = 0 \), the same is true with any \( \delta > 0 \). Now for \( N > 0 \) we have

\[ \left( \sum_{k=-(p+1)}^N z^k Q_k \right) (A_0 + zA_1) \]

\[ = z^{-(n+1)} Q_{-(n+1)} A_0 + \sum_{k=-\mu}^N z^k (Q_k A_0 + Q_{k-1} A_1) + z^{N+1} Q_N A_1 \]

\[ = I + z^{N+1} Q_N A_1 \]

by (4.7), (4.3), (4.10), and Lemma 4.4. Now, if \( 0 < |z| < \delta \), we see from (4.12) that \( z^{N+1} Q_N A_1 \to 0 \) as \( N \to \infty \), so for such \( z \) we have \( Q(z) (A_0 + zA_1) = I \) and \( Q(z) = (A_0 + zA_1)^{-1} \) as claimed.
5. SOME EXAMPLES

Let $A_0 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $A_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Then $N_0 = N_{00}$ is spanned by $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $N_1$ is spanned by $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, so $N_1 \cap N_{00} = \{0\}$. But $R_{10} = A_1 N_{00}$ is spanned by $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, so $N_{01}$ is all of $E^2$ and $N_1 \cap N_{01} \neq \{0\}$. This illustrates that condition D may not hold even though $N_1 \cap N_0 = \{0\}$. Note that $N_{10} = E^2$ in this example, so $R_{01} = R_{00} \neq E^2$ and $\lambda = 0$. Since $N_{00} \neq N_{01}$, we have $\mu = 1$. (See Lemma 3.10.)

As another example, take $A_1 = I$ and

$$A_0 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}.$$  

In this case $(A_0 + zA_1)^{-1} - z^{-2}A_0^2 - z^{-2}A_0 + z^{-1}I$ for all $z \neq 0$. The basic subspaces can easily be determined by the reader. Note that here $\mu + 1 = \text{rank } A_1$, but with $\overline{A}_0 = A_0^2$ we have $(\overline{A}_0 + zA_1)^{-1} = -z^{-2}A_0 + z^{-1}I$ and for this $\mu + 1 < \text{rank } A_1$. Thus $\mu$ is not determined merely by the rank of $A_1$.

Finally consider the case

$$A_0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & -1 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}.$$  

We have

$$(A_0 + zA_1)^{-1} = z^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

for which $\mu + 1 = 1$. For this $A_0$ and the previous $\overline{A}_0$ we have rank $A_0 = \text{rank } \overline{A}_0$, and we see that $\mu$ is not determined merely by the rank of $A_0$ either.

REFERENCE


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