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Multicolored parallelisms of Hamiltonian cycles☆

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ABSTRACT

A subgraph in an *edge-colored* graph is *multicolored* if all its edges receive distinct colors. In this paper, we prove that a complete graph on $2m + 1$ vertices K_{2m+1} can be properly edge-colored with $2m + 1$ colors in such a way that the edges of K_{2m+1} can be partitioned into m multicolored *Hamiltonian* cycles.

1. Introduction

A *proper k-edge-coloring* of a graph G is a mapping from $E(G)$ into a set of colors $\{1, 2, \ldots, k\}$ such that incident edges of G receive distinct colors. The *chromatic index* $\chi'(G)$ of a graph G is the minimum number k for which G has a proper k-edge-coloring.

If G has a k-edge-coloring, G is said to be k-edge-colored or *simply* edge-colored. A subgraph in an edge-colored graph is *multicolored* if all its edges receive distinct colors. The following conjecture was posed by Brualdi and Hollingsworth in [2].

Conjecture A (*[2]*). *If K_{2m} is (2m-1)-edge-colored, then the edges of K_{2m} can be partitioned into m multicolored spanning trees except when $m = 2$.*

In [2], they constructed two multicolored spanning trees in K_{2m} for any proper $(2m-1)$-edge-coloring by making use of Rado's theorem [7,8]. In [6], for any $(2m-1)$-edge-coloring of K_{2m} with $m > 2$, Krussel et al. constructed three multicolored spanning trees. In [4], Constantine used a special $(2m-1)$-edge-coloring of K_{2m} to partition the edges of K_{2m} into multicolored isomorphic spanning trees for specific values of m.

Theorem 1.1 (*[4]*). *For $n = 6$, $n = 2^k$ with $k \geq 3$ or $n = 5 \cdot 2^k$ with $k \geq 1$, there exists an $(n-1)$-edge-coloring of K_n such that the edges of K_n can be partitioned into $\frac{n}{2}$ multicolored isomorphic spanning trees.*

In Fig. 1, the ith row denotes the edges of K_6 which are colored with c_i and the jth column denotes the edges of a multicolored spanning tree for $1 \leq i \leq 5$ and $1 \leq j \leq 3$. Therefore, we have a parallelism as defined in Cameron [3], with an additional property due to color. Indeed, it is a double parallelism of K_n, one parallelism is present in the rows of the array (perfect matchings) and the other parallelism is present in the columns that consist of edge disjoint isomorphic

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	T_1	T_2	T_3
c_1:	35	46	12
c_2:	24	15	36
c_3:	25	34	16
c_4:	26	13	45
c_5:	14	23	56

Fig. 1. 3 multicolored isomorphic spanning trees in K_6.

1	5	2	6	3	7	4
5	2	6	3	7	4	1
2	6	3	7	4	1	5
6	3	7	4	1	5	2
3	7	4	1	5	2	6
7	4	1	5	2	6	3
4	1	5	2	6	3	7

Fig. 2. $n = 7$.

spanning trees. Due to this fact, we say that the complete graph K_{2m} admits a *multicolored tree parallelism (MTP)*, if there exists a proper $(2m-1)$-edge-coloring of K_{2m} for which all edges can be partitioned into m isomorphic multicolored spanning trees.

Following the result given in [4], Constantine made the following conjecture.

Conjecture B ([4]). *K_{2m} admits an MTP for each positive integer $m \neq 2$.*

This conjecture was recently proved by Akbari et al. [1].

In this paper, we extend the study of parallelism to the complete graph K_{2m+1} of odd order. Since $\chi'(K_{2m+1}) = 2m+1$, a multicolored subgraph will have $2m+1$ edges. Thus, a natural subgraph to consider is a *Hamiltonian cycle*. A graph G with n vertices has a *multicolored Hamiltonian cycle parallelism (MHCP)* if there exists an n-edge-coloring of G such that the edges can be partitioned into multicolored Hamiltonian cycles. In this paper, we shall prove that for each positive integer m, K_{2m+1} admits an *MHCP*. This result extends earlier work obtained by Constantine [5] which shows that K_{2m+1} admits an MHCP when $2m+1$ is a prime.

2. Preliminaries

It is well-known that $\chi'(K_n) = n$ if n is odd and $\chi'(K_n) = n-1$ if n is even. Also, $\chi'(K_{n,n}) = n$ [9]. To color the edges of K_n when n is odd, the following notion plays an important role. A *latin square* of order n is an $n \times n$ array of n symbols, $1, 2, \ldots, n$, in which each symbol occurs exactly once in each row and each column of the array. A latin square $L = [\ell_{i,j}]$ is *commutative* if $\ell_{i,j} = \ell_{j,i}$ for each pair of distinct i and j, and L is *idempotent* if $\ell_{i,i} = i$, $i = 1, 2, \ldots, n$. It is well-known that an idempotent commutative latin square of order n exists if and only if n is odd. Now, let $V(K_n) = \{v_1, v_2, \ldots, v_n\}$ and let $L = [\ell_{i,j}]$ be an idempotent commutative latin square of order n. Color edge v_iv_j of K_n with color $\ell_{i,j}$ and observe that this produces an n-edge-coloring of K_n.

A similar idea shows that a latin square of order n corresponds to an n-edge-coloring of the complete bipartite graph $K_{n,n}$. For the convenience in the proof of our main result, we shall use a special latin square $M = [m_{i,j}]$ of order odd n which is a circulant latin square with 1st row $(1, \frac{n+3}{2}, 2, \frac{n+5}{2}, 3, \ldots, \frac{n+n}{2}, \frac{n+1}{2})$. Fig. 2 is such a latin square of order 7.

Now, let $\{u_1, u_2, \ldots, u_n\}$ and $\{v_1, v_2, \ldots, v_n\}$ be the two partite sets of $K_{n,n}$ and let $M = [m_{i,j}]$ be a circulant latin square of order n with the first row as described in the preceding paragraph. Color edge u_iv_j of $K_{n,n}$ with color $m_{i,j}$ and observe that the result is a proper n-edge-coloring of $K_{n,n}$ with the extra property that for $1 \leq j \leq n$, the perfect matching $\{u_1v_j, u_2v_{j+1}, u_3v_{j+2}, \ldots, u_nv_{j+n-1}\}$, where the indices of v_i are taken modulo n with $i \in \{1, 2, \ldots, n\}$, is multicolored. We note here that if we permute the entries of M, we obtain another n-edge-coloring of $K_{n,n}$ which has the same property as above.

The following result by Constantine appears in [5].

Theorem 2.1 ([5]). *If n is an odd prime, then K_n admits an MHCP.*

Note that this result can be obtained by using a circulant latin square of order n to color the edges of K_n and the Hamiltonian cycles are corresponding to 1st, 2nd, ..., $(\frac{n-1}{2})$th sub-diagonals, respectively. For example, in K_7, the edges are colored by using Fig. 2, and the three cycles are induced by $\{v_1v_{i+1}, v_2v_{i+2}, \ldots, v_7v_{i+7}\}$ where $V(K_7) = \{v_1, v_2, \ldots, v_7\}$, $i = 1, 2, 3$, and the sub-indices are in $\{1, 2, \ldots, 7\}$.

In what follows, we extend Theorem 2.1 to the case when n is an odd, but not necessarily prime, integer.

3. The main results

We begin this section with some notations. Let $K_{m(n)}$ be the complete m-partite graph in which each partite set is of size n. In what follows, we will let $\mathbb{Z}_k = \{1, 2, \ldots, k\}$ with the usual addition modulo k. For convenience, let $V(K_{m(n)}) = \bigcup_{i=1}^{m} V_i$ where $V_i = \{x_{i,1}, x_{i,2}, \ldots, x_{i,n}\}$. The graph $C_{m(n)}$ is a spanning subgraph of $V(K_{m(n)})$ where $x_{i,j}$ is adjacent to $x_{i+1,k}$ for all $j, k \in \mathbb{Z}_n$ and $i \in \mathbb{Z}_m$ (mod m). Clearly, if K_m can be decomposed into $\frac{m-1}{2}$ Hamiltonian cycles (m is odd), then $K_{m(n)}$ can be decomposed into $\frac{m-1}{2}$ subgraphs, each of which is isomorphic to $C_{m(n)}$.

In order to prove the main theorem, we need the following two lemmas.

Lemma 3.1. *Let p be an odd prime and m be a positive odd integer with $p \le m$. Let $t \in \{1, 2, \ldots, p-1\}$. Then there exists a set $\{S_i = (a_{i,1}, a_{i,2}, \ldots, a_{i,m}) \mid 0 \le i \le p-1\}$ of m-tuples such that*

(1) $S_0 = (0, 0, \ldots, 0, t)$;
(2) $\{a_{i,j} \mid 0 \le i \le p-1\} = \{0, 1, 2, \ldots, p-1\}$ *for each j with $1 \le j \le m$; and*
(3) $p \nmid w_i$ *where* $w_i = \sum_{j=1}^{m} a_{i,j}$ *for each i with $0 \le i \le p-1$.*

Proof. The proof follows by direct constructions depending on the choice of t where $1 \le t \le p-1$. First, we let $S_0 = (0, 0, \ldots, 0, 1)$, $S_1 = (1, 1, \ldots, 1, 2)$, ..., and $S_{p-1} = (p-1, p-1, \ldots, p-1, 0)$ be the p m-tuples. For each i with $0 \le i \le p-1$, let $w_i = \sum_{j=1}^{m} a_{i,j}$ where $S_i = (a_{i,1}, a_{i,2}, \ldots, a_{i,m})$. If for each $0 \le i \le p-1$, $p \nmid w_i$, we do nothing. Otherwise, assume that $p \mid w_j$ for some $j \in \{1, 2, \ldots, p-1\}$, and note that such j is unique. Now, if $j \in \{1, 2, \ldots, p-2\}$, replace S_j and S_{j+1} with $(j, j, \ldots, j, j+1, j+1)$ and $(j+1, j+1, \ldots, j+1, j, j+2)$, respectively. Else, if $j = p-1$, then replace S_{p-2} and S_{p-1} with $(p-2, p-2, \ldots, p-2, p-1, p-1, p-1)$ and $(p-1, p-1, \ldots, p-1, p-2, p-2, 0)$, respectively.

When $t = 1$, clearly, these p m-tuples above satisfies all the three properties. So, in what follows, we consider $2 \le t \le p-1$. Note that we initially use the same m-tuples constructed in the case $t = 1$ and consider that j causing us to adjust entries above.

Case 1. No such j exists.

First, interchange $a_{0,m}$ with $a_{t-1,m}$. If $w_{t-1} \not\equiv 0 \pmod p$, then we are done. On the other hand, suppose $w_{t-1} \equiv 0 \pmod p$. If $w_t \not\equiv 1 \pmod p$, then replace S_{t-1} and S_t with $(t-1, t-1, \ldots, t-1, t, 1)$ and $(t, t, \ldots, t, t-1, t+1)$, respectively. Otherwise, replace S_{t-1} and S_t with $(t-1, t-1, \ldots, t-1, t-1, t+1)$ and $(t, t, \ldots, t, t, 1)$, respectively.

Case 2. $j \in \{1, 2, \ldots, p-2\}$.

First, interchange $a_{0,m}$ with $a_{t-1,m}$. If $w_{t-1} \not\equiv 0 \pmod p$, then we are done. On the other hand, suppose $w_{t-1} \equiv 0 \pmod p$. If $t = j+2$, then replace S_j and S_{j+1} with $(j, j, \ldots, j, j+1, j+1, j+1)$ and $(j+1, j+1, \ldots, j+1, j, j, 1)$, respectively. Otherwise, interchange $a_{t-1,m-1}$ with $a_{t,m-1}$.

Case 3. $j = p-1$.

Interchange $a_{0,m}$ with $a_{t-1,m}$.

Thus, we can construct the desired p m-tuples. ■

Lemma 3.2. *Let v be a composite odd integer and p be the smallest prime with $p|v$. Assume $v = mp$. If K_m admits an MHCP, then $K_{m(p)}$ has an mp-edge-coloring that admits an MHCP.*

Proof. We prove the lemma by giving an mp-edge-coloring φ. Since K_m defined on $\{x_i \mid i \in \mathbb{Z}_m\}$ admits an *MHCP*, let μ be such an edge-coloring using the colors $1, 2, \ldots, m$. Let $V(K_{m(p)}) = \bigcup_{i=1}^{m} V_i$ where $V_i = \{x_{i,j} \mid j \in \mathbb{Z}_p\}$ and $L = [\ell_{h,k}]$ be a circulant latin square of order p as defined before Fig. 2. Now, we have an mp-edge-coloring of $K_{m(p)}$ by letting $\varphi(x_{a,b}x_{c,d}) = \ell_{b,d} + (\mu(x_ax_c) - 1) \cdot p$, where $a, c \in \mathbb{Z}_m$ and $b, d \in \mathbb{Z}_p$. Therefore, the edges in $K_{m(p)}$ joining a vertex of V_a to a vertex of V_c, denoted (V_a, V_c), are colored with the entries in $L + (\mu(x_ax_c) - 1) \cdot p$. It is not difficult to see that φ is a proper edge-coloring of $K_{m(p)}$. Now, it is left to show that the edges of $K_{m(p)}$ can be partitioned into multicolored Hamiltonian cycles.

Let $C = (x_{i_1}, x_{i_2}, \ldots, x_{i_m})$ be a multicolored Hamiltonian cycle in K_m obtained from the *MHCP* of K_m. Define $C_{m(p)}$ to be the subgraph induced by the set of edges in $(V_{i_1}, V_{i_2}), (V_{i_2}, V_{i_3}), \ldots, (V_{i_{m-1}}, V_{i_m}), (V_{i_m}, V_{i_1})$. Then let $S(r_1, r_2, \ldots, r_m)$, where $r_j \in \{0, 1, \ldots, p-1\}$ for $1 \le j \le m$, be the set of perfect matchings in $(V_{i_1}, V_{i_2}), (V_{i_2}, V_{i_3}), \ldots, (V_{i_{m-1}}, V_{i_m})$ and (V_{i_m}, V_{i_1}), respectively, where the perfect matching in $(V_{i_j}, V_{i_{j+1}})$ is the set of edges $x_{i_j,a}x_{i_{j+1},b}$ with $b - a \equiv r_j \pmod p$ for each $j \in \mathbb{Z}_m$. Since these perfect matchings of $(V_{i_j}, V_{i_{j+1}})$ are multicolored, we have that $S(r_1, r_2, \ldots, r_m)$ is a multicolored 2-factor of

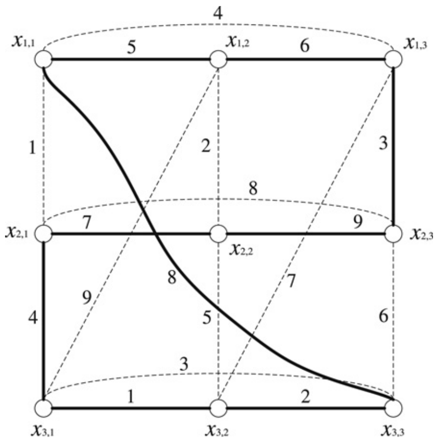

Fig. 3. Two multicolored Hamiltonian cycles.

$C_{m(n)}$. Hence, we can partition the edges of $C_{m(p)}$ into p multicolored 2-factors due to the fact that $r_i \in \{0, 1, \dots, p-1\}$. Note that $S(r_0, r_1, \dots, r_{m-1})$ and $S(r'_0, r'_1, \dots, r'_{m-1})$ are edge-disjoint 2-factors if and only if $r_i \neq r'_i$ for each $i \in \mathbb{Z}_m$.

The proof follows by selecting $(r_0, r_1, \dots, r_{m-1}) \in \mathbb{Z}_p^m$ properly in order that each 2-factor $S(r_0, r_1, \dots, r_{m-1})$ of $C_{m(p)}$ is a Hamiltonian cycle. Observe that if $\sum_{i=0}^{m-1} r_i$ is not a multiple of p (odd prime), then $S(r_0, r_1, \dots, r_{m-1})$ is a Hamiltonian cycle. From Lemma 3.1, let $SS_0, SS_1, \dots, SS_{p-1}$ be the 2-factors of $C_{m(p)}$. This implies that we have a partition of the edges of $C_{m(p)}$ into p edge-disjoint multicolored Hamiltonian cycles. Moreover, since $K_{m(p)}$ can be partitioned into $\frac{m-1}{2}$ copies of $C_{m(p)}$ where each $C_{m(p)}$ arises from a multicolored Hamiltonian cycle in K_m, we have a partition of the edges of $K_{m(p)}$ into $\frac{m-1}{2} \cdot p$ multicolored Hamiltonian cycles. ■

As an example, if $m = p = 3$, then the three multicolored Hamiltonian cycles are $S(0, 0, 1) = (x_{1,1}, x_{2,1}, x_{3,1}, x_{1,2}, x_{2,2}, x_{3,2}, x_{1,3}, x_{2,3}, x_{3,3})$, $S(1, 1, 2) = (x_{1,1}, x_{2,2}, x_{3,3}, x_{1,2}, x_{2,3}, x_{3,1}, x_{1,3}, x_{2,1}, x_{3,2})$, $S(2, 2, 0) = (x_{1,1}, x_{2,3}, x_{3,2}, x_{1,3}, x_{2,2}, x_{3,1}, x_{1,2}, x_{2,1}, x_{3,3})$. In case that $m = 5$ and $p = 3$, then we have 6 multicolored Hamiltonian cycles. For each $C_{5(3)}$, we have three multicolored Hamiltonian cycles of type $S(0, 0, 0, 0, 1)$, $S(1, 1, 1, 2, 2)$, and $S(2, 2, 2, 1, 0)$.

Now, in order to partition the edges of a 9-edge-colored K_9 into 4 Hamiltonian cycles, we combine $S(0, 0, 1)$ with the three cliques (K_3) induced by the three partite sets V_1, V_2 and V_3, to obtain a 4-factor. Since these K_3's can be edge-colored with $\{4, 5, 6\}$, $\{7, 8, 9\}$ and $\{1, 2, 3\}$, respectively, we have an edge-colored 4-factor with each color occurs exactly twice. Thus, if this 4-factor can be partitioned into two multicolored Hamiltonian cycles, then we conclude that K_9 admits an *MHCP*. Fig. 3 shows how this can be done.

Notice that in the induced subgraphs $\langle V_1\rangle$, $\langle V_2\rangle$ and $\langle V_3\rangle$ we have exactly one edge from each graph which is not included in the cycle with solid edges. Therefore, we may first color the edges in $\langle V_1\rangle$, $\langle V_2\rangle$ and $\langle V_3\rangle$, respectively, and then adjust the colors in (V_1, V_2), (V_2, V_3) and (V_3, V_1), respectively, in order to obtain a multicolored Hamiltonian cycle. For example, if the color of $x_{0,0}x_{0,2}$ is 5 instead of 4, then we permute (or interchange) the two entries in

4	6	5
6	5	4
5	4	6

, and thus the latin square used to color (V_2, V_3) becomes

5	6	4
6	4	5
4	5	6

. This is an essential trick we shall use when p is a larger prime.

Theorem 3.3. *For each odd integer $v \geq 3$, K_v admits an MHCP.*

Proof. The proof is by induction on v. By Theorem 2.1, the assertion is true for v is a prime. Therefore, we assume that v is a composite odd integer and the assertion is true for each odd order $u < v$. Let p be the smallest prime such that $v = p \cdot m$ and $V(K_v) = \bigcup_{i=1}^m V_i$ where $V_i = \{x_{i,j} \mid j \in \mathbb{Z}_p\}$, $i \in \mathbb{Z}_m$. By induction, K_m admits an *MHCP* and hence $K_{m(p)}$ can be partitioned into $\frac{m-1}{2}$ $C_{m(p)}$'s each of which admits an *MHCP*. Moreover, by Lemma 3.2, each *MHCP* of $C_{m(p)}$ contains a multicolored Hamiltonian cycle $S(0, 0, \dots, 0, 1)$. Here, the edge-coloring φ of $K_{m(p)}$ is induced by the edge-coloring μ of K_m defined as in Lemma 3.2. That is, if v_iv_j is an edge of K_m with color $\mu(v_iv_j) = t \in \mathbb{Z}_m$, then the colors of the edges in (V_i, V_j) are assigned by using $M + (t-1)p$ where M is a circulant latin square of order p as defined before Fig. 2. We note here again that permuting the entries of a latin square $M + (t-1)p$ gives another edge-coloring, but the edge-coloring is still proper.

So, in order to obtain an *MHCP* of K_v, we first give a v-edge-coloring of K_v and then adjust the coloring if it is necessary. Since $K_{m(p)}$ has an mp-edge-coloring φ, the edge-coloring π of K_v can be defined as follows: (a) $\pi|_{K_{m(p)}} = \varphi$ and (b) $\pi|_{\langle V_i\rangle} = \psi_i$, $i = 1, 2, \dots, m$, where ψ_i is an p-edge-coloring of K_p such that K_p can be partitioned into $\frac{p-1}{2}$ multicolored Hamiltonian cycles. Moreover, the images of ψ_i are $1 + (t-1)p, 2 + (t-1)p, \dots, p + (t-1)p$ where $t \in \mathbb{Z}_m$ and t is the color not occurring in the edges incident to $v_i \in V(K_m)$. (Here, the colors used to color the edges of K_m are $1, 2, 3, \dots, m$.)

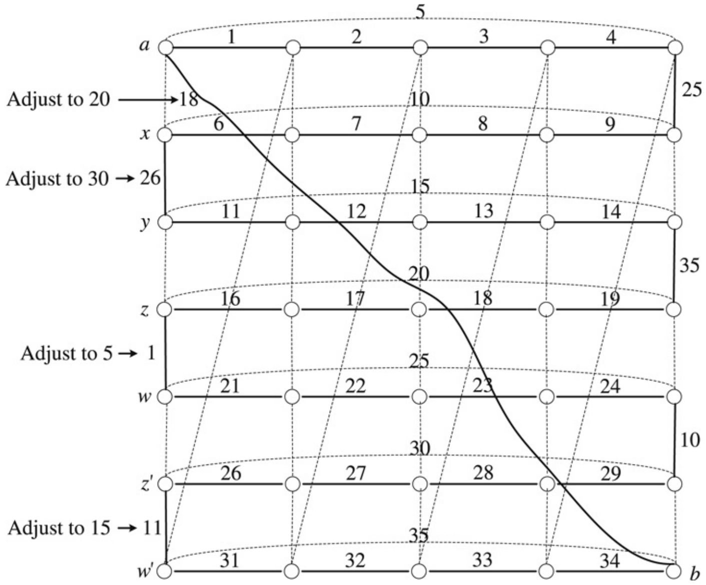

Fig. 4. $E^{(1)} \cup 7D^{(1)}$ in K_{35}.

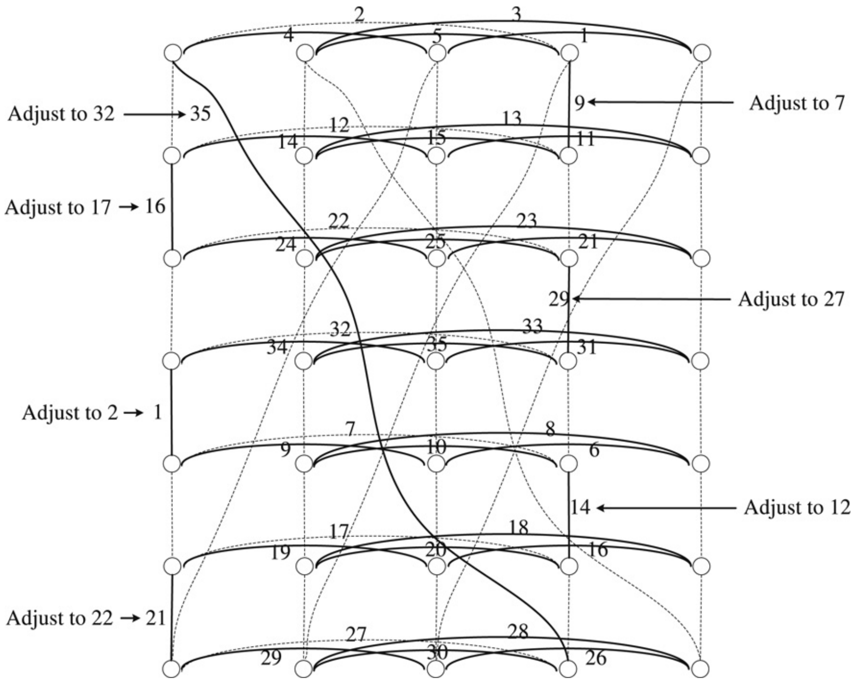

Fig. 5. $E^{(2)} \cup 7D^{(2)}$ in K_{35}.

It is not difficult to check that π is a v-edge-coloring of K_v. We shall revise π by permuting the colors in (V_i, V_{i+1}) for some i and finally obtain the edge-coloring we need.

Let the edges of the K_p induced by V_1 be partitioned into $\frac{p-1}{2}$ multicolored Hamiltonian cycles $D^{(1)}, D^{(2)}, \ldots, D^{(\frac{p-1}{2})}$, and x_{1,t_i} is the neighbor with the larger index t_i of $x_{1,1}$ in $D^{(i)}$. Hence, the m copies of K_p each induces by V_i can be partitioned into m copies of $D^{(1)}, D^{(2)}, \ldots$, and $D^{(\frac{p-1}{2})}$. For convenience, denote them as $mD^{(i)}$, $i = 1, 2, \ldots, \frac{p-1}{2}$. Now, let the edges of $K_{m(p)}$ be partitioned into $C^{(1)}_{m(p)}, C^{(2)}_{m(p)}, \ldots, C^{(\frac{m-1}{2})}_{m(p)}$. By Lemma 3.1, we can let each of $C^{(1)}_{m(p)}, C^{(2)}_{m(p)}, \ldots, C^{(\frac{p-1}{2})}_{m(p)}$ contains a multicolored Hamiltonian cycle $E^{(1)}, E^{(2)}, \ldots, E^{(\frac{p-1}{2})}$ of type $S(0, 0, \ldots, 0, p+1-t_i)$. Since $m \geq p$, we consider the 4-factors $E^{(i)} \cup mD^{(i)}$ where $i = 1, 2, \ldots, \frac{p-1}{2}$. Starting from $i = 1$, we shall partition the edges of $E^{(1)} \cup mD^{(1)}$ into two Hamiltonian cycles such that both of them are multicolored. By the idea explained in Fig. 3, we first obtain two Hamiltonian cycles from $E^{(1)} \cup mD^{(1)}$ by a similar way, see Fig. 4 for example. For the purpose of obtaining multicolored Hamiltonian cycles, we adjust the colors by permuting them in the latin square for (V_i, V_{i+1}) to make sure the first cycle does contain each color exactly once. Then, the second one is clearly multicolored. Now, following the same process, we partition the edges of $E^{(2)} \cup mD^{(2)}, \ldots$, and $E^{(\frac{p-1}{2})} \cup mD^{(\frac{p-1}{2})}$ into two multicolored Hamiltonian cycles, respectively. We remark here that if permuting entries of a latin square is necessary, then we can keep doing the same trick since $C^{(1)}_{m(p)}, C^{(2)}_{m(p)}, \ldots, C^{(\frac{m-1}{2})}_{m(p)}$ are edge-disjoint subgraphs of $K_{m(p)}$. (The permutations are carried out independently.) This implies that after all the permutations are done, we obtain a v-edge-coloring of K_v such that K_v can be partitioned into $\frac{v-1}{2}$ multicolored Hamiltonian cycles. ■

In conclusion, we use Figs. 4 and 5 to explain how our idea works. In Fig. 4, $t_1 = 5$. The edge xy was colored with 26 originally by using the circulant latin square of order 5 mentioned before Fig. 2. But, 26 occurs in the Hamiltonian cycle with solid edges already. Therefore, we use (26, 30) to permute the square to obtain the edge-coloring we would like to have. After adjusting the colors of $zw, z'w'$ and ab, respectively, we have two multicolored Hamiltonian cycles as desired. In Fig. 5, $t_2 = 4$. For convenience, we reset $V_1, V_3, V_5, V_7, V_2, V_4, V_6$ from top to down. Following the same process, we also have two multicolored Hamiltonian cycles.

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