Available online at www.sciencedirect.com
ScienceDirect
Applied Mathematics Letters 20 (2007) 622–628
Applied Mathematics Letters
www.elsevier.com/locate/aml

On Monch type multi-valued maps and fixed points

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Received 1 September 2005; received in revised form 12 June 2006; accepted 28 June 2006

Abstract

In this work, some fixed point theorems for a new class of Chandrabhan maps are proved which in turn include the fixed point theorems of Monch, Sadovskii, Darbo, Krasnoselskii, Dhage, and Covitz and Nadler as special cases.

Keywords: Nonlinear operators; Fixed point

1. Introduction

Fixed point theory for multi-valued mappings is an important topic of multi-valued analysis and finds several applications to differential and integral inclusions, control theory and optimization. The multi-valued analogue of the Schauder fixed point theorem due to Himmelberg [6] is useful for proving the existence theorems for such problems in the multi-valued analysis. Generalizing the fixed point theorem of Schauder [1], Monch [8] proved a fixed point theorem for a new class of single-valued mappings called the Monch mappings hereafter, and applied it to the nonlinear boundary value problems of ordinary differential equations in Banach spaces for proving the existence of solutions. In this work we investigate a new class of multi-valued mappings in Banach spaces possessing the fixed point property which again include the multi-valued analogue of the Monch fixed point theorem as a special case.

Let X be a Banach space and let $\mathcal{P}(X)$ denote the class of all subsets of X. Define

$$\mathcal{P}_p(X) = \{A \subset X \mid A \text{ is non-empty and has a property } p\}. \tag{1.1}$$

Thus, $\mathcal{P}_{bd}(X)$, $\mathcal{P}_{cl}(X)$, $\mathcal{P}_{cv}(X)$, and $\mathcal{P}_{cp}(X)$ denote the classes of all bounded, closed, convex and compact subsets of X respectively. Similarly, $\mathcal{P}_{cl,cv,bd}(X)$ and $\mathcal{P}_{cp,cv}(X)$ denote respectively the classes of closed, convex and bounded, and compact, convex subsets of X respectively. For any $A, B \in \mathcal{P}_p(X)$, let us define

$$A \pm B = \{a \pm b \mid a \in A, b \in B\}$$
$$\lambda A = \{\lambda a \mid a \in A\}$$

for $\lambda \in \mathbf{R}$. Similarly, define

$$\|A\| = \{\|a\| \mid a \in A\}$$

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0893-9659/$ - see front matter
doi:10.1016/j.aml.2006.06.013

and

$$\|A\|_{\mathcal{P}} = \sup\{\|a\| \mid a \in A\}.$$

Let $A, B \in \mathcal{P}_{cl,bd}(X)$ and let $a \in A$. Then define

$$D(a, B) = \inf\{\|a - b\| \mid b \in B\}$$

and

$$\rho(A, B) = \sup\{D(a, B) \mid a \in A\}.$$

The function $d_H : \mathcal{P}_{cl,bd}(X) \times \mathcal{P}_{cl,bd}(X) \to \mathbf{R}^+$ defined by

$$d_H(A, B) = \max\{\rho(A, B), \rho(B, A)\} \tag{1.2}$$

is metric and is called the Hausdorff metric on X. It is clear that

$$d_H(0, C) = \|C\|_{\mathcal{P}} = \sup\{\|c\| \mid c \in C\}$$

for any $C \in \mathcal{P}_{cl,bd}(X)$.

A correspondence $T : X \to \mathcal{P}_p(X)$ is called a multi-valued operator or multi-valued mapping on X. A point $u \in X$ is called a fixed point of T if $u \in Tu$ and the set of all fixed points of T in X is denoted by $\mathcal{F}_{\mathcal{T}}$. For any $A \subset X$, define $T(A) = \bigcup_{x \in A} Tx$.

Definition 1.1. A multi-valued operator $T : X \to \mathcal{P}_{cl,bd}(X)$ is called Lipschitz if there exists a constant $k > 0$ such that

$$d_H(Tx, Ty) \le k\|x - y\| \tag{1.3}$$

for all $x, y \in X$ and the constant k is called the Lipschitz constant of T on X. If $k < 1$, then T is called a multi-valued contraction on X with the contraction constant k. Similarly, a single-valued mapping $T : X \to X$ is called Lipschitz if there exists a constant $k > 0$ such that

$$\|Tx - Ty\| \le k\|x - y\| \tag{1.4}$$

for all $x, y \in X$ and the constant k is called the Lipschitz constant of T on X. If $k < 1$, then T is called a contraction on X with the contraction constant k.

The following fixed point theorem for multi-valued contraction mappings due to Covitz and Nadler (see Zeidler [9]) is well known in the literature.

Theorem 1.1. *Let X be a complete metric space and let $T : X \to \mathcal{P}_{cl}(X)$ be a multi-valued contraction mapping. Then the set $\mathcal{F}_{\mathcal{T}}$ is non-empty and closed in X.*

Remark 1.1. Note that if the multi-valued map T in the above Theorem 1.1 has compact values, then the set $\mathcal{F}_{\mathcal{T}}$ is non-empty and compact in X.

The multi-valued operator T is called lower semi-continuous (for short l.s.c.) if G is any open subset of X; then the weak inverse of G under T

$$T^{-1(w)}(G) = \left\{x \in X \mid Tx \bigcap G \ne \emptyset\right\}$$

is an open subset of X. Similarly the multi-valued operator T is called upper semi-continuous (for short u.s.c.) if the set

$$T^{-1}(G) = \{x \in X \mid Tx \subset G\}$$

is open in X for every open set G in X. Finally T is called continuous if it is lower as well as upper semi-continuous on X. A multi-valued map $T : X \to \mathcal{P}_{cp}(X)$ is called *compact* if $\overline{T(S)}$ is a compact subset of X for any $S \subset X$. T is called *totally bounded* if for any bounded subset S of X, $T(S) = \bigcup_{x \in S} Tx$ is a totally bounded subset of X.

It is clear that every compact multi-valued operator is totally bounded, but the converse may not be true. However the two notions are equivalent on a bounded subset of X. Finally T is called *completely continuous* if it is upper semi-continuous and totally bounded on X. The details on these terminologies appear in Hu and Papageorgiou [7].

The following multi-valued analogue of the Schauder fixed point theorem for multi-valued compact mappings appears in Himmelberg [6].

Theorem 1.2. *Let C be a closed, convex and bounded subset of a Banach space X and let $T : C \to \mathcal{P}_{cp,cv}(C)$ be an upper semi-continuous and compact multi-valued map. Then T has a fixed point.*

Definition 1.2. A multi-valued map $T : X \to \mathcal{P}_{cl,cv,bd}(X)$ is called *Krasnoselskii type* if T can be decomposed as $T = T_1 + T_2$, where $T_1 : X \to \mathcal{P}_{cl,cv,bd}(X)$ is a multi-valued contraction map and $T_2 : X \to \mathcal{P}_{cp,cv}(X)$ is an upper semi-continuous and totally bounded multi-valued map on X.

Definition 1.3. Let X be a Banach algebra. A multi-valued map $T : S \subset X \to \mathcal{P}_{cl,cv,bd}(X)$ is called *Dhage type* if T can be decomposed as $Tx = T_1x\, T_2x$, $x \in X$, where $T_1 : S \to \mathcal{P}_{cl,cv,bd}(X)$ is a multi-valued Lipschitz map with the Lipschitz constant k and $T_2 : S \to \mathcal{P}_{cp,cv}(X)$ is an upper semi-continuous and totally bounded multi-valued map on X satisfying $M\,k < 1$, where $M = \|T_2(S)\|_{\mathcal{P}}$ (Dhage [5]).

The Kurotowskii and Hausdorff measures α and β of noncompactness of a bounded set S in the Banach space X are the nonnegative real numbers $\alpha(S)$ and $\beta(S)$ defined by

$$\alpha(S) = \inf\left\{ r > 0 : S \subset \bigcup_{i=1}^{n} S_i,\ \text{diam}(S_i) \le r \forall\, i \right\}, \tag{1.5}$$

$$\beta(S) = \inf\left\{ r > 0 : S \subset \bigcup_{i=1}^{n} \mathcal{B}_i(x_i, r),\ \text{for some } x_i \in X \right\}, \tag{1.6}$$

where $\mathcal{B}_i(x_i, r) = \{x \in X \mid d(x, x_i) < r\}$.

The details of Kuratowskii and Hausdorff measures of noncompactness appear in Banas and Goebel [1], Deimling [2], Zeidler [9] and the references therein.

Remark 1.2. It is known that $\beta(S) \le \alpha(S) \le 2\beta(S)$ for every bounded subset S of the Banach space X.

Remark 1.3. It is known that if $T : X \to \mathcal{P}_{cl,bd}(X)$ is a multi-valued contraction with a contraction constant k, then $\beta(T(S)) \le k\,\beta(S)$ for all $S \in \mathcal{P}_{cl,bd}(X)$. Similarly, if T is single-valued contraction on X with contraction k, then $\alpha(T(S)) \le k\alpha(S)$.

Definition 1.4. A multi-valued mapping $T : X \to \mathcal{P}_{bd}(X)$ is called a multi-valued *set-contraction* if $\beta(T(S)) \le k\beta(S)$ for any bounded set $S \subset X$, where $k < 1$. Similarly a multi-valued map $T : X \to \mathcal{P}_{bd}(X)$ is called a *nonlinear $\mathcal{D}$-set-contraction* if there exists a continuous and nondecreasing function $\psi : \mathbf{R}^+ \to \mathbf{R}^+$ such that $\beta(T(S)) \le \psi(\beta(S))$ for some bounded subset S of X, where $\psi(r) < r$ for $r > 0$. Finally, a multi-valued map $T : X \to \mathcal{P}_{bd}(X)$ is called *β-condensing* if for any $S \in \mathcal{P}_{bd}(X)$, we have that $\beta(T(S)) < \beta(S)$ for $\beta(S) > 0$. Each of the above terminologies is also applicable to single-valued mappings T on X with β replaced by α.

It is known that compact, Krasnoselskii and Dhage multi-valued maps are *β-condensing*. Notice also that every *multi-valued contraction* $\Longrightarrow$ *Krasnoselskii* $\Longrightarrow$ *set-contraction* $\Longrightarrow$ *nonlinear $\mathcal{D}$-set-contraction* $\Longrightarrow$ *β-condensing*, but the converse need not be true.

2. Fixed point theory

Definition 2.1. A multi-valued mapping $T : X \to \mathcal{P}_{cl}(X)$ is called a *Monch type map* if A is a countable subset of X; then

$$A \subseteq \overline{\text{conv}}\left(\{x_0\} \bigcup T(A)\right) \Longrightarrow \overline{A} \quad \text{is compact} \tag{2.1}$$

for some $x_0 \in A$.

Definition 2.2. A subset A of X is called countable if there exists a one-to-one correspondence $f : \mathbf{N} \to A$, where $\mathbf{N}$ is the set of natural numbers. The element $a = f(1) \in A$ is called the first element of A. A multi-valued mapping $T : X \to X$ is said to satisfy *Condition* $\mathcal{D}$ if for any countable subset A of X,

$$A \subseteq \overline{\text{conv}}\left(\{a\} \bigcup T(A)\right) \Longrightarrow \overline{A} \quad \text{is compact} \tag{2.2}$$

where a is a first element of A.

Definition 2.3. A multi-valued mapping $T : X \to \mathcal{P}_{cl}(X)$ is called *Chandrabhan* if A is a countable subset of X; then

$$A \subseteq \overline{\text{conv}}\left(C \bigcup T(A)\right) \Longrightarrow \overline{A} \quad \text{is compact} \tag{2.3}$$

where C is a relatively compact subset of X called the *support set* of T in X.

Notice that β*-condensing* $\Longrightarrow$ *Monch type maps* $\Longrightarrow$ *Condition* $\mathcal{D}$ $\Longrightarrow$ *Chandrabhan map*, but the converse may not be true. The first and third implications are obvious. To prove the second, let T be a Monch type multi-valued map on X. Then (2.1) holds for some $x_0 \in X$. Set $S = \{x_0\} \bigcup A$. Then S is countable and we may consider the element x_0 to be first element a of S. Therefore condition (2.2) holds and consequently T satisfies Condition $\mathcal{D}$. A few details on Chandrabhan maps may be found in Dhage [3,4].

Theorem 2.1. *Let X be a Banach space, $K \subset X$ a closed convex set, and let $T : K \to \mathcal{P}_{cp,cv}(K)$ be an upper semi-continuous and Chandrabhan map with support set C in K. Then T has a fixed point.*

Proof. We construct a sequence $\{C_n\}$ of the subsets of K defined by

$$C_{n+1} = \text{conv}\left\{C \bigcup T(C_n)\right\}, \quad C_0 = C. \tag{2.4}$$

Define $C' = \bigcup_{n\geq 0} C_n$ and $C^* = \overline{C'}$. Then C' is a convex subset of K since $C_n \subset C_{n+1}$ for each $n = 0, 1, 2, \ldots$. In addition $C' = \text{conv}\{C \bigcup T(C')\}$. Hence C^* is a closed convex subset of K and $T : C^* \to \mathcal{P}_{cp,cv}(C^*)$. On the other hand, by induction, C_n is compact for each $n \in \mathbf{N}$. So there exists a countable set S_n such that $S_n \subset C_n$ with $\overline{S_n} = \overline{C_n}$ for each $n \in \mathbf{N}$.

Consider the countable set $S = \bigcup_{n\geq 0} S_n$. Then we have $\overline{S} = \overline{C'} = C^*$. Also

$$\overline{\text{conv}}\left(\left\{C \bigcup T(S)\right\}\right) \subseteq \overline{\text{conv}}\left(\left\{C \bigcup T(C')\right\}\right) \subset \overline{S}.$$

Thus $\overline{S} = \overline{C^*}$ and $T : C^* \to \mathcal{P}_{cp,cv}(C^*)$.

Now T is an upper semi-continuous compact, convex-valued multi-valued self-map of a non-empty compact convex set C^*, and hence the desired conclusion follows by an application of Theorem 1.2. □

Corollary 2.1. *Let X be a Banach space and let K be a closed convex subset of X. Let $T : K \to \mathcal{P}_{cp,cv}(K)$ be an upper semi-continuous multi-valued map satisfying for a countable set S of K,*

$$S \subset \overline{\text{conv}}\left\{F \bigcup T(S)\right\} \Rightarrow \overline{S}, \quad \textit{is compact} \tag{2.5}$$

for some finite set F in K. Then T has a fixed point.

Corollary 2.2. *Let X be a Banach space and let K be a closed convex subset of X. Let $T : K \to \mathcal{P}_{cp,cv}(K)$ be an upper semi-continuous multi-valued map satisfying Condition $\mathcal{D}$. Then T has a fixed point.*

As a special case to Theorem 2.1 we obtain

Corollary 2.3. *Let X be a Banach space, K be a closed convex subset of X and let $T : K \to \mathcal{P}_{cp,cv}(K)$ be a multi-valued map. Suppose that any one of the following conditions holds.*

(i) *T is upper semi-continuous and β-condensing.*
(ii) *T is upper semi-continuous and a set-contraction.*

(iii) *T is of Krasnoselskii type (Petrusel and Dhage).*
(iv) *T is Dhage type (if X is a Banach algebra) (Dhage).*
(v) *T is upper semi-continuous and compact (Himmelberg).*

Then T has a fixed point.

When T is a single-valued mapping, Theorem 2.1 reduces to

Theorem 2.2. *Let X be a Banach space, $K \subset X$ a closed convex, and let $T : K \to K$ be a continuous and Chandrabhan map. Then T has a fixed point.*

Note that Theorem 2.2 is new to the literature on multi-valued fixed point theory which again includes the following results as corollaries.

Corollary 2.4. *Let X be a Banach space and let K be a closed convex subset of X. Let $T : K \to K$ be a continuous map satisfying for a countable set S in K,*

$$S \subset \overline{\text{conv}} \left\{ F \bigcup T(S) \right\} \Rightarrow \overline{S}, \quad \text{is compact}$$

for some finite set F in K. Then T has a fixed point.

Corollary 2.5. *Let X be a Banach space and let K be a closed convex subset of X. Let $T : K \to K$ be a continuous map satisfying Condition $\mathcal{D}$. Then T has a fixed point.*

Corollary 2.5 contains the following result due to Monch [8] as special case.

Corollary 2.6. *Let X be a Banach space and let K be a closed convex subset of X. Let $T : K \to K$ be a continuous Monch map. Then T has a fixed point.*

The above corollary again includes the following known results in the fixed point theory for single-valued mappings in Banach spaces. See Zeidler [9] and the references therein.

Corollary 2.7. *Let X be a Banach space and let K be a closed, convex set, and bounded subset of X and let $T : K \to K$ be a single-valued map. Suppose that any one of the following conditions holds.*

(i) *T is continuous and α-condensing (Sadovskii).*
(ii) *T is continuous and a set-contraction (Darbo).*
(iii) *T is a Krasnoselskii map (Krasnoselskii).*
(iv) *T is a Dhage map (if X is a Banach algebra) (Dhage).*
(v) *T is a compact and continuous map (Schauder).*

Then T has a fixed point.

3. Leray–Schauder type fixed point theory

Next, we prove a Leray–Schauder type multi-valued fixed point theorem corresponding to Theorem 2.1.

Theorem 3.1. *Let X be a Banach space, $K \subset X$ a closed convex subset, and $U \subset K$ an open and bounded set in K. Let $T : \overline{U} \to \mathcal{P}_{cp,cv}(K)$ be upper semi-continuous and Chandrabhan map with support set C in $\overline{U}$. In addition assume that*

$$x \notin (1-\lambda)\overline{\text{conv}}(C) + \lambda T(x) \quad \text{for all } x \in \partial U \text{ and } \lambda \in [0, 1]. \tag{3.1}$$

Then T has a fixed point.

Proof. If $U = K$, then the conclusion follows directly from Theorem 2.1. Assume $U \neq K$ so that $\partial U \neq \emptyset$. Define the multi-valued homotopy $H : \overline{U} \times [0, 1] \to \mathcal{P}_{cp,cv}(K)$ by

$$H(x, \lambda) = (1 - \lambda)\overline{\text{conv}}(C) + \lambda T(x) \tag{3.2}$$

and let

$$\Sigma = \{x \in \overline{U} : x \in H(x, \lambda) \text{ for some } \lambda \in [0, 1]\}. \tag{3.3}$$

Since H is upper semi-continuous, Σ is closed. Condition (3.1) guarantees that Σ and ∂U are disjoint. So from Urysohn's lemma it follows that there is a continuous function $v : \overline{U} \to [0, 1]$ such that $v(x) = 0$ on ∂U and $v(x) = 1$ on Σ. Define a set

$$D = \overline{\text{conv}}\left(\left\{C \bigcup T(\overline{U})\right\}\right) \tag{3.4}$$

and define a multi-valued map $\widehat{T} : D \to \mathcal{P}_{cp}(D)$ by

$$\widehat{T}(x) = \begin{cases} H(x, v(x)) & \text{for } x \in \overline{U} \\ \text{conv}\{C\} & \text{for } x \notin \overline{U}. \end{cases} \tag{3.5}$$

It is easy to check that $\widehat{T}$ is an upper semi-continuous multi-valued compact and convex-valued self-map of D. Now we prove that $\widehat{T}$ is a multi-valued Chandrabhan map on D. Let S be a countable set of D with $\overline{S} = \overline{\text{conv}}\{C \bigcup T(S)\}$. Using (3.5), we obtain

$$\overline{\text{conv}}\left\{C \bigcup \widehat{T}(S)\right\} = \overline{\text{conv}}\left\{C \bigcup T\left(S \bigcap \overline{U}\right)\right\}.$$

Since T is Chandrabhan, $S \bigcap \overline{U}$ is relatively compact. Now by Mazur's lemma, the entire set $\overline{S}$ is compact. Therefore we may apply Theorem 2.1 to deduce that the multi-valued map $\widehat{T}$ has a fixed point. Since $C \subset U$, we have that $x \in \overline{U}$ and $x \in H(x, v(x))$. This shows that $x \in \Sigma$ with $v(x) = 1$. As a result $x \in T(x)$. □

As a special case of Theorem 3.1 we obtain

Corollary 3.1. *Let X be a Banach space, $K \subset X$ a closed convex subset, and $U \subset K$ an open bounded set in K and let $T : \overline{U} \to \mathcal{P}_{cp,cv}(K)$ be a multi-valued map. Suppose that any one of the following conditions holds.*

(i) *T is upper semi-continuous and β-condensing.*
(ii) *T is upper semi-continuous and a set-contraction.*
(iii) *T is of Krasnoselskii type.*
(iv) *T is of Dhage type (if X is a Banach algebra).*
(v) *T is upper semi-continuous and compact.*

In addition assume that

$$x \notin (1 - \lambda)\overline{\text{conv}}(C) + \lambda T(x) \quad \text{for all } x \in \partial U \text{ and } \lambda \in [0, 1].$$

where C is a relatively compact set in U. Then T has a fixed point.

An interesting corollary to Theorem 3.1 in the applicable form is

Theorem 3.2. *Let $\mathcal{B}_r(0)$ and $\overline{\mathcal{B}_r(0)}$ be respectively the open and closed balls in a closed convex subset K of a Banach space X centered at origin 0 of radius r. Let $T : \overline{\mathcal{B}_r(0)} \to \mathcal{P}_{cp,cv}(K)$ be an upper semi-continuous Chandrabhan map with support set $C = \{0\}$. In addition assume that*

$$\lambda x \notin T(x), \quad \lambda > 1 \tag{3.6}$$

for all $x \in X$ with $\|x\| = r$. Then T has a fixed point in $\overline{\mathcal{B}_r(0)}$.

Corollary 3.2. *Let $\mathcal{B}_r(0)$ and $\overline{\mathcal{B}_r(0)}$ be respectively the open and closed balls in a closed convex subset K of a Banach space X centered at origin 0 of radius r. Let $T : \overline{\mathcal{B}_r(0)} \to K$ be a continuous Chandrabhan map with support set $C = \{0\}$. In addition assume that*

$$\lambda x \neq T(x), \quad \lambda > 1 \tag{3.7}$$

for all $x \in X$ with $\|x\| = r$. Then T has a fixed point in $\overline{\mathcal{B}_r(0)}$.

Finally, we remark that our Theorem 3.2 and Corollary 3.2 have some nice applications respectively to differential and integral inclusions and equations in Banach spaces for proving the existence of the solutions. Some of the results in this direction will be reported elsewhere.

Acknowledgement

The author is grateful to the referee for giving some useful suggestions for the improvement of this work.

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