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Some identities on the Bernoulli, Euler and Genocchi polynomials via power sums and alternate power sums

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ABSTRACT

In this paper, by the generating function method, we establish various identities concerning the (higher order) Bernoulli polynomials, the (higher order) Euler polynomials, the Genocchi polynomials and the degenerate higher order Bernoulli polynomials. Particularly, some of these identities are also related to the power sums and alternate power sums. It can be found that, many well known results, especially the multiplication theorems, and some symmetric identities demonstrated recently, are special cases of our results.

1. Introduction

For a real or complex parameter α, the higher order Bernoulli polynomials $B_n^{(\alpha)}(x)$ and the higher order Euler polynomials $E_n^{(\alpha)}(x)$, each of degree n in x as well as in α, are defined by the following generating functions (for details, see [11, Section 2.8] and [13, Section 1.6]):

$$\left(\frac{t}{e^t-1}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} B_n^{(\alpha)}(x)\frac{t^n}{n!}, \quad (|t|<2\pi), \tag{1.1}$$

$$\left(\frac{2}{e^t+1}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} E_n^{(\alpha)}(x)\frac{t^n}{n!}, \quad (|t|<\pi). \tag{1.2}$$

(In fact, the order α in Eqs. (1.1) and (1.2) can also be p-adic or indeterminate, as occurs in the paper of Adelberg [2], and the only necessary condition is that α should lie in a commutative ring with unity.) Clearly, for all nonnegative integers n, the classical Bernoulli polynomials $B_n(x)$ and the classical Euler polynomials $E_n(x)$ are given by

$$B_n(x) := B_n^{(1)}(x) \quad \text{and} \quad E_n(x) := E_n^{(1)}(x), \tag{1.3}$$

respectively. Moreover, the classical Bernoulli numbers B_n and the classical Euler numbers E_n are given by

$$B_n := B_n(0) \quad \text{and} \quad E_n := 2^n E_n\left(\frac{1}{2}\right), \tag{1.4}$$

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respectively. These polynomials and numbers have numerous important applications in combinatorics, number theory and numerical analysis. Therefore they have been studied extensively over the last two centuries.

From definition (1.3) and generating functions (1.1) and (1.2), it is easily observed that

$$B_n(x+y)=\sum_{k=0}^{n}\binom{n}{k}B_k(x)y^{n-k}, \tag{1.5}$$

$$E_n(x+y)=\sum_{k=0}^{n}\binom{n}{k}E_k(x)y^{n-k}. \tag{1.6}$$

These two identities will be frequently made use of in the next two sections. Additionally, we have $B_n^{(0)}(x)=E_n^{(0)}(x)=x^n$.

The power sums and the alternate power sums are respectively defined by

$$S_k(n)=\sum_{i=0}^{n}i^k,\qquad T_k(n)=\sum_{i=0}^{n}(-1)^i i^k, \tag{1.7}$$

and their exponential generating functions are

$$\sum_{k=0}^{\infty}S_k(n)\frac{t^k}{k!}=\frac{e^{(n+1)t}-1}{e^t-1},\qquad \sum_{k=0}^{\infty}T_k(n)\frac{t^k}{k!}=\frac{1-(-e^t)^{n+1}}{1+e^t}. \tag{1.8}$$

The following are some special values:

$$\begin{gathered} S_k(0)=T_k(0)=\delta_{0,k},\\ S_k(1)=0^k+1^k=\delta_{0,k}+1,\qquad T_k(1)=0^k-1^k=\delta_{0,k}-1, \end{gathered} \tag{1.9}$$

where $\delta_{i,j}$ is the Kronecker delta defined by $\delta_{i,i}=1$ and $\delta_{i,j}=0$ for $i\neq j$.

It is well known that the power sums and the alternate power sums are closely related to the Bernoulli polynomials and the Euler polynomials, respectively, as follows (see [1, Eq. (23.1.4)]):

$$S_k(n)=\sum_{i=0}^{n}i^k=\frac{B_{k+1}(n+1)-B_{k+1}}{k+1},$$

$$(-1)^nT_k(n)=\sum_{i=0}^{n}(-1)^{n-i}i^k=\frac{E_k(n+1)+(-1)^nE_k(0)}{2},$$

where n and k are nonnegative integers. Therefore, it will be instructive and interesting to do some further research on the relations between these famous combinatorial sequences. Now, let us briefly introduce some results of this subject.

In [12], Namias derived the following two recurrence relations for the Bernoulli numbers B_k:

$$B_m=\frac{1}{2(1-2^m)}\sum_{k=0}^{m-1}2^k\binom{m}{k}B_k,$$

$$B_m=\frac{1}{3(1-3^m)}\sum_{k=0}^{m-1}3^k\binom{m}{k}B_k(1+2^{m-k}).$$

He conjectured that an infinite number of such recurrence relations can be obtained.

Subsequently, the formula

$$B_m=\frac{1}{a(1-a^m)}\sum_{k=0}^{m-1}a^k\binom{m}{k}B_k\sum_{i=1}^{a-1}i^{m-k} \tag{1.10}$$

where m and a are positive integers with $a>1$, was proved by Deeba and Rodriguez [7], Gessel [8] and Howard [9]. Howard also established similar recurrences for the Genocchi numbers [9] and the degenerate Bernoulli numbers [10]. Recently, Howard and Cenkci further extended the study to the higher order degenerate Bernoulli numbers [4].

Recurrence (1.10) can be rewritten as the identity below:

$$B_n=\sum_{k=0}^{n}\binom{n}{k}a^{k-1}B_kS_{n-k}(a-1).$$

Tuenter [15] found that this identity is a special case of the following symmetric one

$$\sum_{k=0}^{n}\binom{n}{k}a^{k-1}B_kb^{n-k}S_{n-k}(a-1)=\sum_{k=0}^{n}\binom{n}{k}b^{k-1}B_ka^{n-k}S_{n-k}(b-1), \tag{1.11}$$

where a and b are positive integers and n is a nonnegative integer. Most recently, Tuenter's result was generalized to the higher order Bernoulli polynomials by Yang [17] and to the degenerate Bernoulli polynomials by Young [18].

From the results referred to above, we can see that most of them are related to the Bernoulli numbers and their various generalizations. Thus, it is natural to consider the problem of whether the Euler numbers and their generalizations satisfy similar identities. In Section 2, we will discuss this problem in detail, by means of the expansions of the hyperbolic cotangent and of the hyperbolic tangent. In Section 3, we present some mixed type identities, i.e., identities involving both the Bernoulli polynomials and the Euler polynomials. Additionally, we list explicitly there the well known multiplication formulas and several similar ones. For completeness, we demonstrate the corresponding identities for the Genocchi polynomials in Section 4. Finally, in Section 5, we study briefly the higher order degenerate Bernoulli polynomials.

2. Identities related to the Euler polynomials

The study of Section 2 is based on the generating function

$$g(t) = \frac{e^{abxt}(1-(-e^{bt})^a)e^{abyt}}{(e^{at}+1)^m(e^{bt}+1)^m}.$$

2.1. The first part for the Euler polynomials

In this subsection, we focus on some identities containing the (higher order) Euler polynomials and the alternate power sums.

Theorem 2.1. *For integers $n \geq 0$, $a \geq 1$ and $b \geq 1$, if a and b have the same parity, then we have*

$$\sum_{k=0}^{n}\binom{n}{k}a^{n-k}b^kE_{n-k}^{(m)}(bx)\sum_{i=0}^{k}\binom{k}{i}T_i(a-1)E_{k-i}^{(m-1)}(ay) = \sum_{k=0}^{n}\binom{n}{k}b^{n-k}a^kE_{n-k}^{(m)}(ax)\sum_{i=0}^{k}\binom{k}{i}T_i(b-1)E_{k-i}^{(m-1)}(by). \quad (2.1)$$

Proof. We first use (1.2) and (1.8) to expand $g(t)$ as

$$\begin{aligned} g(t) &= \frac{1}{2^{2m-1}}\left(\frac{2}{e^{at}+1}\right)^m e^{abxt}\left(\frac{1-(-e^{bt})^a}{1+e^{bt}}\right)\left(\frac{2}{e^{bt}+1}\right)^{m-1}e^{abyt} \\ &= \frac{1}{2^{2m-1}}\left(\sum_{n=0}^{\infty}E_n^{(m)}(bx)\frac{(at)^n}{n!}\right)\left(\sum_{n=0}^{\infty}T_n(a-1)\frac{(bt)^n}{n!}\right)\left(\sum_{n=0}^{\infty}E_n^{(m-1)}(ay)\frac{(bt)^n}{n!}\right) \\ &= \frac{1}{2^{2m-1}}\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}a^{n-k}b^kE_{n-k}^{(m)}(bx)\sum_{i=0}^{k}\binom{k}{i}T_i(a-1)E_{k-i}^{(m-1)}(ay)\right)\frac{t^n}{n!}. \end{aligned} \quad (2.2)$$

Note that if a and b have the same parity, then $g(t)$ is symmetric in a and b. Thus, we may also expand $g(t)$ as

$$g(t) = \frac{1}{2^{2m-1}}\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}b^{n-k}a^kE_{n-k}^{(m)}(ax)\sum_{i=0}^{k}\binom{k}{i}T_i(b-1)E_{k-i}^{(m-1)}(by)\right)\frac{t^n}{n!}.$$

Equating coefficients of $t^n/n!$ in the right-hand sides of the last two equations gives the identity of the theorem. □

Remark. It should be noticed that the variables x and y in the right-hand side of identity (2.1) can be exchanged, and similarly for many results in the current paper. However, for simplicity, we will not discuss this case further.

Putting $m = 1$ and $y = 0$ in Theorem 2.1 yields the corollary below.

Corollary 2.2. *For integers $n \geq 0$, $a \geq 1$ and $b \geq 1$, if a and b have the same parity, then we have*

$$\sum_{k=0}^{n}\binom{n}{k}a^kb^{n-k}E_k(bx)T_{n-k}(a-1) = \sum_{k=0}^{n}\binom{n}{k}b^ka^{n-k}E_k(ax)T_{n-k}(b-1). \quad (2.3)$$

Corollary 2.3. *For any nonnegative integer n and any positive odd integer a, we have*

$$E_n(ax) = \sum_{k=0}^{n}\binom{n}{k}a^kE_k(x)T_{n-k}(a-1). \quad (2.4)$$

For any nonnegative integer n and any positive even integer a, we have

$$E_n(ax) - E_n\left(ax + \frac{a}{2}\right) = \sum_{k=0}^{n} \binom{n}{k} \left(\frac{a}{2}\right)^k E_k(2x) T_{n-k}(a-1). \tag{2.5}$$

Proof. (2.4) can be derived from (2.3) by putting $b = 1$. To prove (2.5), note that when $b = 2$, the right-hand side of identity (2.3) turns into

$$\begin{aligned}\sum_{k=0}^{n} \binom{n}{k} 2^k a^{n-k} E_k(ax) T_{n-k}(1) &= 2^n \sum_{k=0}^{n} \binom{n}{k} \left(\frac{a}{2}\right)^{n-k} E_k(ax)(0^{n-k} - 1^{n-k}) \\ &= 2^n \sum_{k=0}^{n} \binom{n}{k} E_k(ax) \left(0^{n-k} - \left(\frac{a}{2}\right)^{n-k}\right).\end{aligned}$$

Making use of identity (1.6) and combining with the left-hand side of (2.3), one can obtain the desired result immediately. □

According to [16], the hyperbolic cotangent satisfies

$$\coth z = \frac{e^{2z} + 1}{e^{2z} - 1} = \sum_{n=-1}^{\infty} \frac{2^n (B_{n+1} + B_{n+1}(1))}{(n+1)!} z^n = \sum_{n=0}^{\infty} \frac{2^{n-1}(B_n + B_n(1))}{n!} z^{n-1}, \tag{2.6}$$

from which we can obtain the following theorem.

Theorem 2.4. *For integers $n \geq 0$, $a \geq 1$ and $b \geq 1$, if a is odd and b is even, then we have*

$$\begin{aligned}&\sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k E_{n-k}^{(m)}(bx) \sum_{i=0}^{k} \binom{k}{i} T_i(a-1) E_{k-i}^{(m-1)}(ay) \\ &= -2 \sum_{l=0, l\neq n}^{n+1} \binom{n+1}{l} \frac{B_{n+1-l}}{n+1} a^{n-l} \sum_{k=0}^{l} \binom{l}{k} b^{n-k} a^k E_{l-k}^{(m)}(ax) \sum_{i=0}^{k} \binom{k}{i} T_i(b-1) E_{k-i}^{(m-1)}(by).\end{aligned}$$

Proof. When a and b have different parity, $g(t)$ is not symmetric in a and b, so we have

$$\begin{aligned}g(t) &= \frac{1}{2^{2m-1}} \left(\frac{2}{e^{bt} + 1}\right)^m e^{abxt} \left(\frac{1 - (-e^{at})^b}{1 + e^{at}}\right) \left(\frac{1 - (-e^{bt})^a}{1 - (-e^{at})^b}\right) \left(\frac{2}{e^{at} + 1}\right)^{m-1} e^{abyt} \\ &= \frac{1}{2^{2m-1}} \left(\sum_{n=0}^{\infty} E_n^{(m)}(ax) \frac{(bt)^n}{n!}\right) \left(\sum_{n=0}^{\infty} T_n(b-1) \frac{(at)^n}{n!}\right) \left(-\sum_{n=0}^{\infty} \frac{B_n + B_n(1)}{n!} (abt)^{n-1}\right) \\ &\quad \times \left(\sum_{n=0}^{\infty} E_n^{(m-1)}(by) \frac{(at)^n}{n!}\right).\end{aligned}$$

Equating coefficients of $t^n/n!$ in the last equation and Eq. (2.2), and making use of the identity $B_n(1) - B_n = \delta_{n,1}$ for $n \geq 0$, we can obtain the final result. □

Setting $m = 1$ and $y = 0$ in Theorem 2.4 gives the following corollary.

Corollary 2.5. *For integers $n \geq 0$, $a \geq 1$ and $b \geq 1$, if a is odd and b is even, then we have*

$$\sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k E_{n-k}(bx) T_k(a-1) = -2 \sum_{l=0, l\neq n}^{n+1} \binom{n+1}{l} \frac{B_{n+1-l}}{n+1} a^{n-l} \sum_{k=0}^{l} \binom{l}{k} b^{n-k} a^k E_{l-k}(ax) T_k(b-1).$$

Corollary 2.6. *For any nonnegative integer n and any positive even integer b, we have*

$$E_n(bx) = -2 \sum_{l=0, l\neq n}^{n+1} \binom{n+1}{l} \frac{B_{n+1-l}}{n+1} \sum_{k=0}^{l} \binom{l}{k} b^{n-k} E_{l-k}(x) T_k(b-1), \tag{2.7}$$

$$E_n(2x) = 2^{n+1} \sum_{l=0, l\neq n}^{n+1} \binom{n+1}{l} \frac{B_{n+1-l}}{n+1} \left(E_l\left(x + \frac{1}{2}\right) - E_l(x)\right), \tag{2.8}$$

$$E_n(0) = 2 \sum_{l=0, l\neq n}^{n+1} \binom{n+1}{l} \frac{B_{n+1-l}}{n+1} (2^{n-l} E_l - 2^n E_l(0)). \tag{2.9}$$

Proof. The substitution $a = 1$ in Corollary 2.5 gives identity (2.7). By putting $b = 2$ in (2.7) and taking into account Eq. (1.6), one has (2.8) as an immediate consequence. Finally, it can be found that the $x = 0$ case of (2.8) is identity (2.9). □

According to [16], the hyperbolic tangent satisfies

$$\tanh z = \frac{e^{2z} - 1}{e^{2z} + 1} = \sum_{n=1}^{\infty} \frac{2^{n+1}(2^{n+1} - 1)B_{n+1}}{(n+1)!} z^n. \tag{2.10}$$

Since $E_n(0) = \frac{2(1-2^{n+1})}{n+1} B_{n+1}$ for $n \geq 0$ (see [1, Eq. (23.1.20)] and [14, Eq. (41)]), then

$$\tanh z = -\sum_{n=1}^{\infty} E_n(0) \frac{(2z)^n}{n!}.$$

Similarly to Theorem 2.4, the following result can be established.

Theorem 2.7. *For integers $n \geq 1$, $a \geq 1$ and $b \geq 1$, if a is even and b is odd, then we have*

$$\begin{aligned} &\sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k E_{n-k}^{(m)}(bx) \sum_{i=0}^{k} \binom{k}{i} T_i(a-1) E_{k-i}^{(m-1)}(ay) \\ &= \sum_{l=0}^{n-1} \binom{n}{l} E_{n-l}(0) a^{n-l} \sum_{k=0}^{l} \binom{l}{k} b^{n-k} a^k E_{l-k}^{(m)}(ax) \sum_{i=0}^{k} \binom{k}{i} T_i(b-1) E_{k-i}^{(m-1)}(by). \end{aligned} \tag{2.11}$$

The following are special cases of Theorem 2.7.

Corollary 2.8. *For integers $n \geq 1$, $a \geq 1$ and $b \geq 1$, if a is even and b is odd, then we have*

$$\sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} E_k(bx) T_{n-k}(a-1) = \sum_{k=0}^{n-1} \binom{n}{k} b^k a^{n-k} E_k(ax)(E_{n-k}(0) - T_{n-k}(b-1)). \tag{2.12}$$

Proof. Putting $m = 1$ and $y = 0$ in Theorem 2.7 gives

$$\begin{aligned} \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} E_k(bx) T_{n-k}(a-1) &= \sum_{l=0}^{n-1} \binom{n}{l} E_{n-l}(0) a^{n-l} \sum_{k=0}^{l} \binom{l}{k} b^{n-l+k} a^{l-k} E_k(ax) T_{l-k}(b-1) \\ &= \sum_{k=0}^{n-1} \binom{n}{k} b^k a^{n-k} E_k(ax) \sum_{l=k}^{n-1} \binom{n-k}{l-k} b^{n-l} E_{n-l}(0) T_{l-k}(b-1). \end{aligned}$$

Making use of (2.4), one has identity (2.12) finally. □

Corollary 2.9. *For any positive integer n and any positive odd integer b, we have*

$$E_n(bx) - E_n\left(bx + \frac{b}{2}\right) = \sum_{k=0}^{n-1} \binom{n}{k} \left(\frac{b}{2}\right)^k E_k(2x)(E_{n-k}(0) - T_{n-k}(b-1)), \tag{2.13}$$

$$E_n(x) - E_n\left(x + \frac{1}{2}\right) = \sum_{k=0}^{n-1} \binom{n}{k} 2^{-k} E_k(2x) E_{n-k}(0), \tag{2.14}$$

$$E_n = -\sum_{k=1}^{n-1} \binom{n}{k} 2^k E_k(0) E_{n-k}(0). \tag{2.15}$$

Proof. When $a = 2$, identity (2.12) reduces to (2.13). Putting $b = 1$ in (2.13) yields identity (2.14). The further substitution $x = 0$ in (2.14) gives identity (2.15). □

2.2. The second part for the Euler polynomials

The identities demonstrated in this subsection do not contain the alternate power sums, but the readers can find that they are associated with the multiplication theorem for the Euler polynomials.

Theorem 2.10. *For integers $n \geq 0$, $a \geq 1$ and $b \geq 1$, if a and b have the same parity, then we have*

$$\sum_{k=0}^{n}\binom{n}{k}a^k b^{n-k}\sum_{i=0}^{a-1}(-1)^i E_k^{(m)}\left(bx+\frac{b}{a}\mathrm{i}\right)E_{n-k}^{(m-1)}(ay)=\sum_{k=0}^{n}\binom{n}{k}b^k a^{n-k}\sum_{i=0}^{b-1}(-1)^i E_k^{(m)}\left(ax+\frac{a}{b}\mathrm{i}\right)E_{n-k}^{(m-1)}(by).$$

Proof. We expand $g(t)$ as follows:

$$\begin{aligned}
g(t) &= \frac{1}{2^{2m-1}}\left(\frac{2}{\mathrm{e}^{at}+1}\right)^m \mathrm{e}^{abxt}\left(\frac{1-(-\mathrm{e}^{bt})^a}{1+\mathrm{e}^{bt}}\right)\left(\frac{2}{\mathrm{e}^{bt}+1}\right)^{m-1}\mathrm{e}^{abyt}\\
&= \frac{1}{2^{2m-1}}\left(\frac{2}{\mathrm{e}^{at}+1}\right)^m \mathrm{e}^{abxt}\left(\sum_{i=0}^{a-1}(-1)^i\mathrm{e}^{bti}\right)\left(\frac{2}{\mathrm{e}^{bt}+1}\right)^{m-1}\mathrm{e}^{abyt}\\
&= \frac{1}{2^{2m-1}}\left(\sum_{i=0}^{a-1}(-1)^i\left(\frac{2}{\mathrm{e}^{at}+1}\right)^m \mathrm{e}^{\left(bx+\frac{b}{a}\mathrm{i}\right)at}\right)\left(\frac{2}{\mathrm{e}^{bt}+1}\right)^{m-1}\mathrm{e}^{abyt}\\
&= \frac{1}{2^{2m-1}}\left(\sum_{i=0}^{a-1}(-1)^i\sum_{n=0}^{\infty}E_n^{(m)}\left(bx+\frac{b}{a}\mathrm{i}\right)\frac{(at)^n}{n!}\right)\left(\sum_{n=0}^{\infty}E_n^{(m-1)}(ay)\frac{(bt)^n}{n!}\right)\\
&= \frac{1}{2^{2m-1}}\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}a^k b^{n-k}\sum_{i=0}^{a-1}(-1)^i E_k^{(m)}\left(bx+\frac{b}{a}\mathrm{i}\right)E_{n-k}^{(m-1)}(ay)\right)\frac{t^n}{n!}.
\end{aligned} \tag{2.16}$$

According to the conditions of the theorem, $g(t)$ is symmetric in a and b, so we also have

$$g(t)=\frac{1}{2^{2m-1}}\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}b^k a^{n-k}\sum_{i=0}^{b-1}(-1)^i E_k^{(m)}\left(ax+\frac{a}{b}\mathrm{i}\right)E_{n-k}^{(m-1)}(by)\right)\frac{t^n}{n!}.$$

By equating coefficients of $t^n/n!$ in the right-hand sides of the last two equations, the identity can be obtained. □

When $m = 1$ and $y = 0$, Theorem 2.10 yields the following corollary.

Corollary 2.11. *For integers $n \geq 0$, $a \geq 1$ and $b \geq 1$, if a and b have the same parity, then we have*

$$a^n\sum_{i=0}^{a-1}(-1)^i E_n\left(bx+\frac{b}{a}\mathrm{i}\right)=b^n\sum_{i=0}^{b-1}(-1)^i E_n\left(ax+\frac{a}{b}\mathrm{i}\right).$$

Corollary 2.12. *For any nonnegative integer n and any positive odd integer a, we have*

$$E_n(ax)=a^n\sum_{i=0}^{a-1}(-1)^i E_n\left(x+\frac{1}{a}\mathrm{i}\right). \tag{2.17}$$

For any nonnegative integer n and any positive even integer a, we have

$$E_n(ax)-E_n\left(ax+\frac{a}{2}\right)=\left(\frac{a}{2}\right)^n\sum_{i=0}^{a-1}(-1)^i E_n\left(2x+\frac{2}{a}\mathrm{i}\right). \tag{2.18}$$

Proof. Putting $b = 1$ and $b = 2$ in Corollary 2.11 will give (2.17) and (2.18), respectively. It should be noticed that (2.17) is one of the two formulas of the multiplication theorem for the Euler polynomials [1, Eq. (23.1.10)]. □

Theorem 2.13. *For integers $n \geq 0$, $a \geq 1$ and $b \geq 1$, if a is odd and b is even, then we have*

$$\begin{aligned}
&\sum_{k=0}^{n}\binom{n}{k}a^k b^{n-k}\sum_{i=0}^{a-1}(-1)^i E_k^{(m)}\left(bx+\frac{b}{a}\mathrm{i}\right)E_{n-k}^{(m-1)}(ay)\\
&\quad=-2\sum_{l=0,l\neq n}^{n+1}\binom{n+1}{l}\frac{B_{n+1-l}}{n+1}b^{n-l}\sum_{k=0}^{l}\binom{l}{k}b^k a^{n-k}\sum_{i=0}^{b-1}(-1)^i E_k^{(m)}\left(ax+\frac{a}{b}\mathrm{i}\right)E_{l-k}^{(m-1)}(by).
\end{aligned}$$

Proof. By means of (2.6), $g(t)$ has the following expansion:

$$g(t) = \frac{1}{2^{2m-1}} \left(\frac{2}{e^{bt}+1}\right)^m e^{abxt} \left(\frac{1-(-e^{at})^b}{1+e^{at}}\right) \left(\frac{1-(-e^{bt})^a}{1-(-e^{at})^b}\right) \left(\frac{2}{e^{at}+1}\right)^{m-1} e^{abyt}$$
$$= \frac{1}{2^{2m-1}} \left(-\sum_{n=0}^{\infty} \frac{B_n + B_n(1)}{n!} (abt)^{n-1}\right) \left(\sum_{i=0}^{b-1} (-1)^i \sum_{n=0}^{\infty} E_n^{(m)} \left(ax + \frac{a}{b} i\right) \frac{(bt)^n}{n!}\right) \left(\sum_{n=0}^{\infty} E_n^{(m-1)}(by) \frac{(at)^n}{n!}\right).$$

Thus, it suffices to identify coefficients of $t^n/n!$ in the last equation and Eq. (2.16) and make use of the identity $B_n(1) - B_n = \delta_{n,1}$. $\square$

To obtain the corollary below, we should substitute $m = 1$ and $y = 0$ into Theorem 2.13, just as what we have done for the theorems above.

Corollary 2.14. *For integers $n \ge 0$, $a \ge 1$ and $b \ge 1$, if a is odd and b is even, then we have*

$$\sum_{i=0}^{a-1} (-1)^i E_n \left(bx + \frac{b}{a} i\right) = -2 \sum_{l=0, l\neq n}^{n+1} \binom{n+1}{l} \frac{B_{n+1-l}}{n+1} b^n a^{-l} \sum_{i=0}^{b-1} (-1)^i E_l \left(ax + \frac{a}{b} i\right).$$

Putting $a = 1$ in Corollary 2.14 yields Corollary 2.15, which will further reduce to (2.8) by the substitution $b = 2$.

Corollary 2.15. *For any nonnegative integer n and any positive even integer b, we have*

$$E_n(bx) = -2 \sum_{l=0, l\neq n}^{n+1} \binom{n+1}{l} \frac{B_{n+1-l}}{n+1} b^n \sum_{i=0}^{b-1} (-1)^i E_l \left(x + \frac{1}{b} i\right). \tag{2.19}$$

Similarly to Theorem 2.13, the next theorem holds.

Theorem 2.16. *For integers $n \ge 1$, $a \ge 1$ and $b \ge 1$, if a is even and b is odd, then we have*

$$\sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} \sum_{i=0}^{a-1} (-1)^i E_k^{(m)} \left(bx + \frac{b}{a} i\right) E_{n-k}^{(m-1)}(ay)$$
$$= \sum_{l=0}^{n-1} \binom{n}{l} E_{n-l}(0) b^{n-l} \sum_{k=0}^{l} \binom{l}{k} b^k a^{n-k} \sum_{i=0}^{b-1} (-1)^i E_k^{(m)} \left(ax + \frac{a}{b} i\right) E_{l-k}^{(m-1)}(by).$$

The substitutions $m = 1$ and $y = 0$ in Theorem 2.16 give us the corollary below.

Corollary 2.17. *For integers $n \ge 1$, $a \ge 1$ and $b \ge 1$, if a is even and b is odd, then we have*

$$\sum_{i=0}^{a-1} (-1)^i E_n \left(bx + \frac{b}{a} i\right) = \sum_{l=0}^{n-1} \binom{n}{l} E_{n-l}(0) b^n a^{-l} \sum_{i=0}^{b-1} (-1)^i E_l \left(ax + \frac{a}{b} i\right).$$

3. Mixed type identities

Now, let us turn to the study of some mixed type identities, i.e., identities which contain both the Bernoulli polynomials and the Euler polynomials. It can be found that some elegant relations between the Bernoulli polynomials and the Euler polynomials are special cases of the results obtained here.

3.1. The first part for mixed type identities

In this subsection, we will establish some identities from the following generating function:

$$h(t) = \frac{t^m e^{abxt} (1 - (-e^{bt})^a) e^{abyt}}{(e^{at} - 1)^m (e^{bt} + 1)^m}.$$

Theorem 3.1. *For integers $n \geq 1$, $a \geq 1$ and $b \geq 1$, if a is even, then we have*

$$\sum_{k=0}^{n}\binom{n}{k}a^{n-k}b^{k}B_{n-k}^{(m)}(bx)\sum_{i=0}^{k}\binom{k}{i}T_i(a-1)E_{k-i}^{(m-1)}(ay)$$
$$=-\frac{n}{2}\sum_{k=0}^{n-1}\binom{n-1}{k}b^{n-1-k}a^{k+1}E_{n-1-k}^{(m)}(ax)\sum_{i=0}^{k}\binom{k}{i}S_i(b-1)B_{k-i}^{(m-1)}(by)$$

and

$$\sum_{k=0}^{n}\binom{n}{k}a^{k}b^{n-k}\sum_{i=0}^{a-1}(-1)^{i}B_k^{(m)}\left(bx+\frac{b}{a}\mathrm{i}\right)E_{n-k}^{(m-1)}(ay)=-\frac{n}{2}\sum_{k=0}^{n-1}\binom{n-1}{k}b^{k}a^{n-k}\sum_{i=0}^{b-1}E_k^{(m)}\left(ax+\frac{a}{b}\mathrm{i}\right)B_{n-1-k}^{(m-1)}(by).$$

Proof. On the one hand,

$$h(t) = \frac{1}{2^{m-1}a^m}\left(\frac{at}{\mathrm{e}^{at}-1}\right)^m \mathrm{e}^{abxt}\left(\frac{1-(-\mathrm{e}^{bt})^a}{1+\mathrm{e}^{bt}}\right)\left(\frac{2}{\mathrm{e}^{bt}+1}\right)^{m-1}\mathrm{e}^{abyt} \tag{3.1}$$

$$= \frac{1}{2^{m-1}a^m}\left(\sum_{n=0}^{\infty}B_n^{(m)}(bx)\frac{(at)^n}{n!}\right)\left(\sum_{n=0}^{\infty}T_n(a-1)\frac{(bt)^n}{n!}\right)\left(\sum_{n=0}^{\infty}E_n^{(m-1)}(ay)\frac{(bt)^n}{n!}\right). \tag{3.2}$$

On the other hand, since a is even, we also have

$$h(t) = -\frac{t}{2^m a^{m-1}}\left(\frac{2}{\mathrm{e}^{bt}+1}\right)^m \mathrm{e}^{abxt}\left(\frac{\mathrm{e}^{abt}-1}{\mathrm{e}^{at}-1}\right)\left(\frac{at}{\mathrm{e}^{at}-1}\right)^{m-1}\mathrm{e}^{abyt}$$

$$= -\frac{t}{2^m a^{m-1}}\left(\sum_{n=0}^{\infty}E_n^{(m)}(ax)\frac{(bt)^n}{n!}\right)\left(\sum_{n=0}^{\infty}S_n(b-1)\frac{(at)^n}{n!}\right)\left(\sum_{n=0}^{\infty}B_n^{(m-1)}(by)\frac{(at)^n}{n!}\right). \tag{3.3}$$

Equating coefficients of $t^n/n!$ gives the first identity.

Now, following (3.1), we expand $h(t)$ as

$$h(t) = \frac{1}{2^{m-1}a^m}\left(\sum_{i=0}^{a-1}(-1)^i\left(\frac{at}{\mathrm{e}^{at}-1}\right)^m \mathrm{e}^{\left(bx+\frac{b}{a}\mathrm{i}\right)at}\right)\left(\frac{2}{\mathrm{e}^{bt}+1}\right)^{m-1}\mathrm{e}^{abyt}$$

$$= \frac{1}{2^{m-1}a^m}\left(\sum_{i=0}^{a-1}(-1)^i\sum_{n=0}^{\infty}B_n^{(m)}\left(bx+\frac{b}{a}\mathrm{i}\right)\frac{(at)^n}{n!}\right)\left(\sum_{n=0}^{\infty}E_n^{(m-1)}(ay)\frac{(bt)^n}{n!}\right). \tag{3.4}$$

When a is even, in view of (3.3), $h(t)$ can also be expanded as

$$h(t) = -\frac{t}{2^m a^{m-1}}\left(\sum_{i=0}^{b-1}\left(\frac{2}{\mathrm{e}^{bt}+1}\right)^m \mathrm{e}^{\left(ax+\frac{a}{b}\mathrm{i}\right)bt}\right)\left(\frac{at}{\mathrm{e}^{at}-1}\right)^{m-1}\mathrm{e}^{abyt}$$

$$= -\frac{t}{2^m a^{m-1}}\left(\sum_{i=0}^{b-1}\sum_{n=0}^{\infty}E_n^{(m)}\left(ax+\frac{a}{b}\mathrm{i}\right)\frac{(bt)^n}{n!}\right)\left(\sum_{n=0}^{\infty}B_n^{(m-1)}(by)\frac{(at)^n}{n!}\right).$$

Thus identifying coefficients of $t^n/n!$ yields the second identity. □

When $m = 1$ and $y = 0$, Theorem 3.1 reduces to the corollary below.

Corollary 3.2. *For integers $n \geq 1$, $a \geq 1$ and $b \geq 1$, if a is even, then we have*

$$\sum_{k=0}^{n}\binom{n}{k}a^{n-k}b^{k}B_{n-k}(bx)T_k(a-1) = -\frac{n}{2}\sum_{k=0}^{n-1}\binom{n-1}{k}b^{n-1-k}a^{k+1}E_{n-1-k}(ax)S_k(b-1),$$
$$\sum_{i=0}^{a-1}(-1)^iB_n\left(bx+\frac{b}{a}\mathrm{i}\right) = -\frac{n}{2}\left(\frac{b}{a}\right)^{n-1}\sum_{i=0}^{b-1}E_{n-1}\left(ax+\frac{a}{b}\mathrm{i}\right).$$

Corollary 3.3. *For any positive integer n and any positive even integer a, we have*

$$E_{n-1}(ax) = -\frac{2}{n}\sum_{k=0}^{n}\binom{n}{k}a^{n-k-1}B_{n-k}(x)T_k(a-1) \tag{3.5}$$

$$= -\frac{2}{n}a^{n-1}\sum_{i=0}^{a-1}(-1)^i B_n\left(x+\frac{1}{a}\mathrm{i}\right), \tag{3.6}$$

$$E_{n-1}(x) = \frac{2^n}{n}\left(B_n\left(\frac{x+1}{2}\right) - B_n\left(\frac{x}{2}\right)\right). \tag{3.7}$$

Proof. Putting $b = 1$ in Corollary 3.2 gives identities (3.5) and (3.6). Putting $a = 2$ in (3.5), making use of Eqs. (1.5) and (1.9) and replacing x by $x/2$, one can further obtain identity (3.7).

Note that (3.6) is another formula of the multiplication theorem for the Euler polynomials (see Eq. (2.17) of Corollary 2.12 and [1, Eq. (23.1.10)]). Additionally, (3.7) is a special case of (3.6) also and is one of the well known relations between the Bernoulli polynomials and the Euler polynomials [1, Eq. (23.1.27)]. □

Corollary 3.4. *For any positive integer n and any positive even integer a, we have*

$$E_{n-1}(ax) + E_{n-1}\left(ax+\frac{a}{2}\right) = -\frac{2}{n}\sum_{k=0}^{n}\binom{n}{k}\left(\frac{a}{2}\right)^{n-k-1}B_{n-k}(2x)T_k(a-1) \tag{3.8}$$

$$= -\frac{2}{n}\left(\frac{a}{2}\right)^{n-1}\sum_{i=0}^{a-1}(-1)^i B_n\left(2x+\frac{2}{a}\mathrm{i}\right), \tag{3.9}$$

$$B_n(2x+1) - B_n(2x) = \frac{n}{2}(E_{n-1}(2x) + E_{n-1}(2x+1)). \tag{3.10}$$

Proof. (3.8) and (3.9) can be obtained from Corollary 3.2 by the substitution $b = 2$. (3.10) is the $a = 2$ case of (3.8) and (3.9). □

Putting $x = 0$ in (3.10), one has the relation $B_n(1) - B_n(0) = \frac{n}{2}(E_{n-1}(0) + E_{n-1}(1))$. Since $B_n(1) - B_n(0) = \delta_{n,1}$, then $E_{n-1}(0) + E_{n-1}(1) = 2\delta_{n,1}$, and vice versa.

Corollary 3.5. *For any positive integers n and b, we have*

$$B_n(bx) - B_n\left(bx+\frac{b}{2}\right) = -\frac{n}{2}\sum_{k=0}^{n-1}\binom{n-1}{k}\left(\frac{b}{2}\right)^k E_k(2x)S_{n-1-k}(b-1) \tag{3.11}$$

$$= -\frac{n}{2}\left(\frac{b}{2}\right)^{n-1}\sum_{i=0}^{b-1}E_{n-1}\left(2x+\frac{2}{b}\mathrm{i}\right). \tag{3.12}$$

Proof. These two identities come from Corollary 3.2 by putting $a = 2$. (3.7) and (3.10) can also be obtained from this corollary by further putting $b = 1$ and $b = 2$ respectively. □

Similarly to Theorem 3.1, we have the next result.

Theorem 3.6. *For integers $n \geq 0$, $a \geq 1$ and $b \geq 1$, if a is odd, then we have*

$$\sum_{k=0}^{n}\binom{n}{k}a^{n-k}b^k B_{n-k}^{(m)}(bx)\sum_{i=0}^{k}\binom{k}{i}T_i(a-1)E_{k-i}^{(m-1)}(ay)$$
$$= \sum_{l=0, l\neq n-1}^{n}\binom{n}{l}B_{n-l}a^{n-l}\sum_{k=0}^{l}\binom{l}{k}b^{n-k-1}a^k E_{l-k}^{(m)}(ax)\sum_{i=0}^{k}\binom{k}{i}S_i(b-1)B_{k-i}^{(m-1)}(by)$$

and

$$\sum_{k=0}^{n}\binom{n}{k}a^k b^{n-k}\sum_{i=0}^{a-1}(-1)^i B_k^{(m)}\left(bx+\frac{b}{a}\mathrm{i}\right)E_{n-k}^{(m-1)}(ay)$$
$$= \sum_{l=0, l\neq n-1}^{n}\binom{n}{l}B_{n-l}b^{n-l-1}\sum_{k=0}^{l}\binom{l}{k}b^k a^{n-k}\sum_{i=0}^{b-1}E_k^{(m)}\left(ax+\frac{a}{b}\mathrm{i}\right)B_{l-k}^{(m-1)}(by).$$

Proof. When a is odd, we can expand $h(t)$ into two other forms:

$$\begin{aligned} h(t) &= \frac{t}{2^m a^{m-1}}\left(\frac{2}{e^{bt}+1}\right)^m e^{abxt}\left(\frac{e^{abt}-1}{e^{at}-1}\right)\left(\frac{1-(-e^{bt})^a}{e^{abt}-1}\right)\left(\frac{at}{e^{at}-1}\right)^{m-1} e^{abyt} \\ &= \frac{t}{2^m a^{m-1}}\left(\sum_{n=0}^{\infty}\frac{B_n+B_n(1)}{n!}(abt)^{n-1}\right)\left(\sum_{n=0}^{\infty}E_n^{(m)}(ax)\frac{(bt)^n}{n!}\right)\left(\sum_{n=0}^{\infty}S_n(b-1)\frac{(at)^n}{n!}\right) \\ &\quad \times \left(\sum_{n=0}^{\infty}B_n^{(m-1)}(by)\frac{(at)^n}{n!}\right) \qquad (3.13) \\ &= \frac{t}{2^m a^{m-1}}\left(\sum_{n=0}^{\infty}\frac{B_n+B_n(1)}{n!}(abt)^{n-1}\right)\left(\sum_{i=0}^{b-1}\sum_{n=0}^{\infty}E_n^{(m)}\left(ax+\frac{a}{b}i\right)\frac{(bt)^n}{n!}\right) \\ &\quad \times \left(\sum_{n=0}^{\infty}B_n^{(m-1)}(by)\frac{(at)^n}{n!}\right). \qquad (3.14) \end{aligned}$$

Thus, the theorem can be established by comparing coefficients of $t^n/n!$ in the right sides of Eqs. (3.2) and (3.13) and Eqs. (3.4) and (3.14), respectively. □

Putting $m = 1$ and $y = 0$ in Theorem 3.6 gives the following corollary.

Corollary 3.7. *For integers $n \geq 0$, $a \geq 1$ and $b \geq 1$, if a is odd, then we have*

$$\sum_{k=0}^{n}\binom{n}{k}a^{n-k}b^k B_{n-k}(bx)T_k(a-1) = \sum_{l=0,l\neq n-1}^{n}\binom{n}{l}B_{n-l}a^{n-l}\sum_{k=0}^{l}\binom{l}{k}b^{n-k-1}a^k E_{l-k}(ax)S_k(b-1)$$

and

$$\sum_{i=0}^{a-1}(-1)^i B_n\left(bx+\frac{b}{a}i\right) = \sum_{l=0,l\neq n-1}^{n}\binom{n}{l}B_{n-l}b^{n-1}a^{-l}\sum_{i=0}^{b-1}E_l\left(ax+\frac{a}{b}i\right).$$

The next corollary contains some further results.

Corollary 3.8. *For any nonnegative integer n and positive integer b, we have*

$$B_n(bx) = \sum_{l=0,l\neq n-1}^{n}\binom{n}{l}B_{n-l}\sum_{k=0}^{l}\binom{l}{k}b^{n-k-1}E_{l-k}(x)S_k(b-1) \qquad (3.15)$$

$$= \sum_{l=0,l\neq n-1}^{n}\binom{n}{l}B_{n-l}b^{n-1}\sum_{i=0}^{b-1}E_l\left(x+\frac{1}{b}i\right), \qquad (3.16)$$

$$B_n(x) = \sum_{l=0,l\neq n-1}^{n}\binom{n}{l}B_{n-l}E_l(x), \qquad (3.17)$$

$$B_n(2x) = 2^{n-1}\left(B_n(x)+B_n\left(x+\frac{1}{2}\right)\right), \qquad (3.18)$$

$$B_n = \sum_{l=0,l\neq n-1}^{n}\binom{n}{l}B_{n-l}E_l(0) = 2^{n-1}B_n + 2^{n-1}B_n\left(\frac{1}{2}\right). \qquad (3.19)$$

Proof. Identities (3.15) and (3.16) can be derived from Corollary 3.7 by the substitution $a = 1$. The $b = 1$ case of (3.15) is (3.17). Putting $b = 2$ in (3.15) and making use of (3.17) yield (3.18). By setting $x = 0$ in (3.17) and (3.18), one can finally obtain (3.19).

Note that (3.17) is the main result of [5] and has been discussed again in [14]. Additionally, the last three identities can also be obtained from (3.16). □

3.2. The second part for mixed type identities

From the discussions in Sections 2.1, 2.2 and 3.1, it can be found that many results are derived from the same generating function with same conditions, such as Theorems 2.1 and 2.10, Theorems 2.4 and 2.13 and Theorems 2.7 and 2.16. Thus, they can be combined to form new identities, and as a consequence, links can also be established between the corresponding corollaries.

For example, from the proof of Theorem 3.1, the following result holds.

Theorem 3.9. *For integers $n \geq 0$, $a \geq 1$ and $b \geq 1$, we have*

$$\sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k B_{n-k}^{(m)}(bx) \sum_{i=0}^{k} \binom{k}{i} T_i(a-1) E_{k-i}^{(m-1)}(ay) = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} \sum_{i=0}^{a-1} (-1)^i B_k^{(m)}\left(bx + \frac{b}{a}\mathrm{i}\right) E_{n-k}^{(m-1)}(ay).$$

When $m = 1$ and $y = 0$, Theorem 3.9 reduces to the corollary below.

Corollary 3.10. *For integers $n \geq 0$, $a \geq 1$ and $b \geq 1$, we have*

$$\sum_{k=0}^{n} \binom{n}{k} \left(\frac{b}{a}\right)^k B_{n-k}(bx) T_k(a-1) = \sum_{i=0}^{a-1} (-1)^i B_n\left(bx + \frac{b}{a}\mathrm{i}\right).$$

The next theorem can be viewed as a "dual" result of Theorem 3.9.

Theorem 3.11. *For integers $n \geq 0$, $a \geq 1$ and $b \geq 1$, we have*

$$\sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k E_{n-k}^{(m)}(bx) \sum_{i=0}^{k} \binom{k}{i} S_i(a-1) B_{k-i}^{(m-1)}(ay) = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} \sum_{i=0}^{a-1} E_k^{(m)}\left(bx + \frac{b}{a}\mathrm{i}\right) B_{n-k}^{(m-1)}(ay).$$

Proof. Let

$$\begin{aligned} f(t) &= \frac{t^{m-1} e^{abxt} (e^{abt} - 1) e^{abyt}}{(e^{at} + 1)^m (e^{bt} - 1)^m} \\ &= \frac{1}{2^m b^{m-1}} \left(\frac{2}{e^{at} + 1}\right)^m e^{abxt} \left(\frac{e^{abt} - 1}{e^{bt} - 1}\right) \left(\frac{bt}{e^{bt} - 1}\right)^{m-1} e^{abyt}. \end{aligned}$$

Then $f(t)$ can be expanded in two ways:

$$\begin{aligned} f(t) &= \frac{1}{2^m b^{m-1}} \left(\sum_{n=0}^{\infty} E_n^{(m)}(bx) \frac{(at)^n}{n!}\right) \left(\sum_{n=0}^{\infty} S_n(a-1) \frac{(bt)^n}{n!}\right) \left(\sum_{n=0}^{\infty} B_n^{(m-1)}(ay) \frac{(bt)^n}{n!}\right) \\ &= \frac{1}{2^m b^{m-1}} \left(\sum_{i=0}^{a-1} \sum_{n=0}^{\infty} E_n^{(m)}\left(bx + \frac{b}{a}\mathrm{i}\right) \frac{(at)^n}{n!}\right) \left(\sum_{n=0}^{\infty} B_n^{(m-1)}(ay) \frac{(bt)^n}{n!}\right). \end{aligned}$$

Equating coefficients of $t^n/n!$ gives the desired result. $\square$

Putting $m = 1$ and $x = 0$ in Theorem 3.11 yields Corollary 3.12.

Corollary 3.12. *For integers $n \geq 0$, $a \geq 1$ and $b \geq 1$, we have*

$$\sum_{k=0}^{n} \binom{n}{k} \left(\frac{b}{a}\right)^k E_{n-k}(bx) S_k(a-1) = \sum_{i=0}^{a-1} E_n\left(bx + \frac{b}{a}\mathrm{i}\right).$$

At the end of this subsection, we would like to present explicitly the multiplication theorems for both the Euler polynomials and the Bernoulli polynomials, in a symmetric way.

Theorem 3.13. *For any nonnegative integer n and any positive odd integer a, we have*

$$E_n(ax) = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} E_{n-k}(x) T_k(a-1) = \sum_{i=0}^{a-1} (-1)^i a^n E_n\left(x + \frac{\mathrm{i}}{a}\right). \tag{3.20}$$

For any positive integer n and any positive even integer a, we have

$$-\frac{an}{2}E_{n-1}(ax)=\sum_{k=0}^{n}\binom{n}{k}a^{n-k}B_{n-k}(x)T_k(a-1)=\sum_{i=0}^{a-1}(-1)^i a^n B_n\left(x+\frac{i}{a}\right). \tag{3.21}$$

This is the multiplication theorem for the Euler polynomials. Identity (3.20) comes from Eqs. (2.4) and (2.17); while identity (3.21) comes from Eqs. (3.5) and (3.6).

Theorem 3.14. *For any nonnegative integer n and any positive integer a, we have*

$$aB_n(ax)=\sum_{k=0}^{n}\binom{n}{k}a^{n-k}B_{n-k}(x)S_k(a-1)=\sum_{i=0}^{a-1}a^n B_n\left(x+\frac{i}{a}\right). \tag{3.22}$$

The above is the multiplication theorem for the Bernoulli polynomials and can be obtained from the two main theorems of Yang [17].

Besides these familiar equations, we demonstrate here some other results.

Theorem 3.15. *For any nonnegative integer n and any positive integer a, we have*

$$-2\sum_{k=0}^{n}\binom{n}{k}\frac{a^{k+1}}{k+1}E_{k+1}(0)B_{n-k}(ax)=\sum_{k=0}^{n}\binom{n}{k}a^{n-k}E_{n-k}(x)S_k(a-1)=\sum_{i=0}^{a-1}a^n E_n\left(x+\frac{i}{a}\right).$$

Proof. It can be verified that the generating functions of these three members are

$$\frac{e^{at}-1}{e^t-1}\frac{2}{e^{at}+1}e^{axt}.$$

Note that Corollary 3.12 has already indicated the equivalence of the last two members. □

Theorem 3.16. *For any positive integer n and any positive even integer a, we have*

$$\sum_{k=1}^{n}\binom{n}{k}a^kE_k(0)E_{n-k}(ax)=\sum_{k=0}^{n}\binom{n}{k}a^{n-k}E_{n-k}(x)T_k(a-1)=\sum_{i=0}^{a-1}(-1)^i a^n E_n\left(x+\frac{i}{a}\right).$$

For any nonnegative integer n and any positive odd integer a, we have

$$\sum_{k=0,k\neq 1}^{n}\binom{n}{k}a^kB_kE_{n-k}(ax)=\sum_{k=0}^{n}\binom{n}{k}a^{n-k}B_{n-k}(x)T_k(a-1)=\sum_{i=0}^{a-1}(-1)^i a^n B_n\left(x+\frac{i}{a}\right).$$

Theorem 3.16 follows from Corollaries 2.8, 2.17 and 3.7. It can also be verified by the generating function method. The readers may compare it with Theorem 3.13.

4. Identities related to the Genocchi polynomials

The Genocchi polynomials $G_n(x)$ are defined by means of the generating function

$$\sum_{n=0}^{\infty}G_n(x)\frac{t^n}{n!}=\frac{2t}{e^t+1}e^{xt},\quad (|t|<\pi). \tag{4.1}$$

For $x=0$, we have the Genocchi numbers G_n, i.e., $G_n:=G_n(0)$. The readers can find in [6, p. 49] some special values of G_n.

The generating function of the Genocchi polynomials is similar to those of the Bernoulli polynomials and the Euler polynomials, so it may be expected that the Genocchi polynomials satisfy similar identities as those established in Sections 2 and 3.

To establish the corresponding identities for the Genocchi polynomials, let us take

$$p(t)=\frac{2abte^{abxt}(1-(-e^{bt})^a)}{(e^{at}+1)(e^{bt}+1)}.$$

Then the following theorem holds.

Theorem 4.1. *For integers $n \geq 0$, $a \geq 1$ and $b \geq 1$, if a and b have the same parity, then we have*

$$\sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k+1} G_k(bx) T_{n-k}(a-1) = \sum_{k=0}^{n} \binom{n}{k} b^k a^{n-k+1} G_k(ax) T_{n-k}(b-1), \tag{4.2}$$

$$b \sum_{i=0}^{a-1} (-1)^i a^n G_n\left(bx + \frac{b}{a} \mathrm{i}\right) = a \sum_{i=0}^{b-1} (-1)^i b^n G_n\left(ax + \frac{a}{b} \mathrm{i}\right). \tag{4.3}$$

Proof. We expand $p(t)$ in two ways:

$$p(t) = b\left(\frac{2at}{\mathrm{e}^{at}+1}\right) \mathrm{e}^{abxt} \left(\frac{1-(-\mathrm{e}^{bt})^a}{\mathrm{e}^{bt}+1}\right) = b\left(\sum_{n=0}^{\infty} G_n(bx)\frac{(at)^n}{n!}\right)\left(\sum_{n=0}^{\infty} T_n(a-1)\frac{(bt)^n}{n!}\right)$$

$$= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k+1} G_k(bx) T_{n-k}(a-1)\right) \frac{t^n}{n!} \tag{4.4}$$

$$= b \sum_{i=0}^{a-1} (-1)^i \left(\frac{2at}{\mathrm{e}^{at}+1}\right) \mathrm{e}^{\left(bx+\frac{b}{a}\mathrm{i}\right)at} = \sum_{n=0}^{\infty} \left(b \sum_{i=0}^{a-1} (-1)^i G_n\left(bx + \frac{b}{a}\mathrm{i}\right) a^n\right) \frac{t^n}{n!}. \tag{4.5}$$

If a and b have the same parity, then $p(t)$ is symmetric in a and b. From this fact we can obtain the theorem. □

Theorem 4.1 can reduce to the next two corollaries.

Corollary 4.2. *For any nonnegative integer n and any positive odd integer a, we have*

$$aG_n(ax) = \sum_{k=0}^{n} \binom{n}{k} a^k G_k(x) T_{n-k}(a-1) = \sum_{i=0}^{a-1} (-1)^i a^n G_n\left(x + \frac{\mathrm{i}}{a}\right), \tag{4.6}$$

$$G_n = \sum_{k=0}^{n} \binom{n}{k} a^{k-1} G_k T_{n-k}(a-1). \tag{4.7}$$

Proof. Putting $b = 1$ in Theorem 4.1 yields (4.6). Putting further $x = 0$ in (4.6) yields (4.7). It should be noticed that, from identity (4.7), we can obtain the following recurrence for the Genocchi numbers:

$$(a - a^n) G_n = \sum_{k=1}^{n-1} \binom{n}{k} a^k G_k T_{n-k}(a-1),$$

which was first given by Howard [9, Theorem 6].

Corollary 4.3. *For any nonnegative integer n and any positive even integer a, we have*

$$G_n(ax) - G_n\left(ax + \frac{a}{2}\right) = \sum_{k=0}^{n} \binom{n}{k} \left(\frac{a}{2}\right)^{k-1} G_k(2x) T_{n-k}(a-1)$$

$$= \sum_{i=0}^{a-1} (-1)^i \left(\frac{a}{2}\right)^{n-1} G_n\left(2x + \frac{2}{a}\mathrm{i}\right).$$

Proof. By computing the generating function, one can verify that

$$G_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} G_k(x) y^{n-k}. \tag{4.8}$$

Thus, the first equation can be obtained by putting $b = 2$ in (4.2) and then making use of identity (4.8). With the substitution $b = 2$ in (4.3), the second identity of the corollary can also be established. □

Theorem 4.4. *For integers $n \geq 0$, $a \geq 1$ and $b \geq 1$, if a is odd and b is even, then we have*

$$\sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k+1} G_k(bx) T_{n-k}(a-1) = -2 \sum_{l=0, l\neq n}^{n+1} \binom{n+1}{l} \frac{B_{n+1-l}}{n+1} b^{n-l} \sum_{k=0}^{l} \binom{l}{k} b^k a^{n-k+1} G_k(ax) T_{l-k}(b-1),$$

$$b \sum_{i=0}^{a-1} (-1)^i a^n G_n\left(bx + \frac{b}{a}\mathrm{i}\right) = -2 \sum_{l=0, l\neq n}^{n+1} \binom{n+1}{l} \frac{B_{n+1-l}}{n+1} a^{n+1-l} \sum_{i=0}^{b-1} (-1)^i b^n G_l\left(ax + \frac{a}{b}\mathrm{i}\right).$$

Proof. When a is odd and b is even, $p(t)$ can be expanded as

$$p(t) = a\left(\frac{2bt}{e^{bt}+1}\right)e^{abxt}\left(\frac{1-(-e^{bt})^a}{1-(-e^{at})^b}\right)\left(\frac{1-(-e^{at})^b}{e^{at}+1}\right)$$

$$= a\left(-\sum_{n=0}^{\infty}\frac{B_n+B_n(1)}{n!}(abt)^{n-1}\right)\left(\sum_{n=0}^{\infty}G_n(ax)\frac{(bt)^n}{n!}\right)\left(\sum_{n=0}^{\infty}T_n(b-1)\frac{(at)^n}{n!}\right) \tag{4.9}$$

$$= a\left(-\sum_{n=0}^{\infty}\frac{B_n+B_n(1)}{n!}(abt)^{n-1}\right)\left(\sum_{i=0}^{b-1}(-1)^i\sum_{n=0}^{\infty}G_n\left(ax+\frac{a}{b}i\right)\frac{(bt)^n}{n!}\right). \tag{4.10}$$

Equating coefficients of $t^n/n!$ in (4.4), (4.5), (4.9) and (4.10) gives us the desired identities. □

Corollary 4.5. *For any nonnegative integer n and any positive even integer b, we have*

$$G_n(bx) = -2\sum_{l=0,l\neq n}^{n+1}\binom{n+1}{l}\frac{B_{n+1-l}}{n+1}b^{n-l}\sum_{k=0}^{l}\binom{l}{k}b^{k-1}G_k(x)T_{l-k}(b-1) \tag{4.11}$$

$$= -2\sum_{l=0,l\neq n}^{n+1}\binom{n+1}{l}\frac{B_{n+1-l}}{n+1}\sum_{i=0}^{b-1}(-1)^ib^{n-1}G_l\left(x+\frac{i}{b}\right), \tag{4.12}$$

$$G_n(2x) = \sum_{l=0,l\neq n}^{n+1}\binom{n+1}{l}\frac{B_{n+1-l}}{n+1}2^n\left(G_l\left(x+\frac{1}{2}\right)-G_l(x)\right). \tag{4.13}$$

Proof. The substitution $a = 1$ in Theorem 4.4 gives identities (4.11) and (4.12), and these two identities have (4.13) as a common special case. □

Similarly to Theorem 4.4, we have the next result.

Theorem 4.6. *For integers $n \geq 1$, $a \geq 1$ and $b \geq 1$, if a is even and b is odd, then we have*

$$\sum_{k=0}^{n}\binom{n}{k}a^kb^{n-k+1}G_k(bx)T_{n-k}(a-1) = \sum_{k=0}^{n-1}\binom{n}{k}b^ka^{n-k+1}G_k(ax)(E_{n-k}(0)-T_{n-k}(b-1)),$$

$$b\sum_{i=0}^{a-1}(-1)^ia^nG_n\left(bx+\frac{b}{a}i\right) = \sum_{l=0}^{n-1}\binom{n}{l}E_{n-l}(0)a^{n-l+1}\sum_{i=0}^{b-1}(-1)^ib^nG_l\left(ax+\frac{a}{b}i\right).$$

Proof. When a is even and b is odd, we can expand $p(t)$ as

$$p(t) = a\left(\frac{2bt}{e^{bt}+1}\right)e^{abxt}\left(\frac{1-(-e^{bt})^a}{1-(-e^{at})^b}\right)\left(\frac{1-(-e^{at})^b}{e^{at}+1}\right)$$

$$= a\left(\sum_{n=1}^{\infty}E_n(0)\frac{(abt)^n}{n!}\right)\left(\sum_{n=0}^{\infty}G_n(ax)\frac{(bt)^n}{n!}\right)\left(\sum_{n=0}^{\infty}T_n(b-1)\frac{(at)^n}{n!}\right). \tag{4.14}$$

Comparing coefficients of $t^n/n!$ in (4.4) and (4.14), we have

$$\sum_{k=0}^{n}\binom{n}{k}a^kb^{n-k+1}G_k(bx)T_{n-k}(a-1) = \sum_{l=0}^{n-1}\binom{n}{l}E_{n-l}(0)b^{n-l}\sum_{k=0}^{l}\binom{l}{k}b^ka^{n-k+1}G_k(ax)T_{l-k}(b-1)$$

$$= \sum_{k=0}^{n-1}\binom{n}{k}b^ka^{n-k+1}G_k(ax)\sum_{l=k}^{n-1}\binom{n-k}{l-k}E_{n-l}(0)b^{n-l}T_{l-k}(b-1).$$

Using (2.4), the first identity of the theorem can be obtained. For the second one, it suffices to note that

$$p(t) = a\left(\sum_{n=1}^{\infty}E_n(0)\frac{(abt)^n}{n!}\right)\left(\sum_{n=0}^{\infty}\sum_{i=0}^{b-1}(-1)^ib^nG_n\left(ax+\frac{a}{b}i\right)\frac{t^n}{n!}\right)$$

and extract coefficients as before. □

Corollary 4.7. *For any positive integer n and any positive even integer a, we have*

$$\sum_{k=0}^{n}\binom{n}{k}a^kG_k(x)T_{n-k}(a-1)=\sum_{k=0}^{n-1}\binom{n}{k}a^{n-k+1}E_{n-k}(0)G_k(ax) \tag{4.15}$$

$$=\sum_{i=0}^{a-1}(-1)^ia^nG_n\left(x+\frac{i}{a}\right), \tag{4.16}$$

$$G_n(x)-G_n\left(x+\frac{1}{2}\right)=\sum_{k=0}^{n-1}\binom{n}{k}2^{-k+1}E_{n-k}(0)G_k(2x). \tag{4.17}$$

Proof. Putting $b=1$ in Theorem 4.6 and in view of (1.9), one has (4.15) and (4.16). The $a=2$ case of (4.15) is identity (4.17).

Corollary 4.8. *For any positive integer n and any positive odd integer b, we have*

$$G_n(bx)-G_n\left(bx+\frac{b}{2}\right)=\sum_{k=0}^{n-1}\binom{n}{k}\left(\frac{b}{2}\right)^{k-1}G_k(2x)(E_{n-k}(0)-T_{n-k}(b-1)). \tag{4.18}$$

Proof. Identity (4.18) can be derived by setting $a=2$ in the first identity of Theorem 4.6. (4.17) is a special case of (4.18) also. Additionally, the readers may compare this corollary with Corollary 4.3. □

Analogous to (4.6), (4.15) and (4.16), we can obtain the following result.

Theorem 4.9. *For any positive integer n and any positive integer a, we have*

$$-2a\sum_{k=1}^{n}\binom{n}{k}a^kE_k(0)B_{n-k}(ax)=\sum_{k=0}^{n}\binom{n}{k}a^kG_k(x)S_{n-k}(a-1)=\sum_{i=0}^{a-1}a^nG_n\left(x+\frac{i}{a}\right).$$

It can be found that there are few references on the higher order Genocchi polynomials, and we do not intend to study them either. The readers can do this as an exercise.

5. Identities related to the higher order degenerate Bernoulli polynomials

In [3] Carlitz defined the higher order degenerate Bernoulli polynomials $\beta_n^{(k)}(\lambda,x)$ by means of the generating function

$$\sum_{n=0}^{\infty}\beta_n^{(k)}(\lambda,x)\frac{t^n}{n!}=\left(\frac{t}{(1+\lambda t)^{\mu}-1}\right)^k(1+\lambda t)^{\mu x} \tag{5.1}$$

for each integer k, where $\lambda\neq 0$ and $\lambda\mu=1$. We often write $\beta_n^{(k)}(\lambda)$ for $\beta_n^{(k)}(\lambda,0)$, and refer to $\beta_n^{(k)}(\lambda)$ as the higher order degenerate Bernoulli numbers; we further call $\beta_n(\lambda):=\beta_n^{(1)}(\lambda)$ the degenerate Bernoulli numbers. Since $(1+\lambda t)^{\mu}\to \mathrm{e}^t$ as $\lambda\to 0$ it is evident that $\beta_n^{(k)}(0,x)=B_n^{(k)}(x)$, i.e., the higher order Bernoulli polynomials.

The generalized falling factorial $(x|\lambda)_n$ with increment λ is defined by $(x|\lambda)_n=\prod_{j=0}^{n-1}(x-j\lambda)$ for positive integer n, with the convention $(x|\lambda)_0=1$; we may also write

$$\sigma_k(\lambda,n)=\sum_{i=0}^{n}(i|\lambda)_k$$

and call $\sigma_k(\lambda,n)$ a generalized falling factorial sum. From [18, Eq. (2.2)], we know that

$$(x|\lambda)_n=\sum_{k=0}^{n}s(n,k)x^k\lambda^{n-k}$$

where the integers $s(n,k)$ are the Stirling numbers of the first kind. Therefore, $(x|\lambda)_n$ is a polynomial in λ and x of degree n, and since $\lambda\neq 0$, then $(x|\lambda)_n=\lambda^n(\lambda^{-1}x|1)_n=\lambda^n(\lambda^{-1}x)_n$, where $(x)_n$ is the usual falling factorial. Moreover, we have $(x|0)_n=x^n$ and

$$\sigma_k(0,n)=\sum_{i=0}^{n}(i|0)_k=\sum_{i=0}^{n}i^k=S_k(n).$$

By some computation, the generating function of $\sigma_k(\lambda, n)$ can be obtained, i.e.,

$$\sum_{k=0}^{\infty} \sigma_k(\lambda, n)\frac{t^k}{k!} = \frac{(1+\lambda t)^{(n+1)\mu} - 1}{(1+\lambda t)^{\mu} - 1}.$$

Now, let

$$q(t) = \frac{t^{2m-1}(1+\lambda t)^{ab\mu x}((1+\lambda t)^{ab\mu} - 1)(1+\lambda t)^{ab\mu y}}{((1+\lambda t)^{a\mu} - 1)^m((1+\lambda t)^{b\mu} - 1)^m}.$$

Similarly to Theorems 2.1 and 2.10, we can establish the following results.

Theorem 5.1. *For integers $n \geq 0$, $a \geq 1$ and $b \geq 1$, we have*

$$\sum_{k=0}^{n} \binom{n}{k} a^{n-k}b^{k+1}\beta_{n-k}^{(m)}(b\lambda, bx) \sum_{i=0}^{k} \binom{k}{i} \sigma_i(a\lambda, a-1)\beta_{k-i}^{(m-1)}(a\lambda, ay)$$
$$= \sum_{k=0}^{n} \binom{n}{k} b^{n-k}a^{k+1}\beta_{n-k}^{(m)}(a\lambda, ax) \sum_{i=0}^{k} \binom{k}{i} \sigma_i(b\lambda, b-1)\beta_{k-i}^{(m-1)}(b\lambda, by).$$

Proof. As before, $q(t)$ can be expanded as

$$q(t) = \frac{1}{a^m b^{m-1}} \left(\frac{at}{(1+\lambda t)^{a\mu} - 1}\right)^m (1+\lambda t)^{ab\mu x} \left(\frac{(1+\lambda t)^{ab\mu} - 1}{(1+\lambda t)^{b\mu} - 1}\right) \left(\frac{bt}{(1+\lambda t)^{b\mu} - 1}\right)^{m-1} (1+\lambda t)^{ab\mu y}$$
$$= \frac{1}{a^m b^{m-1}} \left(\sum_{n=0}^{\infty} \beta_n^{(m)}\left(\frac{1}{a}\lambda, bx\right)\frac{(at)^n}{n!}\right) \left(\sum_{n=0}^{\infty} \sigma_n\left(\frac{1}{b}\lambda, a-1\right)\frac{(bt)^n}{n!}\right) \left(\sum_{n=0}^{\infty} \beta_n^{(m-1)}\left(\frac{1}{b}\lambda, ay\right)\frac{(bt)^n}{n!}\right). \tag{5.2}$$

Since $q(t)$ is symmetric in a and b, then we can obtain an identity, from which the desired identity can be established by further replacing λ with $ab\lambda$. □

Corollary 5.2. *For integers $n \geq 0$, $a \geq 1$ and $b \geq 1$, we have*

$$\sum_{k=0}^{n} \binom{n}{k} a^{n-k}b^{k+1}\beta_{n-k}(b\lambda, bx)\sigma_k(a\lambda, a-1) = \sum_{k=0}^{n} \binom{n}{k} b^{n-k}a^{k+1}\beta_{n-k}(a\lambda, ax)\sigma_k(b\lambda, b-1),$$
$$\sum_{k=0}^{n} \binom{n}{k} a^{n-k}b^{k+1}B_{n-k}^{(m)}(bx) \sum_{i=0}^{k} \binom{k}{i} S_i(a-1)B_{k-i}^{(m-1)}(ay) = \sum_{k=0}^{n} \binom{n}{k} b^{n-k}a^{k+1}B_{n-k}^{(m)}(ax) \sum_{i=0}^{k} \binom{k}{i} S_i(b-1)B_{k-i}^{(m-1)}(by).$$

Proof. Putting $m = 1$ and $y = 0$ in Theorem 5.1 gives the first identity, which is a main result of Young [18, Theorem 3.1]. Putting $\lambda = 0$ in Theorem 5.1 gives the second identity, which is a main result of Yang [17, Theorem 1]. Note that Tuenter's result (1.11) is a common special case of these two identities. □

Theorem 5.3. *For integers $n \geq 0$, $a \geq 1$ and $b \geq 1$, we have*

$$\sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k+1} \sum_{i=0}^{a-1} \beta_k^{(m)}\left(b\lambda, bx + \frac{b}{a}\mathrm{i}\right) \beta_{n-k}^{(m-1)}(a\lambda, ay)$$
$$= \sum_{k=0}^{n} \binom{n}{k} b^k a^{n-k+1} \sum_{i=0}^{b-1} \beta_k^{(m)}\left(a\lambda, ax + \frac{a}{b}\mathrm{i}\right) \beta_{n-k}^{(m-1)}(b\lambda, by).$$

Proof. From (5.2), $q(t)$ can also be expanded as

$$q(t) = \frac{1}{a^m b^{m-1}} \left(\sum_{i=0}^{a-1}\sum_{n=0}^{\infty} \beta_n^{(m)}\left(\frac{1}{a}\lambda, bx + \frac{b}{a}\mathrm{i}\right)\frac{(at)^n}{n!}\right) \left(\sum_{n=0}^{\infty} \beta_n^{(m-1)}\left(\frac{1}{b}\lambda, ay\right)\frac{(bt)^n}{n!}\right).$$

Thus the theorem can be obtained by the symmetry of $q(t)$ and the substitution $ab\lambda$ for λ. □

This theorem has the following special cases.

Corollary 5.4. *For integers $n \geq 0$, $a \geq 1$ and $b \geq 1$, we have*

$$a^n b \sum_{i=0}^{a-1} \beta_n \left(b\lambda, bx + \frac{b}{a}\mathrm{i} \right) = b^n a \sum_{i=0}^{b-1} \beta_n \left(a\lambda, ax + \frac{a}{b}\mathrm{i} \right), \tag{5.3}$$

$$a^{1-n} \beta_n(a\lambda, ax) = \sum_{i=0}^{a-1} \beta_n \left(\lambda, x + \frac{\mathrm{i}}{a} \right), \tag{5.4}$$

$$\sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k+1} \sum_{i=0}^{a-1} B_k^{(m)} \left(bx + \frac{b}{a}\mathrm{i} \right) B_{n-k}^{(m-1)}(ay) = \sum_{k=0}^{n} \binom{n}{k} b^k a^{n-k+1} \sum_{i=0}^{b-1} B_k^{(m)} \left(ax + \frac{a}{b}\mathrm{i} \right) B_{n-k}^{(m-1)}(by). \tag{5.5}$$

Proof. Putting $m = 1$ and $y = 0$ in Theorem 5.3 yields identity (5.3). The $b = 1$ case of (5.3) is (5.4). Finally, putting instead $\lambda = 0$ in Theorem 5.3 yields (5.5).

It should be noticed that (5.4) is the multiplication formula for the degenerate Bernoulli polynomials (see [3, Eq. (5.5)] and [18, Eq. (3.11)]) and (5.5) is another main theorem of Yang [17, Theorem 2]. □

The higher order degenerate Euler polynomials $\varepsilon_n^{(k)}(\lambda, x)$ are defined by means of the generating function

$$\sum_{n=0}^{\infty} \varepsilon_n^{(k)}(\lambda, x) \frac{t^n}{n!} = \left(\frac{2}{(1+\lambda t)^{\mu} + 1} \right)^k (1+\lambda t)^{\mu x}, \tag{5.6}$$

where $\lambda\mu = 1$. When $\lambda \to 0$, $\varepsilon_n^{(k)}(\lambda, x)$ will reduce to the higher order Euler polynomials, i.e., $\varepsilon_n^{(k)}(0, x) = E_n^{(k)}(x)$.

Now, let us define the generalized alternate falling factorial sums by

$$\tau_k(\lambda, n) = \sum_{i=0}^{n} (-1)^i (i|\lambda)_k.$$

Then

$$\tau_k(0, n) = \sum_{i=0}^{n} (-1)^i (i|0)_k = \sum_{i=0}^{n} (-1)^i i^k = T_k(n).$$

By some computation, we can also determine the generating function of $\tau_k(\lambda, n)$, as follows:

$$\sum_{k=0}^{\infty} \tau_k(\lambda, n) \frac{t^k}{k!} = \frac{1 - (-(1+\lambda t)^{\mu})^{n+1}}{1 + (1+\lambda t)^{\mu}}.$$

From the generating functions (5.1) and (5.6), we have

$$\sum_{n=0}^{\infty} \frac{\beta_n(\lambda) + \beta_n(\lambda, 1)}{n!} (2z)^{n-1} = \frac{(1+\lambda \cdot 2z)^{\mu} + 1}{(1+\lambda \cdot 2z)^{\mu} - 1},$$

$$-\sum_{n=1}^{\infty} \varepsilon_n(\lambda, 0) \frac{(2z)^n}{n!} = \frac{(1+\lambda \cdot 2z)^{\mu} - 1}{(1+\lambda \cdot 2z)^{\mu} + 1},$$

which are analogues of the hyperbolic cotangent and the hyperbolic tangent, respectively. See Eqs. (2.6) and (2.10).

Based on the definitions and generating functions above, we can establish various identities concerning the higher order degenerate Euler polynomials, just as what we have done in Sections 2 and 3 for the higher order Euler polynomials. However, we chose not to present them, and they are left to the interested readers.

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