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Edges not contained in triangles and the distribution of contractible edges in a 4-connected graph

Kiyoshi Ando[a], Yoshimi Egawa[b]

[a]*Department of Information and Communication Engineering, The University of Electro-Communications, 1-5-1 Chofugaoka, Chofu-city, Tokyo 182-8585, Japan*

[b]*Department of Mathematical Information Science, Tokyo University of Science, 1-3 Kagurazaka, Shinjuku-ku, Tokyo 162-8601, Japan*

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Abstract

We prove results concerning the distribution of 4-contractible edges in a 4-connected graph G in connection with the edges of G not contained in a triangle. As a corollary, we show that if G is 4-regular 4-connected graph, then the number of 4-contractible edges of G is at least one half of the number of edges of G not contained in a triangle.

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1. Introduction

In this paper, we consider only finite undirected simple graphs with no loops and no multiple edges.

Let $G = (V(G), E(G))$ be a graph. For $e \in E(G)$, we let $V(e)$ denote the set of endvertices of e. For $x \in V(G)$, $N_G(x)$ denotes the neighborhood of x and $\deg_G(x)$ denotes the degree of x; thus $\deg_G(x) = |N_G(x)|$. For $X \subseteq V(G)$, we let $N_G(X) = \bigcup_{x \in X} N_G(x)$. If there is no ambiguity, we write $N(x)$, $\deg(x)$ and $N(X)$ for $N_G(x)$, $\deg_G(x)$ and $N_G(X)$, respectively. For an integer $i \geqslant 0$, we let $V_i(G)$ denote the set of vertices x of G with $\deg(x) = i$. For $X \subseteq V(G)$, the subgraph induced by X in G is denoted by $G[X]$. A subset S of $V(G)$ is called a *cutset* if $G - S$ is disconnected. A cutset with cardinality i is simply referred to as an *i-cutset*. For an integer $k \geqslant 1$, we say that G is k-connected if $|V(G)| \geqslant k + 1$ and G has no $(k-1)$-cutset.

Let G be a 4-connected graph. A 4-cutset S of G is said to be *trivial* if there exists $z \in V_4(G)$ such that $N(z) = S$; otherwise it is said to be *nontrivial*. For $e \in E(G)$, we let G/e denote the graph obtained from G by contracting e into one vertex (and replacing each resulting pair of double edges by a simple edge). We say that e is 4-*contractible* or 4-*noncontractible* according as G/e is 4-connected or not. Note that if $|V(G)| \geqslant 6$, then $e \in E_n(G)$ if and only if there exists a 4-cutset S such that $V(e) \subseteq S$. A 4-noncontractible edge $e = ab$ is said to be *trivially* 4-noncontractible if there exists $z \in V_4(G)$ such that $za, zb \in E(G)$. We let $E_c(G)$, $E_n(G)$ and $E_{tn}(G)$ denote the set of 4-contractible edges, the set of 4-noncontractible edges and the set of trivially 4-noncontractible edges, respectively. Thus $e \in E_{tn}(G)$ if

E-mail address: ando@ice.uec.ac.jp (K. Ando).

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and only if there exists a trivial 4-cutset S such that $V(e) \subseteq S$. Finally we let $\tilde{E}(G)$ denote the set of those edges of G which are not contained in a triangle. Note that $\tilde{E}(G) \cap E_{\rm n}(G) \subseteq E_{\rm n}(G) - E_{\rm tn}(G)$.

The following characterization of 4-connected graphs with $E_{\rm c}(G) = \emptyset$ was obtained by Fontet [3] and independently by Martinov [6].

Theorem A. *Let G be a 4-connected graph of order n, and suppose that G has no 4-contractible edge. Then one of the following holds*:

(1) *G is the square of the cycle of order n; i.e., we can write $V(G) = \{v_1, v_2, \ldots, v_n\}$ so that $E(G) = \{v_i v_j | i - j \in \{\pm 1, \pm 2\} (\text{mod } n)\}$; or*
(2) *there exists a 3-regular graph H such that G is the line graph of H.*

(*It is easy to see that if a 4-connected graph satisfies (1) or (2), then G has no 4-contractible edge.*)

From Theorem A, we see that if G is a 4-connected graph with $E_{\rm c}(G) = \emptyset$, then G is 4-regular and each edge of G is contained in a triangle. Thus if a 4-connected graph G satisfies $V(G) - V_4(G) \neq \emptyset$ or $\tilde{E}(G) \neq \emptyset$, then G has a 4-contractible edge. Further it is natural to expect that under the same assumption, there is a 4-contractible edge in the neighborhood of each vertex in $V(G) - V_4(G)$ and also in the neighborhood of each edge in $\tilde{E}(G)$. As for the distribution of contractible edges in the neighborhood of a vertex with degree at least 5, the following result was proved in [1].

Theorem B. *Let G be a 4-connected graph with $V(G) - V_4(G) \neq \emptyset$, and let $u \in V(G) - V_4(G)$. Then there exists $e \in E_{\rm c}(G)$ such that either e is incident with u or at least one of the endvertices of e is adjacent to u. Further if $G[N_G(u) \cap V_4(G)]$ is not a path of order 4 (length 3), then there are two such 4-contractible edges.*

In this paper, we prove the following theorem concerning the local distribution of contractible edges in the neighborhood of an edge not contained in a triangle.

Theorem 1. *Let G be a 4-connected graph with $\tilde{E}(G) \neq \emptyset$, and let $uv \in \tilde{E}(G)$. Suppose that $uv \in E_{\rm n}(G)$ and let S be a 4-cutset with $u, v \in S$, and let A be the vertex set of a component of $G - S$. Then there exists $e \in E_{\rm c}(G)$ such that either e is incident with u or there exists $a \in N_G(u) \cap (S \cup A) \cap V_4(G)$ such that e is incident with a.*

We also prove a somewhat global result. To state our result, we need some more definitions.

Throughout this and the next paragraph, we let G be a 4-connected graph. Let $\tilde{V}$ denote the set of those vertices of G which are incident with an edge in $\tilde{E}(G) \cap E_{\rm n}(G)$, and let $\tilde{G}$ denote the spanning subgraph of G with edge set $\tilde{E}(G) \cap E_{\rm n}(G)$; that is to say, $\tilde{V} = \bigcup_{e \in \tilde{E}(G) \cap E_{\rm n}(G)} V(e)$ and $\tilde{G} = (V(G), \tilde{E}(G) \cap E_{\rm n}(G))$. Set

$$\mathscr{L} = \{(S, A) | S \text{ is a 4-cutset}, \ A \text{ is the union of the vertex sets of some components of } G - S, \emptyset \neq A \neq V(G) - S\}, \tag{1.1}$$

$$\mathscr{L}_0 = \{(S, A) \in \mathscr{L} | S \text{ is a nontrivial 4-cutset}\}. \tag{1.2}$$

For $(S, A) \in \mathscr{L}$, we let $\bar{A} = V(G) - S - A$. Thus if $(S, A) \in \mathscr{L}$, then $(S, \bar{A}) \in \mathscr{L}$ and $N_G(A) - A = N_G(\bar{A}) - \bar{A} = S$.

Now take $(S_1, A_1), \ldots, (S_k, A_k) \in \mathscr{L}$ so that for each $e \in \tilde{E}(G) \cap E_{\rm n}(G)$, there exists S_i such that $V(e) \subseteq S_i$. We choose $(S_1, A_1), \ldots, (S_k, A_k)$ so that k is minimum and so that $(|A_1|, \ldots, |A_k|)$ is lexicographically minimum, subject to the condition that k is minimum (thus if $\tilde{E}(G) \cap E_{\rm n}(G) = \emptyset$, then $k = 0$). Note that the minimality of k implies that for each $1 \leqslant i \leqslant k$, we have $E(G[S_i]) \cap (\tilde{E}(G) \cap E_{\rm n}(G)) \neq \emptyset$ and hence $(S_i, A_i) \in \mathscr{L}_0$. Set $\mathscr{S} = \{S_1, \ldots, S_k\}$. Further set

$$\mathscr{K} = \{(u, S, A) | u \in \tilde{V}, S \in \mathscr{S}, (S, A) \in \mathscr{L}_0, \text{ there exists } e \in \tilde{E}(G) \cap E_{\rm n}(G) \text{ such that } u \in V(e) \subseteq S\}. \tag{1.3}$$

We define two subsets $\mathscr{K}^*$ and $\mathscr{K}_0$ of $\mathscr{K}$. Let $\mathscr{K}^*$ be the set of those members (u, S, A) of $\mathscr{K}$ for which A is minimal; that is to say,

$$\mathscr{K}^* = \{(u, S, A) \in \mathscr{K} \mid \text{there is no } (v, T, B) \in \mathscr{K} \text{ with } v = u \text{ and } (T, B) \neq (S, A) \text{ such that } B \subseteq A\}. \tag{1.4}$$

Finally, let $\mathscr{K}_0$ be the set of those members (u, S, A) of $\mathscr{K}^*$ which satisfy one of the following two conditions:

(1) $\deg(u) \geqslant 5$; or
(2) $\deg(u) = 4$, $|N(u) \cap A| = 1$ and, if we write $N(u) \cap A = \{a\}$, then $ua \in E_c(G)$.

We can now state our result.

Theorem 2. *Let G be a 4-connected graph, and let $\mathscr{K}_0$ be as above. Then we can assign to each $(u, S, A) \in \mathscr{K}_0$ a 4-contractible edge $\varphi(u, S, A)$ having the property stated in Theorem 1, so that for each $e \in E_c(G)$ there are at most two members (u, S, A) of $\mathscr{K}_0$ such that $\varphi(u, S, A) = e$.*

As an application of Theorems 1 and 2, we obtain the following corollary concerning the number of contractible edges.

Corollary 3. *Let G be a 4-regular 4-connected graph. Then $|E_c(G)| \geqslant |\tilde{E}(G)|/2$.*

The bound $|\tilde{E}(G)|/2$ of Corollary 3 is sharp. To see this, let $\ell \geqslant 2$ be an integer, and define a graph G of order 8ℓ as follows:

$$V(G) = \{a_i, b_i, c_i, d_i, t_i, u_i, v_i, w_i \mid 1 \leqslant i \leqslant \ell\},$$

$$E(G) = \{a_ib_i, b_ic_i, c_id_i, d_ia_i, t_ia_i, t_ib_i, u_ic_i, u_id_i, v_ib_i, v_ic_i, w_id_i, w_ia_i, t_iu_i, v_iw_i, v_it_{i+1}, w_iu_{i+1} \mid 1 \leqslant i \leqslant \ell\}$$

(indices are to be read modulo ℓ). Then G is 4-regular 4-connected, and $\tilde{E}(G) = \{t_iu_i, v_iw_i, v_it_{i+1}, w_iu_{i+1} \mid 1 \leqslant i \leqslant \ell\}$, $E_c(G) = \{v_it_{i+1}, w_iu_{i+1} \mid 1 \leqslant i \leqslant \ell\}$. Thus $|E_c(G)| = 2\ell = |\tilde{E}(G)|/2$.

For a 4-connected graph G which is not necessarily 4-regular, we can show that if $|\tilde{E}(G)| \geqslant 16$, then $|E_c(G)| \geqslant (|\tilde{E}(G)| + 8)/4$. However, the verification of this statement involves lengthy calculations, and will thus be discussed in a separate paper.

The organization of this paper is as follows. Section 2 contains preliminary results. We prove Theorem 1 in Sections 3 and 4, Theorem 2 in Section 5 through 7, and Corollary 3 in Section 8. We remark that Proposition 3.1, which is proved in Section 3, may be of independent interest in connection with Theorem B.

2. 4-Cutsets

Throughout the rest of this paper, we let G denote a 4-connected graph with $\tilde{E}(G) \neq \emptyset$ (note that in proving Theorem 2 and Corollary 3, we may clearly assume $\tilde{E}(G) \neq \emptyset$). Thus $|V(G)| \geqslant 8$. We write V_4 for $V_4(G)$. Also let $\mathscr{L}$, $\mathscr{L}_0$ be as in the two paragraphs preceding the statement of Theorem 2 (see (1.1) and (1.2)).

In this section, we prove preliminary results concerning the contractibility of edges. We start with four easy lemmas concerning 4-cutsets.

Lemma 2.1. *Let $(S, A), (T, B) \in \mathscr{L}_0$, and suppose that $A \cap B = \emptyset$ and $A \cap \bar{B} = \emptyset$. Then $S \cap T = \emptyset$, and $|S \cap B| = |S \cap \bar{B}| = |A \cap T| = |\bar{A} \cap T| = 2$.*

B	$\bar{A} \cap B$	$S \cap B$	$A \cap B$
T	$\bar{A} \cap T$	$S \cap T$	$A \cap T$
$\bar{B}$	$\bar{A} \cap \bar{B}$	$S \cap \bar{B}$	$A \cap \bar{B}$
	$\bar{A}$	S	A

Proof. Since $A \cap B = A \cap \bar{B} = \emptyset$ and S is a nontrivial 4-cutset, $|A \cap T| = |A| \geqslant 2$. Set $Q = (S \cap T) \cup (S \cap \bar{B}) \cup (\bar{A} \cap T)$. If $\bar{A} \cap \bar{B} \neq \emptyset$, then Q separates $\bar{A} \cap \bar{B}$ and $A \cup B$, and hence $|Q| \geqslant 4$, which implies $|S \cap \bar{B}| = |Q| - |S \cap T| - |\bar{A} \cap T| \geqslant 4 - |S \cap T| - |\bar{A} \cap T| = |A \cap T| \geqslant 2$. If $\bar{A} \cap \bar{B} = \emptyset$, then since T is a nontrivial 4-cutset, we have $|S \cap \bar{B}| = |\bar{B}| \geqslant 2$. Thus $|S \cap \bar{B}| \geqslant 2$ in either case. Similarly $|S \cap B| \geqslant 2$. Since $|S| = 4$, this implies $|S \cap B| = |S \cap \bar{B}| = 2$ and $S \cap T = \emptyset$. If $\bar{A} \cap B$ or $\bar{A} \cap \bar{B}$, say $\bar{A} \cap \bar{B}$, is nonempty, then we have $|S \cap \bar{B}| \geqslant |A \cap T|$, which forces $|A \cap T| = 2$ and hence $|\bar{A} \cap T| = 2$. Thus we may assume $\bar{A} \cap B = \bar{A} \cap \bar{B} = \emptyset$. Then we obtain $|\bar{A} \cap T| = |\bar{A}| \geqslant 2$, which implies $|A \cap T| = |\bar{A} \cap T| = 2$. □

Lemma 2.2. *Let $(S, A), (T, B) \in \mathscr{L}_0$, and suppose that $S \cap T \neq \emptyset$. Then either $A \cap B \neq \emptyset$ and $\bar{A} \cap \bar{B} \neq \emptyset$, or $A \cap \bar{B} \neq \emptyset$ and $\bar{A} \cap B \neq \emptyset$.*

Proof. Suppose that we have $A \cap B = \emptyset$ or $\bar{A} \cap \bar{B} = \emptyset$, and we also have $A \cap \bar{B} = \emptyset$ or $\bar{A} \cap B = \emptyset$. By symmetry, we may assume $A \cap B = \emptyset$ and $A \cap \bar{B} = \emptyset$. But then $S \cap T = \emptyset$ by Lemma 2.1, which contradicts the assumption that $S \cap T \neq \emptyset$. □

Lemma 2.3. *Let $(S, A), (T, B) \in \mathscr{L}$, and suppose that $A \cap B \neq \emptyset$ and $\bar{A} \cap \bar{B} \neq \emptyset$. Then $((S \cap T) \cup (S \cap B) \cup (A \cap T), A \cap B) \in \mathscr{L}$ and $((S \cap T) \cup (S \cap \bar{B}) \cup (A \cap \bar{T}), \bar{A} \cap \bar{B}) \in \mathscr{L}$.*

Proof. Set $R = (S \cap T) \cup (S \cap B) \cup (A \cap T)$ and $Q = (S \cap T) \cup (S \cap \bar{B}) \cup (\bar{A} \cap T)$. Then R separates $A \cap B$ and $\bar{A} \cup \bar{B}$, and Q separates $\bar{A} \cap \bar{B}$ and $A \cup B$. Hence $|R|, |Q| \geqslant 4$. On the other hand, $|R| + |Q| = |S| + |T| = 8$. Consequently $|R| = |Q| = 4$, and hence $(R, A \cap B), (Q, \bar{A} \cap \bar{B}) \in \mathscr{L}$. □

Lemma 2.4. *Let $(S, A) \in \mathscr{L}$.*

(i) *If $W \subseteq S$ and $|W| \leqslant |A|$, then $|N(W) \cap A| \geqslant |W|$. Further if $|W| < |A|$ and $|N(W) \cap A| = |W|$, then $((S - W) \cup (N(W) \cap A), A - (N(W) \cap A)) \in \mathscr{L}$.*
(ii) *If $x \in S$, then $N(x) \cap A \neq \emptyset$. Further if $(S, A) \in \mathscr{L}_0$ and $|N(x) \cap A| = 1$, then $((S - \{x\}) \cup (N(x) \cap A), A - (N(x) \cap A)) \in \mathscr{L}$.*

Proof. Note that (ii) follows from (i) by letting $W = \{x\}$. Thus it suffices to prove (i). Now if $A - (N(W) \cap A) \neq \emptyset$, then $(S - W) \cup (N(W) \cap A)$ separates $A - (N(W) \cap A)$ and $\bar{A} \cup W$. Thus, the desired conclusions follow from the assumption that G is 4-connected. □

In the following four lemmas, we consider edges which are adjacent to the endvertices of an edge contained in two triangles. Recall that $\tilde{V} = \bigcup_{e \in \tilde{E}(G) \cap E_n(G)} V(e)$.

Lemma 2.5. *Let $ab \in E(G)$ with* $\deg(a) = \deg(b) = 4$. *Then $N(a) - \{b\} \neq N(b) - \{a\}$.*

Proof. If $N(a) - \{b\} = N(b) - \{a\}$, then $N(a) - \{b\}$ separates $\{a, b\}$ from the rest, which contradicts the assumption that G is 4-connected. □

Lemma 2.6. *Let u, a, b, w be four distinct vertices with $ua, ub, ab, aw, bw \in E(G)$ and* $\deg(a) = \deg(b) = 4$, *and write $N(a) = \{u, b, w, x\}$ and $N(b) = \{u, a, w, y\}$. Then $x \neq y$, and we have $ax, by \in E_c(G) \cup E_{tn}(G)$ and $a, b \notin \tilde{V}$.*

Proof. By Lemma 2.5, $x \neq y$. In view of the symmetry of the roles of a and b, it suffices to prove $ax \in E_c(G) \cup E_{tn}(G)$ and $a \notin \tilde{V}$. By way of contradiction, suppose that $ax \notin E_c(G) \cup E_{tn}(G)$. Then there exists $(S, A) \in \mathscr{L}_0$ with $a, x \in S$. By Lemma 2.4 (ii), $N(a) \cap A \neq \emptyset$ and $N(a) \cap \bar{A} \neq \emptyset$. Since a vertex in $N(a) \cap A$ and a vertex in $N(a) \cap \bar{A}$ are nonadjacent, this means that one of u and w lies in A and the other one lies in $\bar{A}$. We may assume $u \in A$ and $w \in \bar{A}$. Then $b \in S$. Since $N(b) = \{u, a, w, y\}$, it follows that we have $N(\{a, b\}) \cap A = \{u\}$ or $N(\{a, b\}) \cap \bar{A} = \{w\}$, which contradicts Lemma 2.4 (i). Thus $ax \in E_c(G) \cup E_{tn}(G)$. Now again by way of contradiction, suppose that $a \in \tilde{V}$. Then there exists $e \in \tilde{E}(G) \cap E_n(G)$ such that e is incident with a. Since au, ab, aw are contained in a triangle, $e \neq au, ab, aw$. Hence $e = ax$. But since $\tilde{E}(G) \cap E_n(G) \subseteq E_n(G) - E_{tn}(G)$, this contradicts the earlier assertion that $ax \in E_c(G) \cup E_{tn}(G)$. Thus $a \notin \tilde{V}$. □

Lemma 2.7. *Under the notation of Lemma* 2.6, *suppose that* $\deg(u), \deg(w) \geqslant 5$. *Then* $ax, by \in E_c(G)$.

Proof. Suppose that $ax \in E_n(G)$. Then by Lemma 2.6, $ax \in E_{tn}(G)$, and hence there exists $c \in V_4$ such that $ca, cx \in E(G)$. Then $c \in N(a) - \{x\} = \{u, w, b\}$. Since $\deg(u), \deg(w) \geqslant 5$, this forces $c = b$, which contradicts Lemma 2.5. Thus $ax \in E_c(G)$, and we can similarly show that $by \in E_c(G)$. □

Lemma 2.8. *Under the notation of Lemma* 2.6, *suppose that* $\deg(u) \geqslant 5$ *and* $\deg(w) = 4$. *Then one of the following holds*:

(1) $xw \notin E(G)$ *and* $ax \in E_c(G)$, *or*
(2) $yw \notin E(G)$ *and* $by \in E_c(G)$.

Proof. If $xw, yw \in E(G)$, then $N(\{a, b, w\}) - \{a, b, w\} = \{u, x, y\}$, which contradicts the assumption that G is 4-connected. Thus we have $xw \notin E(G)$ or $yw \notin E(G)$. We may assume that $xw \notin E(G)$. Now suppose that $ax \in E_n(G)$. Then by Lemma 2.6, $ax \in E_{tn}(G)$. Hence there exists $c \in V_4$ such that $ca, cx \in E(G)$. Arguing as in Lemma 2.7, we see that $c = w$. But this contradicts the assumption that $xw \notin E(G)$. □

We now prove two auxiliary results.

Lemma 2.9. *Let* $(P, X) \in \mathscr{L}_0$ *and* $u \in P$. *Suppose that X is minimal*, *subject to the condition that* $u \in P$ (*i.e.*, *there is no* $(R, Z) \in \mathscr{L}_0$ *with* $(P, X) \neq (R, Z)$ *such that* $u \in R$ *and* $Z \subseteq X$). *Then* $ua \in E_c(G) \cup E_{tn}(G)$ *for each* $a \in N(u) \cap X$.

Proof. Let $a \in N(u) \cap X$, and suppose that $ua \in E_n(G) - E_{tn}(G)$. Then there exists $(Q, Y) \in \mathscr{L}_0$ with $u, a \in Q$. Note that $u \in P \cap Q$. Thus in view of Lemma 2.2, we may assume $X \cap Y \neq \emptyset$ and $\bar{X} \cap \bar{Y} \neq \emptyset$. Set $U = (P \cap Q) \cup (P \cap Y) \cup (X \cap Q)$. Then by Lemma 2.3, $(U, X \cap Y) \in \mathscr{L}$. But since $u, a \in (P \cup X) \cap Q \subseteq U$, this implies $(U, X \cap Y) \in \mathscr{L}_0$, which contradicts the minimality of X. □

Lemma 2.10. *Let* $(R, Z) \in \mathscr{L}_0$ *and* $a \in R$. *Suppose that* $|N(a) \cap Z| = 1$, *and write* $N(a) \cap Z = \{x\}$. *Then* $ax \in E_c(G) \cup E_{tn}(G)$.

Proof. Suppose that $ax \in E_n(G) - E_{tn}(G)$. Then there exists $(Q, Y) \in \mathscr{L}_0$ with $a, x \in Q$. By Lemma 2.2, we may assume $Z \cap Y \neq \emptyset$ and $\bar{Z} \cap \bar{Y} \neq \emptyset$. Then by Lemma 2.3, $((R \cap Q) \cup (R \cap Y) \cup (Z \cap Q), Z \cap Y) \in \mathscr{L}$. Hence by Lemma 2.4, $N(a) \cap (Z \cap Y) \neq \emptyset$, which contradicts the assumption that $N(a) \cap Z = \{x\}$. □

The last three lemmas are analogous to Lemmas 2.6 through 2.8.

Lemma 2.11. *Let* u, a, b *be three distinct vertices with* $ua, ub, ab \in E(G)$ *and* $\deg(a) = 4$, *and write* $N(a) = \{u, b, x, y\}$. *Suppose that there exists* $(R, Z) \in \mathscr{L}_0$ *such that* $u, a \in R$, $b, y \in Z$ *and* $x \in \bar{Z}$. *Suppose further that Z is minimal*, *subject to the condition that* $u, a \in R$ *and* $b \in Z$. *Then the following hold.*

(i) $xy \notin E(G)$.
(ii) $ax \in E_c(G) \cup E_{tn}(G)$.
(iii) $ay \in E_c(G) \cup E_{tn}(G)$.
(iv) $a \notin \tilde{V}$.

Proof. Since $x \in \bar{Z}$ and $y \in Z$, we clearly have $xy \notin E(G)$ and, applying Lemma 2.10 to $(R, \bar{Z})$, we obtain $ax \in E_c(G) \cup E_{tn}(G)$. Thus (i) and (ii) are proved. To prove (iii), suppose that $ay \in E_n(G) - E_{tn}(G)$. Then there exists $(Q, Y) \in \mathscr{L}_0$ with $a, y \in Q$. By Lemma 2.2, we may assume $Z \cap Y \neq \emptyset$ and $\bar{Z} \cap \bar{Y} \neq \emptyset$. Set $U = (R \cap Q) \cup (R \cap Y) \cup (Z \cap Q)$. Since $a, y \in U$, it follows from Lemma 2.3 that $(U, Z \cap Y) \in \mathscr{L}_0$. Hence by Lemma 2.4 (ii), $N(a) \cap (Z \cap Y) \neq \emptyset$, which implies $N(a) \cap (Z \cap Y) = \{b\}$. Since $ub \in E(G)$, this forces $u \in (Q \cup Y) \cap R$, and hence $u \in U$. Since $a \in U$ and $b \in Z \cap Y$, this contradicts the minimality of Z, completing the proof of (iii). Now to prove (iv), suppose that $a \in \tilde{V}$. Then there exists $e \in \tilde{E}(G) \cap E_n(G)$such that e is incident with a. Since au, ab are contained in a triangle, $e \neq au, ab$. Consequently $e = ax$ or ay, which contradicts (ii) or (iii). □

Lemma 2.12. *Under the notation of Lemma* 2.11, *suppose that* $\deg(b) \geqslant 5$. *Then* $ax \in E_c(G)$ *or* $ay \in E_c(G)$.

Proof. Suppose that $ax, ay \in E_n(G)$. Then by Lemma 2.11 (ii) and (iii), $ax, ay \in E_{tn}(G)$. Hence there exist $c, c' \in V_4$ such that $ca, cx \in E(G)$ and $c'a, c'y \in E(G)$. Since $\deg(b) \geqslant 5$, $c, c' \neq b$. Since $xy \notin E(G)$ by Lemma 2.11 (i), $c, c' \notin \{x, y\}$. Consequently $\deg(u) = 4$ and $c = c' = u$. But this contradicts Lemma 2.5. □

Lemma 2.13. *Under the notation of Lemma* 2.11, *suppose that* $\deg(b), \deg(u) \geqslant 5$. *Then* $ax, ay \in E_c(G)$.

Proof. Suppose that $ax \in E_n(G)$. Then $ax \in E_{tn}(G)$ by Lemma 2.11 (ii), and hence there exists $c \in V_4$ such that $ca, cx \in E(G)$. Since $\deg(b), \deg(u) \geqslant 5$, $c \neq b, u$. Hence $c = y$. But this contradicts Lemma 2.11 (i). Thus $ax \in E_c(G)$. By means of Lemma 2.11 (iii), we similarly obtain $ay \in E_c(G)$. □

3. Neighborhood of a vertex of degree 5

In this section, we prove a result which shows that Theorem 1 holds if $\deg(u) \geqslant 5$. Specifically, we prove the following proposition in a series of claims.

Proposition 3.1. *Let* $(P, X) \in \mathscr{L}_0$ *and* $u \in P$, *and suppose that* $\deg(u) \geqslant 5$. *Then one of the following holds*:

(1) *there exists* $a \in N(u) \cap X$ *such that* $ua \in E_c(G)$; *or*
(2) *there exists* $a \in N(u) \cap (P \cup X) \cap V_4$ *for which there exists* $e \in E_c(G)$ *such that* e *is incident with* a.

Through this section, let (P, X), u be as in Proposition 3.1. We may assume that X is minimal, subject to the condition that $u \in P$ (i.e., there is no $(R, Z) \in \mathscr{L}_0$ with $(R, Z) \neq (P, X)$ such that $u \in R$ and $Z \subseteq X$).

Claim 3.2. *Suppose that there exists an edge* e *joining a vertex in* $N(u) \cap X \cap V_4$ *and a vertex in* $N(u) \cap (P \cup X) \cap V_4$. *Suppose that* $e \in E_n(G)$, *and write* $e = ab$. *Then* a *or* b, *say* a, *satisfies the following conditions.*

(i) *If we write* $N(a) = \{u, b, x, y\}$, *then* $xy \notin E(G)$.
(ii) $a \notin \tilde{V}$.
(iii) *There exists* $e' \in E_c(G)$ *such that* e' *is incident with* a.

Proof. If $ab \in E_{tn}(G)$, then there exists $w \in V_4$ such that $wa, wb \in E(G)$, and hence the desired conclusions follow from Lemmas 2.6 and 2.8. Thus we may assume that $ab \in E_n(G) - E_{tn}(G)$. Then there exists $(R, Z) \in \mathscr{L}_0$ with $a, b \in R$. We first show that $u \notin R$. Suppose that $u \in R$. Then by Lemma 2.2, we may assume $X \cap Z \neq \emptyset$ and $\bar{X} \cap \bar{Z} \neq \emptyset$. Since $a, b \in (P \cup X) \cap R$, it follows from Lemma 2.3 that $((P \cap R) \cup (P \cap Z) \cup (X \cap R), X \cap Z) \in \mathscr{L}_0$, which contradicts the minimality of X. Thus $u \notin R$. We may assume $u \in Z$. We may also assume that we have chosen (R, Z) so that Z is minimal, subject to the condition that $a, b \in R$ and $u \in Z$. By Lemma 2.4 (i), we have $N(a) \cap Z \neq \{u\}$ or $N(b) \cap Z \neq \{u\}$. We may assume $N(a) \cap Z \neq \{u\}$. Since $N(a) \cap \bar{Z} \neq \emptyset$ by Lemma 2.4 (ii), we have $|N(a) \cap Z| = 2$ and $|N(a) \cap \bar{Z}| = 1$. Write $N(a) \cap Z = \{u, y\}$ and $N(a) \cap \bar{Z} = \{x\}$. Then b, a, u, x, y satisfy the assumptions of Lemmas 2.11 and 2.12 with the roles of b and u replaced by each other. Consequently the desired conclusions follow from (i), (iv) of Lemmas 2.11 and 2.12. □

Claim 3.3. *Let* $a \in X$, *and suppose that* $ua \in E_n(G)$. *Then* $ua \in E_{tn}(G)$.

Proof. This follows from Lemma 2.9. □

Claim 3.4. *Suppose that each edge joining* u *and a vertex in* X *is* 4*-noncontractible, and that there is no edge which joins a vertex in* $N(u) \cap X \cap V_4$ *and a vertex in* $N(u) \cap (P \cup X) \cap V_4$. *Then* $N(u) \cap X \cap V_4 = \emptyset$.

Proof. Suppose that $N(u) \cap X \cap V_4 \neq \emptyset$, and take $a \in N(u) \cap X \cap V_4$. We have $ua \in E_{\mathrm{tn}}(G)$ by Claim 3.3. Hence there exists $b \in V_4$ such that $ub, ab \in E(G)$. From $a \in X$ and $ab \in E(G)$, it follows that $b \in P \cup X$. Thus ab is an edge joining a vertex in $N(u) \cap X \cap V_4$ and a vertex in $N(u) \cap (P \cup X) \cap V_4$, a contradiction. □

Claim 3.5. *Suppose that each edge joining u and a vertex in X is* 4*-noncontractible, and that there is no edge which joins a vertex in $N(u) \cap X \cap V_4$ and a vertex in $N(u) \cap (P \cup X) \cap V_4$. Then there exists $a \in N(u) \cap P \cap V_4$ and $b \in N(u) \cap X$ such that $ab \in E(G)$, $|N(a) \cap X| = 2$ and $|N(a) \cap \bar{X}| = 1$.*

Proof. Take $z \in N(u) \cap X$. Then $uz \in E_{\mathrm{tn}}(G)$ by Claim 3.3, and hence there exists $a_z \in V_4$ such that $a_z u, a_z z \in E(G)$. Since $N(u) \cap X \cap V_4 = \emptyset$ by Claim 3.4, $a_z \in P$. Since $\deg(a_z) = 4$ and $u \in N(a_z) \cap P$, $|N(a_z) \cap X| + |N(a_z) \cap \bar{X}| \leqslant 3$, and hence it follows from Lemma 2.4 (ii) that $1 \leqslant |N(a_z) \cap X| \leqslant 2$. Now by way of contradiction, suppose that the claim is false. Then $|N(a_z) \cap X| = 1$, i.e., $N(a_z) \cap X = \{z\}$. Since $z \in N(u) \cap X$ is arbitrary, this means that $a_y \neq a_z$ for any $y, z \in N(u) \cap X$ with $y \neq z$ and if we set $W = \{a_z | z \in N(u) \cap X\}$, then we have $|W| = |N(u) \cap X|$ and $N(\{u\} \cup W) \cap X = N(u) \cap X$, and hence $|N(\{u\} \cup W) \cap X| = |W| = |\{u\} \cup W| - 1$. In view of Lemma 2.4 (i), this implies $|\{u\} \cup W| \geqslant |X| + 1$, i.e., $|W| \geqslant |X|$. Again fix $z \in N(u) \cap X$. Since $N(a_y) \cap X = \{y\}$ for each $y \in (N(u) \cap X) - \{z\}$, $N(z) \subseteq (P - (W - \{a_z\})) \cup (X - \{z\})$. Consequently $\deg(z) \leqslant |P| - |W| + |X| \leqslant |P| = 4$, which implies $z \in N(u) \cap X \cap V_4$. But this contradicts Claim 3.4, completing the proof. □

Claim 3.6. *Suppose that each edge joining u and a vertex in X is* 4*-noncontractible, and that there is no edge which joins a vertex in $N(u) \cap X \cap V_4$ and a vertex in $N(u) \cap (P \cup X) \cap V_4$. Further let a, b be as in Claim* 3.5*, and write $N(a) \cap X = \{b, y\}$ and $N(a) \cap \bar{X} = \{x\}$. Then $xy \notin E(G)$, $a \notin \tilde{V}$, and $ax, ay \in E_{\mathrm{c}}(G)$.*

Proof. Note that $\deg(b) \geqslant 5$ by Claim 3.4, and $\deg(u) \geqslant 5$ by the assumption of Proposition 3.1. Thus the desired conclusions follow from (i) and (iv) of Lemmas 2.11 and 2.13. □

Proposition 3.1 now follows from Claims 3.2 and 3.6.

4. Non-meshing 4-cutsets

In this section, we prove Theorem 1, and fix notation for the proof of Theorem 2. Following Cheriyan and Thurimella [2] and Jordán [4], for two disjoint 4-cutsets S, T of G, we say that S meshes with T if S intersects with at least two components of $G - T$. It is easy to see that if S meshes with T, then T intersects with every component of $G - S$, and hence T meshes with S and S intersects with every component of $G - T$. Now let $(S_1, A_1), \ldots, (S_k, A_k)$ and $\mathscr{S}$ be as in the paragraph preceding the statement of Theorem 2. Note that the minimality of k implies that $(S_i, A_i) \in \mathscr{L}_0$ for each $1 \leqslant i \leqslant k$. The following claim is virtually proved in Kriesell [5, Lemma 3], but we include its proof for the convenience of the reader.

Claim 4.1. *No two members of $\mathscr{S}$ mesh with each other.*

Proof. Suppose that there exist $i, j\,(i < j)$ such that S_i meshes with S_j. Then $A_i \cap S_j \neq \emptyset$. We first show that $A_i \cap A_j = \emptyset$. Suppose that $A_i \cap A_j \neq \emptyset$. Set $R = (S_i \cap S_j) \cup (S_i \cap A_j) \cup (A_i \cap S_j)$ and $Q = (S_i \cap S_j) \cup (S_i \cap \bar{A}_j) \cup (\bar{A}_i \cap S_j)$. Then $|R| \geqslant 4$, and hence $|S_i \cap \bar{A}_j| = 4 - |S_i \cap S_j| - |S_i \cap A_j| \leqslant |R| - |S_i \cap S_j| - |S_i \cap A_j| = |A_i \cap S_j|$. If $\bar{A}_i \cap \bar{A}_j = \emptyset$, then $|\bar{A}_j| = |S_i \cap \bar{A}_j| + |A_i \cap \bar{A}_j| \leqslant |A_i \cap S_j| + |A_i \cap \bar{A}_j| < |A_i|$, and hence we get a contradiction to the minimality of $(|A_1|, |A_2|, \ldots, |A_k|)$ by replacing (S_i, A_i) and (S_j, A_j) by $(S_j, \bar{A}_j)$ and (S_i, A_i), respectively. Thus $\bar{A}_i \cap \bar{A}_j \neq \emptyset$. Hence $(R, A_i \cap A_j), (Q, \bar{A}_i \cap \bar{A}_j) \in \mathscr{L}$ by Lemma 2.3. Note that each edge contained in $G[S_i]$ or $G[S_j]$ is contained in $G[R]$ or $G[Q]$. Consequently, we get a contradiction by replacing (S_i, A_i) and (S_j, A_j) by $(R, A_i \cap A_j)$ and $(Q, \bar{A}_i \cap \bar{A}_j)$, respectively. Thus $A_i \cap A_j = \emptyset$ as desired, and we similarly obtain $A_i \cap \bar{A}_j = \emptyset$.

Consequently $S_i \cap S_j = \emptyset$ and $|S_i \cap A_j| = |S_i \cap \bar{A}_j| = |A_i \cap S_j| = |\bar{A}_i \cap S_j| = 2$ by Lemma 2.1. Write $A_i \cap S_j = \{a, b\}$. If $ab \in E(G)$, then since $N(a), N(b) \subseteq (A_i \cap S_j) \cup S_i$, ab is contained in a triangle. This means that each edge in $\tilde{E}(G) \cap E_{\mathrm{n}}(G)$ which is contained in $G[S_j]$ is contained in $G[\bar{A}_i \cap S_j]$. Now if $\bar{A}_i \cap A_j \neq \emptyset$, then we get a contradiction by replacing (S_j, A_j) by $((S_i \cap A_j) \cap (\bar{A}_i \cap S_j), \bar{A}_i \cap A_j)$. Thus $\bar{A}_i \cap A_j = \emptyset$, which implies $\bar{A}_i \cap \bar{A}_j \neq \emptyset$ because $E(G[S_j]) \cap (\tilde{E}(G) \cap E_{\mathrm{n}}(G)) \neq \emptyset$. We now get a contradiction to the minimality of k by replacing (S_i, A_i) and (S_j, A_j) by $((S_i \cap \bar{A}_j) \cup (\bar{A}_i \cap S_j), \bar{A}_i \cap \bar{A}_j)$. This completes the proof of Claim 4.1. □

Let $\mathscr{K}$, $\mathscr{K}^*$ and $\mathscr{K}_0$ be as in the paragraph preceding Theorem 2 (see (1.3), (1.4) and conditions (1) and (2) stated at the end of the paragraph).

The following claim immediately follows from the definition of $\mathscr{K}^*$.

Claim 4.2. *Let $u \in \tilde{V}$. Then for each $(u, S, A) \in \mathscr{K}$, there exists a member (v, T, B) of $\mathscr{K}^*$ with $v = u$ and $B \subseteq A$. In particular, there exist at least two members (v, T, B) of $\mathscr{K}^*$ with $v = u$.*

Claim 4.3. *Let $(u, S, A), (v, T, B) \in \mathscr{K}^*$ with $u = v$ and $(S, A) \neq (T, B)$. Then $(S \cup A) \cap B = A \cap (T \cup B) = \emptyset$.*

Proof. If $S = T$, the desired conclusion clearly holds. Thus we may assume that $S \neq T$. By Claim 4.1, we have that $S \cap \bar{B} = T \cap \bar{A} = \emptyset$, $S \cap B = T \cap \bar{A} = \emptyset$, $S \cap \bar{B} = T \cap A = \emptyset$, or $S \cap B = T \cap A = \emptyset$. Suppose that $S \cap \bar{B} = T \cap \bar{A} = \emptyset$. Then since $S \neq T$, we have $A \cap T \neq \emptyset$ and $|(S \cap T) \cup (\bar{A} \cap T) \cup (S \cap \bar{B})| = |T| - |A \cap T| < 4$, and hence $\bar{A} \cap \bar{B} = \emptyset$. Since $S \cap \bar{B} = \emptyset$ and $A \cap T \neq \emptyset$, this implies $\bar{B}$ is a proper subset of A. But since $(u, T, \bar{B}) \in \mathscr{K}$ and $(u, S, A) \in \mathscr{K}^*$, this contradicts the definition of $\mathscr{K}^*$. If $S \cap B = T \cap \bar{A} = \emptyset$ or $S \cap \bar{B} = T \cap A = \emptyset$, then we obtain $B \subseteq A$ or $A \subseteq B$, respectively, and hence we similarly get a contradiction. Thus $S \cap B = T \cap A = \emptyset$. Since $S \neq T$, this also implies $A \cap B = \emptyset$, as desired. □

Recall that $\tilde{G} = (V(G), \tilde{E}(G) \cap E_n(G))$.

Claim 4.4. *Let $u \in \tilde{V}$. Then the following hold.*

(i) *There exists a member (v, T, B) of $\mathscr{K}_0$ with $v = u$.*
(ii) *Suppose that $\deg_G(u) \geqslant 5$, or $\deg_{\tilde{G}}(u) \geqslant 2$, or there exist three members (v, T, B) of $\mathscr{K}^*$ with $v = u$. Then for each $(u, S, A) \in \mathscr{K}^*$, we have $(u, S, A) \in \mathscr{K}_0$. In particular, if $\deg_G(u) = 4$ and $\deg_{\tilde{G}}(u) \geqslant 2$, then $\deg_{\tilde{G}}(u) = 2$ and there exist precisely two members (v, T, B) of $\mathscr{K}_0$ with $v = u$.*

Proof. If $\deg_G(u) \geqslant 5$, the desired conclusion immediately follows from Claim 4.2 and the definition of $\mathscr{K}_0$. Thus we may assume $\deg_G(u) = 4$. We first prove (ii). Thus let u be as in (ii) with $\deg_G(u) = 4$. Then by Lemma 2.4 (ii) and Claim 4.3, it follows that $|N_G(u) \cap A| = 1$ for each $(u, S, A) \in \mathscr{K}^*$, and that for each $a \in N_G(u) - N_{\tilde{G}}(u)$, there exists $(u, S, A) \in \mathscr{K}^*$ such that $a \in A$. Again by Claim 4.3, this implies that for each $(u, S, A) \in \mathscr{K}^*$, $N_G(u) \cap S = N_{\tilde{G}}(u) \cap S$. Note that this also implies that if $\deg_{\tilde{G}}(u) \geqslant 2$, then we have $\deg_{\tilde{G}}(u) = 2$ and there exist precisely two members (v, T, B) of $\mathscr{K}^*$ with $v = u$. Now let $(u, S, A) \in \mathscr{K}^*$, and write $N_G(u) \cap A = \{a\}$. To complete the proof of (ii), it suffices to show that $(u, S, A) \in \mathscr{K}_0$. Suppose that $(u, S, A) \notin \mathscr{K}_0$. Then $ua \in E_n(G)$, and hence $ua \in E_{tn}(G)$ by Lemma 2.10, which implies that there exists $c \in V_4$ such that $cu, ca \in E(G)$. Since $N_G(u) \cap A = \{a\}$, this forces $c \in S$. But since uc is contained in a triangle, $c \notin N_{\tilde{G}}(u)$, which contradicts the earlier assertion that $N_G(u) \cap S = N_{\tilde{G}}(u) \cap S$. Thus (ii) is proved.

We now prove (i). We may assume that there exists $(u, S, A) \in \mathscr{K}^*$ such that $(u, S, A) \notin \mathscr{K}_0$. Then arguing as above, we see that $|N_G(u) \cap (S \cup A)| \geqslant 3$ (note that if $|N_G(u) \cap A| \geqslant 2$, we clearly have $|N_G(u) \cap (S \cup A)| \geqslant 3$). Take $(u, T, B) \in \mathscr{K}^*$ with $B \subseteq \bar{A}$. Then $|N_G(u) \cap B| = 1$. Write $N_G(u) \cap B = \{b\}$. Suppose that $(u, T, B) \notin \mathscr{K}_0$. Then there exists $c' \in V_4$ such that $c'u, c'b \in E(G)$. This in turn implies $|N_G(u) \cap A| = 1$. Write $N_G(u) \cap A = \{a\}$. Then there exists $c \in V_4$ such that $cu, ca \in E(G)$. Since $\deg_G(u) = 4$, $\deg_{\tilde{G}}(u) \geqslant 1$ and $ab \notin E(G)$, this forces $c = c'$. But then applying Lemma 2.6 with a and b replaced by u and c, we obtain $u \notin \tilde{V}$, which contradicts the assumption that $u \in \tilde{V}$. Thus (i) is also proved. □

We can now easily prove Theorem 1.

Proof of Theorem 1. Let u, S, A be as in Theorem 1. Then $(S, A) \in \mathscr{L}_0$. Hence if $\deg_G(u) \geqslant 5$, then the desired conclusion follows from Proposition 3.1. Thus we may assume $\deg_G(u) = 4$. But then from Claim 4.4 (i) and the definition of $\mathscr{K}_0$, we see that there exists $e \in E_c(G)$ such that e is incident with u, as desired. □

5. Definition of $\lambda(u, S, A)$, $\alpha(u, S, A)$ and $\varphi(u, S, A)$

In this section, to each $(u, S, A) \in \mathscr{K}_0$, we assign an edge $\lambda(u, S, A)$, and an endvertex $\alpha(u, S, A)$ of $\lambda(u, S, A)$, and a 4-contractible edge $\varphi(u, S, A)$ incident with $\alpha(u, S, A)$. We start with a claim

Claim 5.1. *Let* $(u, S, A) \in \mathscr{K}_0$, *and set* $W = \{z \in S - \{u\} - N_{\tilde{G}}(u) \mid |N_G(z) \cap A| = 1\}$. *Then* $((S - W) \cup (N_G(W) \cap A), A - (N_G(W) \cap A)) \in \mathscr{L}_0$.

Proof. By the definition of $\mathscr{K}$, there exists $e \in \tilde{E}(G) \cap E_n(G)$ such that $u \in V(e) \subseteq S$. Hence $W \subseteq S - V(e)$, which implies $|W| \leqslant 2$. On the other hand, since $(S, A) \in \mathscr{L}_0$, $|A| \geqslant 2$. Thus $|W| \leqslant |A|$. Suppose that $|W| = |A|$. Then $|W| = |A| = 2$. By Lemma 2.4 (i), $N_G(\{x, z\}) \cap A = A$ for each $x \in V(e)$ and each $z \in W$. Since we also have $N_G(W) \cap A = A$ by Lemma 2.4 (i) and since $|N_G(z) \cap A| = 1$ for each $z \in W$, this means that $N_G(x) \cap A = A$ for each $x \in V(e)$. But then e is contained in a triangle, a contradiction. Thus $|W| < |A|$. Consequently it follows from Lemma 2.4 (i) that $((S - W) \cup (N_G(W) \cap A), A - (N_G(W) \cap A)) \in \mathscr{L}$, which implies the desired conclusion because $V(e) \subseteq S - W$. □

Now let $(u, S, A) \in \mathscr{K}_0$, and let W be as in Claim 5.1. We let $(P_{u,S,A}, X_{u,S,A})$ be a member of $\mathscr{L}_0$ with $u \in P_{u,S,A}$ and $X_{u,S,A} \subseteq A - (N_G(W) \cap A)$ such that $X_{u,S,A}$ is minimal, i.e., there is no $(R, Z) \in \mathscr{L}_0$ with $(R, Z) \neq (P_{u,S,A}, X_{u,S,A})$ such that $u \in R$ and $Z \subseteq X_{u,S,A}$. We remark that we do not require that there should exist an edge $e \in E_n(G)$ with $u \in V(e) \subseteq P_{u,S,A}$. The following claim immediately follows from the definition of $(P_{u,S,A}, X_{u,S,A})$.

Claim 5.2. *Let* $(u, S, A) \in \mathscr{K}_0$. *Let* $z \in S - \{u\} - N_{\tilde{G}}(u)$ *and suppose that* $|N_G(z) \cap A| = 1$. *Then* $z \notin P_{u,S,A}$.

Let again $(u, S, A) \in \mathscr{K}_0$, and let $(P, X) = (P_{u,S,A}, X_{u,S,A})$ be as above. We define the type of (u, S, A) as follows: (u, S, A) is of type 1 if there exists a 4-contractible edge joining u and a vertex in X; (u, S, A) is of type 2 if it is not of type 1 and there exists a 4-contractible edge joining a vertex in $N_G(u) \cap X \cap V_4$ and a vertex in $N_G(u) \cap (P \cup X) \cap V_4$; (u, S, A) is type 3 if it is not of type 1 or 2 but there exists an edge joining a vertex in $N_G(u) \cap X \cap V_4$ and a vertex in $N_G(u) \cap (P \cup X) \cap V_4$; (u, S, A) is type 4 if it is not of type i for any $i = 1, 2, 3$. We let $\mathscr{K}_i$ denote the set of those members of $\mathscr{K}_0$ which are the type i ($i = 1, 2, 3, 4$). The following claim will be used implicitly throughout the rest of this paper.

Claim 5.3. *Let* $(u, S, A) \in \mathscr{K}_0 - \mathscr{K}_1$. *Then* $\deg(u) \geqslant 5$.

Proof. Suppose that $\deg(u) = 4$. Then by the definition of $\mathscr{K}_0$, $|N_G(u) \cap A| = 1$ and, if we write $N_G(u) \cap A = \{a\}$, then $ua \in E_c(G)$. By Lemma 2.4 (ii), $a \in X$. Consequently $(u, S, A) \in \mathscr{K}_1$ by definition, which contradicts the assumption that $(u, S, A) \in \mathscr{K}_0 - \mathscr{K}_1$. □

We first define $\lambda(u, S, A)$. If $(u, S, A) \in \mathscr{K}_1$, let $\lambda(u, S, A)$ be a 4-contractible edge joining u and a vertex in X; if $(u, S, A) \in \mathscr{K}_2$, let $\lambda(u, S, A)$ be a 4-contractible edge joining a vertex in $N_G(u) \cap X \cap V_4$ and a vertex in $N_G(u) \cap (P \cup X) \cap V_4$; if $(u, S, A) \in \mathscr{K}_3$, let $\lambda(u, S, A)$ be an edge joining a vertex in $N_G(u) \cap X \cap V_4$ and a vertex in $N_G(u) \cap (P \cup X) \cap V_4$; if $(u, S, A) \in \mathscr{K}_4$, let $\lambda(u, S, A) = ab$ where a, b are as in Claim 3.5. The following claim follows from the definition of $\lambda(u, S, A)$.

Claim 5.4. *Let* $2 \leqslant i, j \leqslant 4$ *with* $i \neq j$, *and let* $(u_1, S_1, A_1) \in \mathscr{K}_i$ *and* $(u_2, S_2, A_2) \in \mathscr{K}_j$. *Then* $\lambda(u_1, S_1, A_1) \neq \lambda(u_2, S_2, A_2)$.

Claim 5.5. *Let* $(u_1, S_1, A_1), (u_2, S_2, A_2) \in \mathscr{K}_0$ *with* $u_1 = u_2$ *and* $(S_1, A_1) \neq (S_2, A_2)$. *Then* $\lambda(u_1, S_1, A_1) \neq \lambda(u_2, S_2, A_2)$.

Proof. By Claim 4.3, $A_1 \cap A_2 = \emptyset$. Hence $X_{u_1,S_1,A_1} \cap X_{u_2,S_2,A_2} \subseteq A_1 \cap A_2 = \emptyset$. Since at least one of the endvertices of $\lambda(u_j, S_j, A_j)$ is in X_{u_j,S_j,A_j}, this implies $\lambda(u_1, S_1, A_1) \neq \lambda(u_2, S_2, A_2)$. □

Claim 5.6. *Let e be an edge joining two vertices of degree* 4. *Then there exist at most two members* (u, S, A) *of* $\mathscr{K}_2 \cup \mathscr{K}_3$ *for which* $\lambda(u, S, A) = e$.

Proof. Suppose that there exist three members (u_j, S_j, A_j) $(1 \leqslant j \leqslant 3)$ of $\mathscr{K}_2 \cup \mathscr{K}_3$ such that $\lambda(u_j, S_j, A_j) = e$. By Claim 5.5, the u_j are all distinct. But this contradicts Lemma 2.5. □

We now define $\alpha(u, S, A)$. If $(u, S, A) \in \mathscr{K}_1$, let $\alpha(u, S, A) = u$. Now assume $(u, S, A) \in \mathscr{K}_2$. In this case, we let $\alpha(u, S, A)$ be an endvertex of $\lambda(u, S, A)$. If $\lambda(u, S, A)$ has an endvertex in P and there is no $(w, R, Z) \in \mathscr{K}_2$ with $(w, R, Z) \neq (u, S, A)$ such that $\lambda(w, R, Z) = \lambda(u, S, A)$, then we let $\alpha(u, S, A)$ be the endvertex of $\lambda(u, S, A)$ in X. Next assume $(u, S, A) \in \mathscr{K}_3$. In this case, we let $\alpha(u, S, A)$ be an endvertex of $\lambda(u, S, A)$ which satisfies (ii) and (iii) of Claim 3.2. If there is no $(w, R, Z) \in \mathscr{K}_3$ with $(w, R, Z) \neq (u, S, A)$ such that $\lambda(w, R, Z) = \lambda(u, S, A)$, then we choose $\alpha(u, S, A)$ so that it also satisfies (i) of Claim 3.2. Finally if $(u, S, A) \in \mathscr{K}_4$, let $\alpha(u, S, A) = a$, where a is as in Claim 3.5. Note that if $(u_1, S_1, A_1), (u_2, S_2, A_2) \in \mathscr{K}_3$ with $(u_1, S_1, A_1) \neq (u_2, S_2, A_2)$ and $\lambda(u_1, S_1, A_1) = \lambda(u_2, S_2, A_2)$, then $u_1 \neq u_2$ by Claim 5.5, and hence it follows from Lemmas 2.6 and 2.7 that both endvertices of $\lambda(u_1, S_1, A_1)$ satisfy (ii) and (iii) of Claim 3.2. Thus in view of Claim 5.6, we can define $\alpha(u, S, A)$ so that the following claim holds.

Claim 5.7. *Let $(u_1, S_1, A_1), (u_2, S_2, A_2) \in \mathscr{K}_2 \cup \mathscr{K}_3$ with $(u_1, S_1, A_1) \neq (u_2, S_2, A_2)$ and $\lambda(u_1, S_1, A_1) = \lambda(u_2, S_2, A_2)$. Then $\alpha(u_1, S_1, A_1) \neq \alpha(u_2, S_2, A_2)$.*

Finally we define $\varphi(u, S, A)$. If $(u, S, A) \in \mathscr{K}_1 \cup \mathscr{K}_2$, simply let $\varphi(u, S, A) = \lambda(u, S, A)$; if $(u, S, A) \in \mathscr{K}_3$, let $\varphi(u, S, A)$ be a 4-contractible edge incident with $\alpha(u, S, A)$, whose existence is guaranteed by Claim 3.3 (iii) or Lemma 2.7 (it is possible that the other endvertex of $\varphi(u, S, A)$ lies $\overline{X}$); if $(u, S, A) \in \mathscr{K}_4$, let $\varphi(u, S, A) = ax$, where a, x are as in Claim 3.6.

6. Properties of $\lambda(u, S, A)$

We continue with the notation of the preceding section. Our main concern is $\varphi(u, S, A)$ but, in this section, we consider $\lambda(u, S, A)$.

Claim 6.1. *Let $(u, S, A), (v, T, B) \in \mathscr{K}_0 - \mathscr{K}_1$ with $u = v$ and $(S, A) \neq (T, B)$. Then $\lambda(u, S, A)$ and $\lambda(v, T, B)$ do not share an endvertex of degree 4.*

Proof. Suppose that $\lambda(u, S, A)$ and $\lambda(v, T, B)$ share an endvertex a of degree 4. Let $(P, X) = (P_{u,S,A}, X_{u,S,A})$. Then $a \in P \cup X \subseteq S \cup A$. Similarly $a \in T \cup B$. Hence $a \in (S \cup A) \cap (T \cup B) \subseteq S \cap T$ by Claim 4.3. Since $\deg(a) = 4$ and $u \in N_G(a) \cap S \cap T$, $|N_G(a) \cap (A \cup B)| \leqslant 3$. Since $A \cap B = \emptyset$ by Claim 4.3, this together with Lemma 2.4 (ii) implies that we have $|N_G(a) \cap A| = 1$ or $|N(a) \cap B| = 1$. We may assume $|N_G(a) \cap A| = 1$. On the other hand, since ua is contained in a triangle, $a \notin N_{\tilde{G}}(u)$. But since $a \in (P \cup X) \cap S \subseteq P$, this contradicts Claim 5.2. □

Claim 6.2. *Let $(u, S, A), (v, T, B) \in \mathscr{K}_4$ with $(u, S, A) \neq (v, T, B)$. Then $\lambda(u, S, A) \neq \lambda(v, T, B)$.*

Proof. Suppose that $\lambda(u, S, A) = \lambda(v, T, B)$. Let $(P, X) = (P_{u,S,A}, X_{u,S,A})$, and let a, b, x, y be as in Claims 3.5 and 3.6. Then $\lambda(u, S, A) = \lambda(v, T, B) = ab$, and hence $v \in N_G(a) \cap N_G(b)$. In particular $v \in N_G(a) - \{b\} = \{u, x, y\}$. Since we get $xb \notin E(G)$ from $x \in \bar{X}$ and $b \in X$, $v \neq x$. We also have $v \neq u$ by Claim 5.5. Thus $v = y$, and hence $y, a \in P_{v,T,B}$. Consequently $ya \in E_n(G)$, which contradicts Claim 3.6. □

7. Properties of $\varphi(u, S, A)$

In this section, we complete the proof of Theorem 2 by showing that we have $(\varphi(u, S, A), \alpha(u, S, A)) \neq (\varphi(v, T, B), \alpha(v, T, B))$ for any distinct members $(u, S, A), (v, T, B)$ of $\mathscr{K}_0$. The first two claims immediately from Claims 5.5 and 5.7, respectively.

Claim 7.1. *Let $(u, S, A), (v, T, B) \in \mathscr{K}_1$ with $(u, S, A) \neq (v, T, B)$. Then $(\varphi(u, S, A), \alpha(u, S, A)) \neq (\varphi(v, T, B), \alpha(v, T, B))$.*

Claim 7.2. *Let $(u, S, A), (v, T, B) \in \mathscr{K}_2$ with $(u, S, A) \neq (v, T, B)$. Then $(\varphi(u, S, A), \alpha(u, S, A)) \neq (\varphi(v, T, B), \alpha(v, T, B))$.*

Claim 7.3. *Let $(u, S, A) \in \mathscr{K}_2$ and $(v, T, B) \in \mathscr{K}_1$, and suppose that $\varphi(u, S, A) = \varphi(v, T, B)$. Then $v \in P_{u,S,A}$, and there is no $(w, R, Z) \in \mathscr{K}_2$ with $(w, R, Z) \neq (u, S, A)$ such that $\varphi(w, R, Z) = \varphi(u, S, A)$.*

Proof. Write $\varphi(u, S, A) = \varphi(v, T, B) = vb$. Also let vz be an edge in $E(G) \cap E_n(G)$ such that $v, z \in T$. Let $(P, X) = (P_{u,S,A}, X_{u,S,A})$. Suppose that $v \in X$. Then since $vz \in E(G)$, we have $z \in P \cup X$, and hence $z \in (P \cup X) \cap T$. Since $\deg(v) = 4$, it follows from the definition of $\mathscr{K}_0$ that $N(v) \cap B = \{b\}$. Since $u \in N(v) \cap N(b)$, this implies $u \in T$, and hence $u \in P \cap T$. Thus by Lemmas 2.2 and 2.3, there exists a 4-cutset U with $U \supseteq (P \cup X) \cap T$ such that $G - U$ has a component H with $V(H) \subseteq X - (X \cap T) \subseteq X - \{v\}$. But then since $v \in X \cap T \subseteq U$, $z \in (P \cup X) \cap T \subseteq U$ and $vz \in \tilde{E}(G) \cap E_n(G) \subseteq E_n(G) - E_{tn}(G)$, U is a nontrivial 4-cutset, which contradicts the minimality of X because $u \in P \cap T \subseteq U$ (see the remark made in the paragraph preceding Claim 5.2). Thus $v \in P$. Now suppose that there exists $(w, R, Z) \in \mathscr{K}_2$ with $(w, R, Z) \neq (u, S, A)$ such that $\varphi(w, R, Z) = \varphi(u, S, A)$. Then $w \neq u$ by Claim 5.5. Hence applying Lemma 2.6 with $a = v$, we see that $v \notin \tilde{V}$. But this contradicts the assumption that $(v, T, B) \in \mathscr{K}_1$. Thus there is no such (w, R, Z). □

Claim 7.4. *Let $(u, S, A) \in \mathscr{K}_2$ and $(v, T, B) \in \mathscr{K}_1$. Then $(\varphi(u, S, A), \alpha(u, S, A)) \neq (\varphi(v, T, B), \alpha(v, T, B))$.*

Proof. We may assume $\varphi(u, S, A) = \varphi(v, T, B)$. Write $\varphi(u, S, A) = vb$. We have $\alpha(v, T, B) = v$ by definition. On the other hand, in view of Claim 7.3, $\alpha(u, S, A) = b$ by the choice of $\alpha(u, S, A)$ described in Section 5. Thus $\alpha(u, S, A) \neq \alpha(v, T, B)$. □

Claim 7.5. *Let $(u, S, A) \in \mathscr{K}_3 \cup \mathscr{K}_4$ and $(v, T, B) \in \mathscr{K}_1$. Then $\alpha(u, S, A) \neq \alpha(v, T, B)$.*

Proof. By Lemma 2.6, Claim 3.3 or Claim 3.6, $\alpha(u, S, A) \notin \tilde{V}$. On the other hand, $\alpha(v, T, B) = v \in \tilde{V}$. Thus $\alpha(u, S, A) \neq \alpha(v, T, B)$. □

Claim 7.6. *Let $(u, S, A) \in \mathscr{K}_3 \cup \mathscr{K}_4$ and $(v, T, B) \in \mathscr{K}_2$. Then $\varphi(u, S, A) \neq \varphi(v, T, B)$.*

Proof. Suppose that $\varphi(u, S, A) = \varphi(v, T, B)$. Write $\lambda(u, S, A) = ab$ with $\alpha(u, S, A) = a$. Then $\deg(a) = 4$. Also write $\varphi(u, S, A) = \varphi(v, T, B) = ax$. Then $v \in N(a) \cap N(x)$. First assume that there exists $(w, R, Z) \in \mathscr{K}_3$ with $(w, R, Z) \neq (u, S, A)$ such that $\lambda(w, R, Z) = \lambda(u, S, A)$. Then $\deg(b) = 4$. By Claim 5.5, $w \neq u$. Thus $N(a) = \{u, b, w, x\}$. Since $\deg(v) \geqslant 5$ and $\deg(b) = 4$, $v \neq b$. Since $v \in N(a) \cap N(x) \subseteq N(a) - \{x\}$, this implies $v = u$ or w. On the other hand, $\deg(a) = 4$ and a is a common endvertex of $\varphi(v, T, B)$ and $\lambda(u, S, A) = \lambda(w, R, Z)$. Since $\varphi(v, T, B) = \lambda(v, T, B)$, this contradicts Claim 6.1. Next assume that there is no such (w, R, Z). Write $N(a) = \{u, b, x, y\}$. If $(u, S, A) \in \mathscr{K}_3$, then $xy \notin E(G)$ by the choice of $\alpha(u, S, A)$; if $(u, S, A) \in \mathscr{K}_4$, then $xy \notin E(G)$ by Claim 3.6. Thus $xy \notin E(G)$, which implies $v \neq y$. Now if $(u, S, A) \in \mathscr{K}_3$, then $\deg(b) = 4$; if $(u, S, A) \in \mathscr{K}_4$, then $xb \notin E(G)$ by Claim 3.6. In either case, $v \neq b$. Consequently, $v = u$, which again contradicts Claim 6.1. □

Claim 7.7. *Let $(u, S, A), (v, T, B) \in \mathscr{K}_3$ with $(u, S, A) \neq (v, T, B)$. Then $(\varphi(u, S, A), \alpha(u, S, A)) \neq (\varphi(v, T, B), \alpha(v, T, B))$.*

Proof. Suppose that $(\varphi(u, S, A), \alpha(u, S, A)) = (\varphi(v, T, B), \alpha(v, T, B))$. Write $\lambda(u, S, A) = ab$, $\varphi(u, S, A) = \varphi(v, T, B) = ax$, and $N(a) = \{u, b, x, y\}$. Then $\alpha(u, S, A) = \alpha(v, T, B) = a$, and $v \in N(a) - \{x\}$. Since $\deg(a) = 4$ and a is a common endvertex of $\lambda(u, S, A)$ and $\lambda(v, T, B)$, $v \neq u$ by Claim 6.1. Since $\deg(b) = 4$, $v \neq b$. Thus $v = y$, and hence $\lambda(v, T, B) = au$ or ab. On the other hand, since $\deg(u) \geqslant 5$, $\lambda(v, T, B) \neq au$. Consequently $\lambda(v, T, B) = ab$, which contradicts Claim 5.7.

We are now in a position to complete the proof of Theorem 2.

Let $(u, S, A), (v, T, B) \in \mathscr{K}_0$ with $(u, S, A) \neq (v, T, B)$. We aim at showing that $(\varphi(u, S, A), \alpha(u, S, A)) \neq (\varphi(v, T, B), \alpha(v, T, B))$. By Claims 7.1, 7.2 and 7.4 through 7.6, we may assume $(u, S, A), (v, T, B) \in \mathscr{K}_3 \cup \mathscr{K}_4$. In view of Claim 7.7, we may also assume $(u, S, A) \in \mathscr{K}_4$. Suppose that $(\varphi(u, S, A), \alpha(u, S, A)) = (\varphi(v, T, B), \alpha(v, T, B))$. Let $(P, X) = (P_{u,S,A}, X_{u,S,A})$ and let a, b, x, y be as in Claims 3.5 and 3.6. Also let $(Q, Y) = (P_{v,T,B}, X_{v,T,B})$. Note that $N(a) = \{u, b, x, y\}$, and $v \in N(a) - \{x\}$. If $v = y$, then $y, a \in Q$, and hence $ya \in E_n(G)$, which contradicts Claim 3.6. Thus $v \neq y$. We also have $v \neq u$ by Claim 6.1. Consequently $v = b$, which implies $\lambda(b, T, B) = au$ or ay. Now suppose that $(b, T, B) \in \mathscr{K}_3$. Then both endvertices of $\lambda(b, T, B)$ have degree 4. Hence $\lambda(b, T, B) = ay$. But then $ay \in E_n(G)$ by the definition of $\mathscr{K}_3$, which contradicts Claim 3.6. Thus $(b, T, B) \in \mathscr{K}_4$. Applying Claim 3.6 to (Q, Y), we now obtain $b, a \in Q$, $x \in \bar{Y}$ and $y, u \in Y$, regardless of whether $\lambda(b, T, B) = au$ or ay. In particular, $xu \notin E(G)$. Set $U = (P \cap Q) \cup (P \cap Y) \cup (X \cap Q)$. Since $y \in X \cap Y$ and $x \in \bar{X} \cap \bar{Y}$, it follows from Lemma 2.3 that $(U, X \cap Y) \in \mathscr{L}$. Since $u \in P \cap Y \subseteq U$, it follows from the minimality of X that $(U, X \cap Y) \notin \mathscr{L}_0$, i.e., U is a trivial 4-cutset. Hence there exists $c \in V_4$ such that $N(c) = U$. Since $a, b, u \in U$, $c \in N(a) - \{b, u\} = \{x, y\}$. On the other hand, since $xu \notin E(G)$, $c \neq x$. Consequently $c = y$, which implies $y \in N(u) \cap X \cap V_4$. But since $(u, S, A) \in \mathscr{K}_4$, this contradicts Claim 3.4. Thus $(\varphi(u, S, A), \alpha(u, S, A)) \neq (\varphi(v, T, B), \alpha(v, T, B))$, as desired. This completes the proof of Theorem 2. □

8. Number of 4-contractible edges

In this section, we prove Corollary 3.

Let G be a 4-regular 4-connected graph. Let $\mathscr{K}_0$ be as in Section 4. For $u \in \tilde{V}$, let $c(u)$ denote the number of those members (v, T, B) of $\mathscr{K}_0$ for which $v = u$. By Claim 4.4, $c(u) \geqslant \deg_{\tilde{G}}(u)$ for each $u \in \tilde{V}$. Since $|E_c(G)| \geqslant (\sum_{u \in \tilde{V}} c(u))/2$ by Theorem 2, this implies $|E_c(G)| \geqslant (\sum_{u \in \tilde{V}} \deg_{\tilde{G}}(u))/2 = |\tilde{E}(G) \cap E_n(G)|$. Since we clearly have $|E_c(G)| \geqslant |\tilde{E}(G) - E_n(G)|$, we obtain $2|E_c(G)| \geqslant |\tilde{E}(G) \cap E_n(G)| + |\tilde{E}(G) - E_n(G)| = |\tilde{E}(G)|$, as desired. □

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