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Double diffusive Darcy flow induced by a spherical source

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Abstract A simple mathematical theory is proposed for the study of the free convective mass transfer flow induced by a spherical source in an unbounded porous medium assuming the validity of the Darcy flow model. Besides generating heat, the source generates a chemical substance too at a constant rate. Assuming the heat generation rate not excessive, an exact analytical solution is obtained for the flow field using the method of superposition for the determination of the temperature and concentration fields. The significance of the impact of the species concentration gradients upon the thermally driven flow has been highlighted. The results are delineated by comparing them with those of a point source and the evolution of the flow field is contrasted with that due to the presence of a heated sphere. Solutions for sources with spherical geometry could be deduced algebraically from the results of this study.

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1. Introduction

Among the thermal engineering applications which benefit from a better understanding of the fundamentals of heat and fluid flow in a porous medium are thermal insulations, geothermal systems, cooling of nuclear reactors and underground disposal of nuclear waste. In particular, those concerning subsea-bed disposal of nuclear waste attract greater attention in view of their growing relevance to the successful containment of the transport of radio-nuclide into the water columns and several results are available in the literature modeling the situation with heat sources buried in unbounded porous media. In most of the existing published studies [1–7], the source is assumed to be concentrated at one point although in practice, the nuclear waste is first encapsulated in a suitably designed container before being implanted into the sedimentary layer below the subsea-bed [4,5]. Hence, to have a better understanding of the complexity of the situation, a different modeling, other than that of a concentrated point source, may be needed in order to have a comprehensive estimate of the overall convection effects, the most appropriate one with spherical geometry being that of a spherical source, which indeed is the focus of our investigation in the present study. Motivated by the significance of understanding such free convection flows, Hardee [8]

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investigated the phenomenon of buoyancy-driven thermal convection due to the presence of a spherical source in an unbounded porous medium using a boundary-layer approximation while, Hodgkinson [9] investigated the same associated with an idealized waste depository in a permeable hard rock mass. With the exception of these two works no other work seems to be available in the literature on such free convection flows due to the presence of spherical sources in porous media. Of course, a few other papers are available in the literature using spherical geometry [10–17], but they essentially deal with heated spheres only. In this context, the work of Ganapathy [18] deserves mention.

In most practical situations, quite often, species concentration gradients greatly affect the flow and as a result they play a decisive role in the development of the flow field [19]. Even though the importance of this class of problems has been to some extent established in the literature [20–24], yet, those concerning double diffusion due to spherical sources have remained largely overlooked which indeed has motivated our interest in the present topic. The analysis of such flows is essential for the solution of many engineering problems such as the spreading of pollutants created by an exothermic reaction at an underground site. Besides its importance in geophysics, the problem finds its applications in chemical engineering too. Although the broader problem of double diffusion depends on two Rayleigh numbers, one based on the thermal field and the other on the concentration field, yet for simplicity, we formulate the problem in such a way that the impact of the species concentration gradients upon the thermally driven flow is measured by a parameter $N$ [19]. It may be noted that all discussions in this work focus on the case where the net flow is upward.

It may be mentioned here that Jaballah et al. [22], by considering the mixed convection in a channel with porous layers and using a thermal non-equilibrium model have shown that in respect of the thermal equilibrium model, the quantity of heat is preserved within the channel for a fluid injected with at velocity well determined and a heat flow imposed, so that the quantity of heat collected by the solid particles is yielded to the fluid particles without loss which is in conformity with our assumption in the present work.

In conclusion, the main goal of this article is to investigate analytically the effect of species diffusion on the buoyancy induced free convective mass transfer flow from a spherical source in an unbounded porous medium assuming the validity of the Darcy flow model. Assuming the heat generation rate not excessive, an exact analytical solution is obtained for the flow field using the method of superposition for the determination of the temperature and concentration fields around the spherical source. As a summary of what is presented below, the mathematical problem is formulated in Section 2 and the method of solution is presented in Section 3, with a discussion on the flow field in Section 4. Finally, we conclude the study with a review of the results obtained.

2. Mathematical formulation

We consider the free convective mass transfer flow around a continuous spherical source of radius $a$, buried in an unbounded porous medium of low permeability, from which a quantity $pcQ$ of heat is liberated together with a substance at the rate $m$ [kg s$^{-1}$], where $p$ is the fluid density, $c$ the specific heat at constant pressure and $Q$, the thermal energy of the source. The medium is assumed to be rigid, homogeneous and isotropic and the fluid saturating the medium Boussinesq incompressible with the density–temperature–concentration relation:

$$\rho = \rho_\infty \{ 1 - (\beta T + \beta_c C) \},$$

(1)

where $(\beta, \beta_c)$ are respectively the volumetric coefficients of thermal and concentration expansions, $(T, C)$ are the temperature and species concentration in excess of the ambient and the subscript $\infty$ denoting a reference state. An illustration of the physical setup with the spherical source embedded in the unbounded porous medium together with the configuration of interest is given in Fig. 1. A spherical-polar co-ordinate system $(r, \phi, \theta)$ is chosen with the origin at the center of the spherical source and the axis $\phi = 0$ vertically upward and parallel but opposite to the gravity vector. Consequently, as the problem is symmetrical in the angular direction $\theta$, neither $\theta$ nor the $\theta$-component of velocity appears in the analysis. Thus, assuming the Darcy flow model, we have for the conservation of mass, momentum, energy and species concentration in the medium in the absence of dispersion [26]:

$$\frac{\partial (r^2 u \sin \phi)}{\partial r} + \frac{\partial (rv \sin \phi)}{\partial \phi} = 0,$$  \hspace{1cm} \text{(2)}

$$u = -(K/\mu)(\partial P/\partial r + pg \cos \phi),$$  \hspace{1cm} \text{(3)}

$$v = -(K/\mu)(r^{-1}\partial P/\partial \phi - pg \sin \phi),$$  \hspace{1cm} \text{(4)}

$$\alpha \delta T/\partial t + u \delta T/\partial r + (v/r) \delta T/\partial \phi = z \Delta T,$$  \hspace{1cm} \text{(5)}

$$\varepsilon \delta C/\partial t + u \delta C/\partial r + (v/r) \delta C/\partial \phi = D \Delta C,$$  \hspace{1cm} \text{(6)}

where $(u, v)$ are the radial and transverse components of the mean filtration velocity of the fluid, $g$ the acceleration due to gravity, $t$ the time, $K$ the medium permeability, $\mu$ the coefficient of viscosity, $P$ the fluid pressure in excess of its hydrostatic value, $\varepsilon$ the porosity of the porous matrix, $z$ the effective thermal diffusivity of the medium which is equal to the ratio of thermal conductivity $k$ of the porous medium filled with the stagnant fluid divided by the heat capacity $(pc)\varepsilon$ of the

![Figure 1](image-url)

Figure 1  Physical setup of the spherical source embedded in an unbounded porous medium. Configuration of interest: Spherical-polar co-ordinate system $(r, \phi, \theta)$ with the origin at the center of the spherical source.
fluid, $D$ the species diffusivity in the medium when filled with
the fluid, $\sigma$ is the heat capacity ratio given by
\[
\sigma = \varepsilon + (1 - \varepsilon)(\rho_c)\, / (\rho_c),
\]  
(7a)
\[
\Delta = r^{-2}((\partial^2/\partial r^2)(r^2\partial T/\partial r) + (\sin \phi)^{-1}(\partial/\partial \phi)(\sin \phi \partial/\partial \phi))
\]  
(7b)
and the subscripts $s$ and $f$ referring to the solid and fluid phase
respectively. In writing the above equations we assume that the
medium and the saturating fluid are in thermal equilibrium
and that all physical quantities are constant except in the buoy-
cy term. The initial and boundary conditions necessary for
the completion of the mathematical formulation are as follows:
u = 0, \quad \phi = 0, \quad T = 0, \quad C = 0 \text{ at } t = 0,
u = 0, \quad \phi = 0, \quad \phi = 0 \quad \text{as } r \to \infty, \quad \quad \quad \quad \quad (8a - c)
Unlike with the case of a point source, here we do not encounter
any singularity. However, the spherical geometry of the
present configuration requires, for the heat and concentration
balance:
\[
\lim_{a \to 0} (4\pi r^2)((\partial T/\partial r)] = Q,
\]
\[
\lim_{a \to 0} (4\pi r^2)((\partial C/\partial r)] = m.
\]  
(9)
with $\Omega$ representing the radial distance of a fluid particle at $(r, \phi, \theta)$ $(r > a)$ from the point source at $(a, \phi, \theta)$ or the one at $(a, \pi + \phi, \theta)$, either concurrently or otherwise.

3. Method of solution

Providing the heat generation rate not excessive, the time
taken to set up the temperature and consequently the concen-
tration fields will be smaller than that required for the flow
field to fully develop so that the convective terms in the conser-
vation equations of energy and species concentration may be
neglected in preference to the local derivative terms. Conse-
quently, Eqs. (5) and (6) reduce to:
\[
\sigma \partial T/\partial t = \kappa \Delta T,
\]  
(10a)
\[
\partial C/\partial t = \Delta C.
\]  
(10b)
As an exact analytical solution of (10) is directly not plausible
for the case of the continuous spherical source, following the
method adopted by Hicko [4], we determine the solution of
(10) by treating the spherical source in the first instance as
an assemblage of an infinitely large number of instantaneous
point sources of equal strength, each besides generating heat
at the rate $Q/4\pi a^2$ generates a chemical substance too at a con-
stant rate $m/4\pi a^2$ and then apply the method of superposition.
Determines the temperature and concentration fields due to
the presence of an instantaneous spherical source from which
we deduce the solution for the determination of the tempera-
ture and concentration fields around a continuous spherical
source. We first determine the thermal field and using the
same analogy we determine the concentration field. Here we find
it convenient to treat the problem in its dimensional form.

It is well known from the theory of heat conduction that the
temperature distribution due to the presence of an instantan-
eous point heat source placed at the origin of an unbounded
homogeneous porous medium is found from the solution of the
equation:
\[
\sigma \beta T/\beta t = \kappa r^{-2}((\partial^2/\partial r^2)(r^2\beta T/\partial r)) + (Q/4\pi a^2)\beta(t)\beta(r),
\]  
(11)
where $\beta(t)\beta(r)$ is the Direc-Delta function satisfying the relation:
\[
\int_0^\infty \int_0^\infty \beta(t)\beta(r)drdt = 1,
\]  
(11b)
the volume $V$ including the point heat source. With the appli-
cation of Laplace Transform, one obtains from (11) (which we
shall denote by $T_1$):
\[
T_1 = (Q/32\pi a^2)(\pi kt)^{-3/2}\exp[-r^2/4\pi kt]. \quad (\kappa = \kappa/a)
\]  
(12)
Instead, if the source is placed at the location $(a, \phi, \theta)$, then, for
the temperature distribution due to the presence of this
instantaneous point heat source of strength $(Q/4\pi a^2)$ placed
at $(a, \phi, \theta)$ on the surface of the sphere associated with a
surface element $a^2\sin \phi d\phi d\theta$ subject to the same conditions as
before, we have (which we shall denote by $T_2$):
\[
T_2 = (Q/32\pi a^2)(\pi kt)^{-3/2}\exp[-(r^2 + a^2 - 2ar \cdot \cos \phi)/4\pi kt],
\]  
(13)
which on integration over the surface of the sphere yields the
solution for the temperature distribution due to the presence
of an instantaneous spherical source of strength $Q$ (which we
shall denote by $T_3$):
\[
T_3 = Q/(8\pi a^2)^{-1}(\pi kt)^{-1/2}\exp[-(r-a)^2/4\pi kt] - \exp[-(r+a)^2/4\pi kt].
\]  
(14)
The expression on the right side of (14) on integration over a
time interval $t$, finally yields the solution for the temperature
distribution due to the presence of a continuous spherical
source of strength $Q$ embedded in the unbounded porous
medium [27]:
\[
T = Q/(8\pi a^2)^{-1}(\sqrt{\pi kt})\exp[-(r-a)^2/4\pi kt] - \exp[-(r-a)^2/4\pi kt] + (r+a)\exp[(r+a)/2\sqrt{\pi kt}], (\kappa = \kappa/a)
\]  
(15)
With the same analogy, we have for the concentration field:
\[
C = m/(8\pi a^2)^{-1}(\sqrt{\pi kt})\exp[-(r-a)^2/4\pi kt] - \exp[-(r+a)^2/4\pi kt] + (r+a)\exp[(r+a)/2\sqrt{\pi kt}], (\kappa_m = D/c)
\]  
(16)
It is easily verified that (9) is identically satisfied by (15) and
(16).

The flow field:

In view of (2) one may now define a stream function $\psi$ such
that
\[
u = -r^{-2} \cdot \partial \psi/\partial \zeta, \quad \nu = -(r \sin \phi)^{-1} \cdot \partial \psi/\partial \phi,
\]  
(17)
with $\zeta = \cos \phi$. We introduce the non-dimensional quantities
according to [9]:
\[
R = r/L, \quad \tau = kt/L^2, \quad \Psi = \psi/(\pi L^2),
\]  
(18)
where $L$ is the characteristic length scale, and eliminate the
pressure terms in (3) and (4) through cross differentiation.
The physical impact of this scaling is to bring out the main
non-dimensional parameters that determine the characteristics
of the ensuing flow field and heat transfer, besides reducing the
number of free parameters. In view of our hypothesis that the temperature and concentration fields remain unaffected by fluid motion, we find that the flow field is now adequately described by the Darcy–Oberbeck–Boussinesq equation:

\[
\frac{\partial \Psi}{\partial R} + \left[ (1 - \zeta^2) / R^2 \right] \frac{\partial^2 \Psi}{\partial \zeta^2} = R_s \cdot \left( (1 - \zeta^2) R \left[ \frac{\partial \Theta}{\partial R} \right] + N \left( \frac{\partial C^*}{\partial R} \right) \right),
\]

(19)

where

\[ R_s = (\beta g L \sqrt{K}/\nu v) Q \] (the thermal Rayleigh number), \( N = \beta m k / (\beta Q D) \) (sustentation parameter)

and \( \nu \), the kinematic viscosity of the fluid. In writing the above equations we have assumed all quantities are constant except in the buoyancy term and that the fluid flow is governed by the Darcy flow model. In what follows, the asterisk in \( C \) is dropped for convenience. The initial and boundary conditions necessary for the determination of flow field in the non-dimensional form:

\[ U = 0, \quad V = 0, \quad \Theta = 0, \quad C = 0 \text{ at } \tau = 0, \]

\[ U \to 0, \quad V \to 0, \quad \Theta \to 0, \quad C \to 0 \text{ as } R \to \infty, \]

\[ \frac{\partial U}{\partial \varphi} = V = 0 \text{ at } \varphi = 0, \quad \pi \text{ on } R = R_0, \]

(20a–c)

where \( R_0 = (a/L) \) is the non-dimensional radius of the sphere. In this non-dimensional formulation (15) and (16) reduce to:

\[ \Theta = (8\pi R_0)^{-1} \chi(\eta, \eta_0), \]

(21a)

\[ C = (8\pi R_0)^{-1} \chi(A \eta, A \eta_0), \]

(21b)

where \( \eta = R/(2\sqrt{\tau}), \eta_0 = R_0/(2\sqrt{\tau}), A = (\kappa/\kappa_0)^{1/2} \) and

\[ \chi(\eta, \eta_0) = \eta^{-1} \left\{ \left( 1 / \sqrt{\pi} \right) \exp\left( - (\eta - \eta_0)^2 \right) - \exp\left( - (\eta + \eta_0)^2 \right) \right\} - \eta_0 \operatorname{erfc}(\eta - \eta_0) + \eta \operatorname{erfc}(\eta + \eta_0) + (\eta + \eta_0) \operatorname{erfc}(\eta + \eta_0). \]

(21c)

Separation of variables in (19) is achieved by setting

\[ \Psi = (2\pi R_0)^{-1} \tau \cdot (1 - \zeta^2) \cdot f(\eta). \]

(22)

This then leads to an ordinary differential equation of the second order for the determination of \( f \):

\[ \eta^2 f''(\eta) - 2f(\eta) = \sigma(\eta, \eta_0) + (N/A^2) \cdot \sigma(A \eta, A \eta_0), \]

(23)

where

\[ \sigma(\eta, \eta_0) = -\left( \eta / \sqrt{\pi} \right) \left\{ \exp\left( - (\eta - \eta_0)^2 \right) - \exp\left( - (\eta + \eta_0)^2 \right) \right\} - \eta_0 \operatorname{erfc}(\eta - \eta_0) + \eta \operatorname{erfc}(\eta + \eta_0), \]

(24)

and the primes denoting differentiation with respect to \( \eta \). The general solution of (23) is:

\[ f(\eta) = c_1 \eta^2 + c_2 / \eta + (1/3) \left\{ G(\eta, \eta_0) + (N/A^2) G(A \eta, A \eta_0) \right\}, \]

(25)

where

\[ G(\eta, \eta_0) = \left( 1 / \sqrt{\pi} \right) \left\{ (\eta - 1/2) \eta^{-1} \right\} \left\{ \exp\left( - (\eta - \eta_0)^2 \right) - \exp\left( - (\eta + \eta_0)^2 \right) \right\} + \eta \left( \eta \eta_0 \right) \operatorname{erfc}(\eta - \eta_0) + \eta_0 \operatorname{erfc}(\eta + \eta_0) \]

+ \left\{ (1/2) \eta_0 \eta_0 \right\} \left\{ \operatorname{erfc}(\eta - \eta_0) + \operatorname{erfc}(\eta + \eta_0) \right\}. \]

(26a)

The condition that the fluid velocity \( q \to 0 \) as \( R \to \infty \) ensures \( c_1 = 0 \) and the condition that the surface of the spherical source is a natural boundary ensures

\[ \begin{align*}
(2a) & \quad \text{Spherical source } (R_0 = 1). \text{ Transient streamlines } (2\pi)(\Psi/R_0) = \text{const. } (A = 0.4, N = -0.16) \text{ at } \tau = 0.1. \\
(2b) & \quad \text{Spherical source } (R_0 = 1). \text{ Transient streamlines } (2\pi)(\Psi/R_0) = \text{const. } (A = 0.4, N = -0.16) \text{ at } \tau = 0.2. \\
(2c) & \quad \text{Spherical source } (R_0 = 1). \text{ Transient streamlines } (2\pi)(\Psi/R_0) = \text{const. } (A = 0.4, N = -0.16) \text{ at } \tau = 1.0.
\end{align*} \]
\[ c_2 = -\frac{1}{3} \eta_0 \{ G(\eta, \eta_0) + (N/A^2)G(A, A\eta_0) \} \text{ as } R \to R_0. \]

Thus,
\[ \Psi = R_0 \left( 12\pi \eta_0 \right)^{-1} \sqrt{\tau} \left( 1 - \zeta^2 \right) \{ \xi(\eta, \eta_0) + (N/A^2) \cdot \xi(A, A\eta_0) \}, \]

where
\[ \zeta(\eta, \eta_0) = G(\eta, \eta_0) - (\eta_0/\eta)G(\eta_0, \eta_0). \]

4. Discussion

Among the various parameters that have entered into the solution, namely, \( R_0, N \) and \( A \), the parameter \( A \) is of special significance. Since \( A \) is the ratio of two length scales namely, the ratio of thermal penetration length \( O[(\pi/\sigma)^{1/2}] \) to that of species penetration \( O[(Dt/e)^{1/2}] \), values of \( A \) of order less than one imply \( (Dt/e) > (\pi/\sigma) \), so that chemical species diffuse faster than that of heat especially when the buoyancy mechanisms are opposed. Consequently, outside the region in which the thermal effect of the source is felt, species concentration gradients act alone creating a downward flow. Hence, with a view to determine the impact of the parameter \( A \) on the convective flow pattern when the buoyancy mechanisms are opposed, using Eq. (27), we draw the maps of the transient streamlines \( (12\pi \ R_0)(\Psi/R_0) = \text{const.} \) in Fig. 2 for three different times choosing in illustration \( R_0 = 1, A = 0.4 \) and \( N = -0.16 \). The range of values of \( A \) we have chosen is valid for a variety of physical and laboratory models, for instance, in the case of diffusion of methanol and hydrogen in water saturated glass beads, \( (e/r) \approx 1.2, Le \approx 0.3 \) so that \( A \approx 0.36 \). Furthermore, the choice of \( N \) is in conformity with the condition that it must be negative when the buoyancy mechanisms are opposed [19] and for the sake of simplicity we have chosen \( N = -0.16 \) so that \( N/A^2 = -1 \). It is deducible from the figures that there is a bifurcation of the flow field such that, the streamlines close to the source rotate clockwise whereas, those in the opposite region rotate anticlockwise giving life for a downward flow.

Figure 3  (a) Maps of the radial velocity profiles \( 100U/\cos \phi \) for different values of \( \tau (A = 0.4, N = -0.16) \). (b) Maps of the radial velocity profiles \( 100U/\cos \phi \) for different values of \( N(A = 0.4, \tau = 1.0) \). (c) Variations in temperature/concentration fields at different special positions \( R (A = 1) \). Maps represent \( 80\pi \left( \Theta_0 \text{ or } C_0 \right) \).
Further, as soon as the source’s energy is released, there is a concentration of the velocity field around the spherical source with the streamlines of the circulating flow field coming close together in its vicinity and the geometry changes radically as $\tau$ increases. However, at all times, not only the symmetry of the streamlines about the plane $\phi = \pi/2$ is preserved but also the flow field remains laminar and there is no accumulation of heat and concentration into the flow field. The rate of propagation of the circulating flow field is found to be slow in relation to the rate of its growth and the entire process is dominated by viscosity coupled with species and thermal diffusion. As $\tau$ increases, the flow pattern present near the source spreads outward driving the enveloping downward flow to move farther away from the source so that the flow field for large times resembles that due to the presence of a heated sphere. The streamlines of the circulating flow field enclose the stagnation points of the fluid motion (points of max. sphere. The streamlines of the circulating flow field enclose the stagnation points of the fluid motion (points of max. W ¼ 1/2 for a stagnation point at $R = R_0 (R > R_0)$. For the parameters under consideration, these stagnation points are found to be at $R = 1.4, 1.5$ and 2.2 when $\tau = 0.1, 0.2$ and 1.0 respectively. Clearly, the phenomenon described above is a property of the transient state for otherwise the heating effect of the source would be felt throughout the porous space.

The profiles in Fig. 3(a) show that at any time, the radial velocity increases very rapidly in the vicinity of the source and after attaining its maximum value, it starts decreasing at an ever decreasing rate to zero farther away from the source. Similarly, at any special position, the radial velocity increases with time and as a result, the rate of momentum transfer is much quicker and faster in the vicinity of the source than elsewhere, a phenomenon that is more pronounced in respect of positive values of $N$ (Fig. 3b). This is otherwise expected since, relative to the no species generation limit $N = 0$, positive values of $N$ aid convection whereas negative values tend to weaken the same, for here, the generated substance acts as a break on the thermally induced flow. The temperature and concentration profiles drawn in Fig. 3(c) show that at any given time and spatial position in the neighborhood of the source, the attainment of maximum temperature and concentration is achieved much earlier than that of the maximum velocity and consequently, the time taken to set up the thermal and concentration fields is relatively shorter than that required for the flow field to fully develop, a phenomenon that is characteristic of fluid motion at small Rayleigh numbers. Though the magnitude of temperature decreases with distance, at any special position, it is relatively higher than that of radial velocity.

When $N = 0$, we recover the solution for the case of pure thermal convection:}

$$
\Psi = R_0 ((12\pi n_0)^{-1} \sqrt{\tau} (1 - \zeta^2)^2 \xi (\eta, n_0)), \quad (28)
$$

and from the associated maps of the streamlines $(12\pi R_0) (\Psi / R_0) = \text{const.}$ drawn in Fig. 4 with $R_0 = 1$, using (28), it is evident that excepting the downward flow, most of the essential features of the flow remain almost the same as before and the stagnation points in this case are found to be at $R = 1.6, 1.7$ and 2.5 when $\tau = 0.1, 0.2$ and 1.0 respectively. In this context, additional comments are perhaps in order regarding the present work and those of Hardee [8] and Hodgkinson [9]. The boundary-layer solution for the free convection surrounding the spherical heat source obtained by Hardde is primarily concerned with the determination of the physical properties of the medium especially the permeability in some cases and as such there is no discussion on the evolution of the flow field. Using a configuration identical to the one we have in the present paper and with a regular perturbation technique in the limit of small Rayleigh numbers, Hodgkinson obtained an analytical solution for the first convective correction to the flow field assuming the heat generation rate uniform through-

![Figure 4](image-url)  
(a) Spherical source ($R_0 = 1$). Pure thermal convection. Maps of transient streamlines $(12\pi \Psi / R_0) = \text{const.} (\tau = 0.1)$. (b) Spherical source ($R_0 = 1$). Pure thermal convection. Maps of transient streamlines $(12\pi \Psi / R_0) = \text{const.} (\tau = 0.2)$. (c) Spherical source ($R_0 = 1$). Pure thermal convection. Maps of transient streamlines $(12\pi \Psi / R_0) = \text{const.} (\tau = 1.0)$.  

out the volume of the spherical source that decays with time. In order to avoid singularity at the origin, an additional constraint namely, \( (\psi_r)_r \) is finite at the origin, has been imposed. As a result, the stream function is expressed as a combination of two functions, one representing the inner solution valid for \( r < a \) and the other, the outer solution valid for \( r > a \). Consequently, the streamlines drawn in [9] are convex as against the concave ones presented in this paper.

As the radius of the spherical source shrinks, that is, smaller the radius of the sphere, lesser is the concavity of the streamlines in its vicinity and in the limit \( R_0 \to 0 \), the streamlines instead of being concave tend to be convex so that the situation becomes identical with that due to the presence of a point source. With a view to validating this, we set \( R_0 \to 0 \), and obtain from (21) and (27) using l’Hospital rule or otherwise:

\[
\begin{align*}
\Theta &= (4\pi R)^{-1} \text{erfc}\eta, \quad (29a) \\
C &= (4\pi R)^{-1} \text{erfc}(A\eta), \quad (29b) \\
\Psi &= R_0(12\pi)^{-1} \sqrt{\tau} \cdot (1 - \zeta^2) \{\zeta(\eta) + (N/A)\zeta(A\eta)\}, \quad (29c)
\end{align*}
\]

where \( \zeta(\eta) = 3[\eta \text{erfc}\eta + (1/2)\eta^{-1} \text{erf}\eta - (1/\pi) \exp(-\eta^2)] \).

These results were previously obtained by Poulikakos [23] in the limit of small Rayleigh number, for the first convective correction to the flow field, of course, with a different non-dimensionalization scheme for the radial distance. We exemplify the evolution of the flow pattern due to the presence of the point source by drawing the maps of transient streamlines \((12\pi)^{-1}(\Psi/R_0) = \text{const.}\) (Fig. 5) using (29c) for the same set of values of \( s \), choosing in illustration \( N = 0.4 \) and \( A = 0.4 \). A closer look into the maps reveals that, apart from the existence of a downward flow, the streamlines around the point source are convex in the form of toroidal vortices with a small bulging near the origin, whereas, those around the spherical source remain concave. The slight bulging of the streamlines near the origin is essentially due to the impulsive effect of the source on the fluid particles in its vicinity and is created by the fluid particles coming from below in order to replace those which are driven upward under the action of buoyancy. The stagnation points are found to be at \( R = 0.32, 0.48 \) and 0.8 when \( \tau = 0.1, \tau = 0.2 \) and 1.0 respectively. These stagnation points...
are relatively closer to the origin than those of the spherical source.

Lastly, we present a comparison of the present study with the previous one done by the same author on double diffusion from a heated sphere [20]. With the same geometry and scaling, it may be verified for a heated sphere of constant temperature $Q$ and of radius $a$ embedded in an unbounded porous medium:

\[ \Theta = (\eta_0/\eta_0)erfc(\eta - \eta_0), \]

\[ C = (\eta_0/\eta_0)erfc[A(\eta - \eta_0)], \]

\[ \Psi = R_a(2\eta_0)\tau \cdot (1 - e^{-1/\tau}) (\Phi(\eta, \eta_0) + (N/A) \cdot \Phi(A\eta, A\eta_0)), \]

where

\[ \Phi(\eta, \eta_0) = \eta^{-1}[2(\eta_0/\eta) + (\eta^2 - \eta_0^2)erfc(\eta - \eta_0) + (1/2)erfc(\eta - \eta_0) - (1/\sqrt{\pi})(\eta + \eta_0)exp(-\eta^2)]. \]

The maps of streamlines using (27) and (32) at time $\tau = 0.1$ for the two studies with the nodal points of their points of incidence with the $\phi = \pi/2$ axis at identical locations are illustrated in Fig. 6 with $\tau = 1$. While the pictures of the flow pattern are qualitatively preserved, the intensity of the flow field is found to vary. Our computations show that in respect of the heated sphere, the stagnation points are relatively farther away from the origin than those due to the presence of the spherical source at all times. One may then conclude that the steady-state is established earlier in respect of the heated sphere than with the spherical source.

We may mention here the work of Jaballah et al. [25] wherein the authors investigated numerically the steady mixed convection in a horizontal channel saturated with an incompressible fluid and fitted with some porous blocks inserted intermittently in transverse direction relative to the horizontal axis, assuming the validity of the Darcy–Brinkman model, choosing in illustration $D_a = 10^{-4}$, $R_0 = 10$ and $R_a = 104$, where $D_a$ and $R_a$ are respectively the Darcy and Reynolds number. They have shown that compromise values of the coefficient of exchange transfer and the thermal conductivity ratio are necessary for obtaining the highest temperature outlet of the channel. Further, in respect of the thermal equilibrium model, the quantity of heat collected by the solid particles is yielded to the fluid particles without loss. This indeed is the case in the present work since we consider the thermal equilibrium model and unlike the work of Jaballah et al. [25], here we consider the Darcy flow model.

5. Conclusion

Using a thermal equilibrium model an exact analytical solution is obtained for the stream function in its closed form which is applicable for a class of sources with spherical geometry and from which solutions of problems involving point sources could be deduced algebraically as a limiting case. Solutions for the temperature and concentration fields are obtained using the method of superposition. The results are valid in the diffusion dominated regime only and our findings determine the impact of species concentration gradients upon the thermally driven flow. When the two buoyancy mechanisms are opposed, there appear two oppositely rotating vortices leading to a bifurcation of the flow field with a downward flow in the farther region. In the isothermal region of the downward flow, the lower half space is found to be richer in chemical species than the upper one. The neglect of the transport of heat and concentration by fluid motion does not impose any restriction on the range of applicability of the results as long as the heat generation rate is not excessive. Depending upon the warranting situation, with the convective terms in the energy and concentration equations accounted for, the analysis would then be tantamount to a regular perturbation in the limit of small Raleigh number and in which case one may then set $\Theta = \Theta_0 + O(R_a)$, $C = C_0 + O(R_a)$ and $\Psi = (R_a)\Psi_0 + O(R_a^2)$. In such a situation, the results we have presented in (27) would represent the first convective correction to the flow field. Our assumption that the heat generation rate is not excessive and the medium is of low permeability implies $R_a < 1$ though of course, in the absence of instability, our solution is valid for moderately larger values of Rayleigh number also. In respect of the heated sphere, steady-state sets in at an earlier time than in the case of the spherical source. Though it is customary to set either $a$ or $\sqrt{K}$ as a length scale for radial distance, yet, we have chosen $L$ in order to prove that the solutions for point sources could be deduced algebraically from the results of this study. The problems discussed by Bejan [2] and Poulikakos [23] are only the limiting forms of the present work and their solutions are deducible algebraically from (27) in the limit of vanishing radius. Lastly, the results of this study are applicable as well for flows due to spherical sources which generate simultaneously two different chemical components.

References


Double diffusive Darcy flow induced by a spherical source


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