# Shortest Coverings of Graphs with Cycles 

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#### Abstract

It is shown that the edges of a bridgeless graph $G$ can be covered with cycles such that the sum of the lengths of the cycles is at most $|E(G)|+$ $\min \left\{\frac{2}{3}|E(G)|, \frac{7}{3}(|V(G)|-1)\right\}$.


## 1. Introduction

### 1.1. Definitions

All graphs considered are finite, and may contain loops and multiple edges. Let $G$ be a graph. For $S \subseteq V(G)$, we denote by $\omega(S)$ the set of edges of $G$ with exactly one end in $S$. A $k$-cut of $G$ is a set of the form $\omega(S)$ ( $S \subseteq V(G)$ ) with $|\omega(S)|=k$. A bridge is a 1-cut. A cycle in a graph is a connected, 2 -regular subgraph. The length of a cycle is the number of edges it contains. A digon is a cycle of length two. Given the graph $G$, a cycle cover of $G$ is a set $\mathscr{C}$ of cycles of $G$ such that each edge of $G$ belongs to at least one cycle of $\mathscr{C}$. The length of $\mathscr{C}$ is the sum of the lengths of the cycles in $\mathscr{C}$ and is denoted by $l(\mathscr{C})$. It is clear that a graph admits a cycle cover if and only if it contains no bridges. Other definitions for graphs can be found in $[2,3]$.

### 1.2. The Main Results

Itai and Rodeh [12] have shown that every connected bridgeless graph $G$ has a cycle cover of length at most $|E(G)|+2|V(G)| \log |V(G)|$. This upper bound was improved to $\min \{3|E(G)|-6,|E(G)|+6|V(G)|-7\}$ by Itai, Lipton, Papadimitriou, and Rodeh in [11]. The main result of this paper is

Theorem 1. Let $G$ be a bridgeless graph. Then $G$ has a cycle cover $\mathscr{C}$ such that $l(\mathscr{C}) \leqslant \frac{5}{3}|E(G)|$.

Although Theorem 1 appears to be stronger than the previous results only if $G$ has relatively few edges, we shall use Theorem 1 to improve these results for all graphs.

Tireorem 2. Every bridgeless graph $G$ has a cycle cover of length at most $|E(G)|+\frac{7}{3}(|V(G)|-1)$.

### 1.3. Relationship with the Chinese Postman Problem

Itai and Rodeh point out in [12] that one may obtain a lower bound for the length of a shortest cycle cover by considering the Chinese postman problem. That is, given a connected graph $G$, find a closed walk which traverses each edge of $G$ at least once and is as short as possible. An algorithm for finding such a "postman tour" appears in [7]. The problem is equivalent to constructing a graph $H_{0}$ such that:
(i) $H_{0}$ is obtained by replacing each edge of $G$ by one or more parallel edges,
(ii) $H_{0}$ is Eulerian, and
(iii) $\left|E\left(H_{0}\right)\right|$ is as small as possible.

A cycle cover $\mathscr{C}$ for a connected bridgeless graph $G$ easily gives rise to a graph $H$ satisfying (i) and (ii), and such that $l(\mathscr{C})=|E(H)|$, by replacing each edge $e$ of $G$ by a number of parallel edges equal to the number of cycles of $\mathscr{C}$ which contain $e$. Thus $l(\mathscr{C}) \geqslant\left|E\left(H_{0}\right)\right|$. Indeed, it would seem at first sight that the two problems were equivalent, since a cycle decomposition of $H_{0}$ should give rise to a cycle cover of $G$. This is not necessarily the case, however, since it is possible that every cycle decomposition of $H_{0}$ contains digons which correspond to single edges in $G$. (We shall henceforth refer to such digons of $H_{0}$ as forbidden digons.) For example, if $G$ is a cubic graph, then every $H_{0}$ satisfying (i)-(iii) is obtained by replacing each edge of some 1 -factor of $G$ by 2 parallel edges. Thus if $\mathscr{C}$ is a cycle cover of $G$, then $l(\mathscr{C}) \geqslant\left|E\left(H_{0}\right)\right|=\frac{4}{3}|E(G)|$. If $G$ is the Petersen graph, however, then a shortest cycle cover of $G$ has length 21 (see [12]), and $\frac{4}{3}|E(G)|=20$. On the
other hand one can prove that for planar graphs such a situation cannot occur.

Proposition 1. For every connected bridgeless planar graph $G, a$ shortest cycle cover has length equal to the length of a shortest postman tour.

To prove this (see also [10]) we need some further definitions. Let $v$ be a vertex of a loopless Eulerian graph $H$. A transition at $v$ is a pair of edges incident to $v$. A set of transitions for $v$ is a partition $T(v)$ of $\omega(\{v\})$ into transitions. If $T(v)$ is defined for every vertex $v$ of $H$ of degree greater than 2, the resulting family $\mathscr{E}$ of transitions is a transition system for $H$. The system is non-separating if the graph obtained from $H$ by deleting any one transition of $\mathscr{E}$ is connected. Note that if $H$ has no cut-vertices, every transition system for $H$ is non-separating.

A cycle decomposition $\mathscr{C}$ of $H$ is compatible with $\mathscr{E}$ if no cycle of $\mathscr{C}$ contains a transition of $\mathscr{F}$. It is clear that given $\mathscr{F}$, a necessary condition for the existence of a cycle decomposition which is compatible with $\mathscr{E}$ is that $\mathscr{E}$ be non-separating. Fleischner has shown

Theorem 3 [9]. Let $H$ be a planar loopless Eulerian graph and $\mathscr{E}$ be a non-separating system of transitions for $H$. Then $H$ has a cycle decomposition which is compatible with $\mathscr{G}$.

Proof of Proposition 1. It is easy to see that we may assume that $G$ has no cut-vertices. Let $H_{0}$ be a graph satisfying (i)-(iii). It follows from (iii) that $H_{0}$ is obtained by replacing each edge of $G$ by at most 2 parallel edges. Whenever $e_{1}$ and $e_{2}$ are two parallel edges of $H_{0}$ which correspond to a single edge of $G$, let $\left\{e_{1}, e_{2}\right\}$ be a transition at $v$ for each vertex $v$ incident with both $e_{1}$ and $e_{2}$. This family of transitions can be extended to a system of transitions $\mathscr{E}^{-}$for $H_{0}$. Since $H_{0}$ has no cut-vertices, $\mathscr{C}$ is non-separating. By Fleischner's theorem, $H_{0}$ has a cycle decomposition $\mathscr{C}$ which is compatible with $\mathscr{E}$, and hence does not contain any forbidden digons. Thus $\mathscr{C}$ gives rise to a cycle cover of $G$ of length $\left|E\left(H_{0}\right)\right|$.

## 2. $\mathbb{Z}_{2}$-Flows and $\mathbb{Z}_{2}$-Cycles

### 2.1. Definition

Let $k \geqslant 1$ and consider the additive group $\left(\mathbb{Z}_{2}\right)^{k}$. A $\left(\mathbb{Z}_{2}\right)^{k}$-flow of the graph $G$ is a mapping $\phi$ from $E(G)$ to $\left(\mathbb{Z}_{2}\right)^{k}$, such that: $\forall v \in V(G)$, $\sum_{e \in \omega(1 v)} \phi(e)=0$. (The summation and zero symbols refer to the structure of the group $\left(\mathbb{Z}_{2}\right)^{k}$.)

### 2.2. Elementary properties

(1) If $\phi$ is a $\left(\mathbb{Z}_{2}\right)^{k}$-flow of $G$, for any $S \subseteq V(G)$ we have $\sum_{e \in \omega(S)} \phi(e)=0$.
(2) The support of the $\left(\mathbb{Z}_{2}\right)^{k}$-flow $\phi$, denoted by $\sigma(\phi)$, is the set of edges $e \in E(G)$ such that $\phi(e) \neq 0$.
Then for $F \subseteq E(G)$ the following properties are equivalent:
(i) $F$ is the support of some $\mathbb{Z}_{2}$-flow of $G$.
(ii) Each vertex of $G$ is incident to an even number of edges of $F$.
(iii) $F$ can be partitioned into cycles of $G$.

A subset $F$ of $E(G)$ satisfying (i)-(iii) will be called a $\mathbb{Z}_{2}$-cycle of $G$.
(3) It easily follows from (1) (or (2)) that if $e$ is a bridge of $G$, $\phi(e)=0$ for any $\left(\mathbb{Z}_{2}\right)^{k}$-flow $\phi$. A $\left(\mathbb{Z}_{2}\right)^{k}$-flow $\phi$ is said to be nowhere-zero if $\sigma(\phi)=E(G)$. Thus if a graph has a nowhere-zero $\left(\mathbb{Z}_{2}\right)^{k}$-flow, it has no bridges.
(4) Let $\phi$ be a $\left(\mathbb{Z}_{2}\right)^{k}$-flow. Let $\phi_{1}, \ldots, \phi_{k}$ be $\mathbb{Z}_{2}$-flows such that $\forall e \in E(G): \phi(e)=\left(\phi_{1}(e), \ldots, \phi_{k}(e)\right)$. We shall write $\phi=\left(\phi_{1}, \ldots, \phi_{k}\right)$. Then $\phi$ is nowhere-zero if and only if $\bigcup_{i=1}^{k} \sigma\left(\phi_{i}\right)=E(G)$. In this case, for every $i \in\{1, \ldots, k\}$, there exists a partition $P_{i}=\left\{C_{i}^{1}, \ldots, C_{i}^{r_{i}}\right\}$ of $\sigma\left(\phi_{i}\right)$ into cycles of $G$, and $\bigcup_{i=1}^{k} P_{i}$ is a cycle cover of $G$ of length $\sum_{i=1}^{k}\left|\sigma\left(\phi_{i}\right)\right|$. This number will be denoted by $l(\phi)$. Conversely, if $\mathscr{C}=\left\{C_{1}, \ldots, C_{k}\right\}$ is a cycle cover of $G$, let $\phi_{i}(i=1, \ldots, k)$ be the unique $\mathbb{Z}_{2}$-flow such that $\sigma\left(\phi_{i}\right)=C_{i}$. Then $\phi=\left(\phi_{1}, \ldots, \phi_{k}\right)$ is a nowhere-zero $\left(\mathbb{Z}_{2}\right)^{k}$-flow and

$$
l(\mathscr{C})=\sum_{i=1}^{k}\left|\sigma\left(\phi_{i}\right)\right|=l(\phi) .
$$

We conclude that the minimum of $l(\mathscr{C})$ over the set of cycle covers $\mathscr{C}$ of $G$ is equal to the minimum of $l(\phi)$ over the set of nowhere-sero $\left(\mathbb{Z}_{2}\right)^{k}$-flows $\phi$ of $G(k \geqslant 1)$.

Remark. If $G$ is bridgeless, $G$ has a cycle cover and hence $G$ has a newhere-zero $\left(\mathbb{Z}_{2}\right)^{k}$-flow for some $k \geqslant 1$ (see the above discussion).
(5) Let $z=\left(z_{1}, \ldots, z_{k}\right) \in\left(\mathbb{Z}_{2}\right)^{k}$.

The (Hamming) weight $w(z)$ of $z$ is the number of nonzero components of $z$, that is, $w(z)=\left|\left\{i \in\{1, \ldots, k\}: z_{i}=1\right\}\right|$. Let $\phi$ be a nowhere-zero $\left(\mathbb{Z}_{2}\right)^{k}$-flow of G. A straightforward counting argument yields $l(\phi)=\sum_{e \in E(G)} w(\phi(e))$.

### 2.3. The Double-Cover Conjecture

A cycle double-cover of $G$ is a cycle cover $\mathscr{C}$ of $G$ such that each edge appears in exactly two cycles of $\mathscr{C}$. The double-cover conjecture asserts that
every bridgeless graph has a cycle double cover [17, Conjecture 3.3]. We shall denote by $D_{k}(k \geqslant 2)$ the subset of $\left(\mathbb{Z}_{2}\right)^{k}$ consisting of the elements of wcight 2. It is easy to show (see the above discussion in $2.2(4)$ ) that $G$ has a cycle double-cover iff it has a $\left(\mathbb{Z}_{2}\right)^{k}$-flow with all edge-values in $D_{k}$ for some $k \geqslant 2$ (such a flow will be called a $D_{k}$ flow).

Remark. If a graph has a $D_{k}$ flow it has a $D_{k^{\prime}}$-flow for every $k^{\prime} \geqslant k$. Hence the double-cover conjecture can be formulated as follows:

For every bridgeless graph $G$, there exists a $k \geqslant 2$ such that $G$ has a $D_{k}$-flow.
(DCC)

This conjecture is clearly related to the shortest cycle cover problem. In fact Itai and Rodeh rediscover an equivalent form of the (DCC) in [12, Problem (ii)].
We have the following result:
Proposition 2. If $G$ has a $D_{k}$ flow $(k \geqslant 2$ ), it has a cycle cover $\mathscr{C}$ with $l(\mathscr{C}) \leqslant(2(k-1) / k)|E(G)|$
Pronf. Let $\phi=\left(\phi_{1}, \ldots, \phi_{k}\right)$ be a $D_{k}$-flow of $G$. Clearly $l(\phi)=$ $\sum_{e \in E(G)} w(\phi(e))=2|E(G)|$.
On the other hand, $l(\phi)=\sum_{i=1}^{k}\left|\sigma\left(\phi_{i}\right)\right|$. We may assume without loss of generality that $\forall i \in\{1, \ldots, k-1\},\left|\sigma\left(\phi_{k}\right)\right| \geqslant\left|\sigma\left(\phi_{i}\right)\right|$. Consider now the $\left(\mathbb{Z}_{2}\right)^{k-1}$ flow $\phi^{\prime}=\left(\phi_{1}, \ldots, \phi_{k-1}\right)$. It is clearly nowhere-zero. Moreover

$$
l\left(\phi^{\prime}\right)=l(\phi)-\left|\sigma\left(\phi_{k}\right)\right| \leqslant \frac{k-1}{k} l(\phi)=\frac{2(k-1)}{k}|E(G)| .
$$

This completes the proof.

### 2.4. Some Consequences of Proposition 2

(1) For $k=2$ the situation is quite simple. If $G$ has a $D_{2}$-flow, it has a cycle cover $\mathscr{C}$ with $l(\mathscr{C})=|E(G)|$, i.e., $E(G)$ can be partitioned into cycles, and conversely.
(2) For $k=3$, we obtain that if $G$ has a $D_{3}$-flow it has a cycle cover $\mathscr{C}$ with $l(\mathscr{C}) \leqslant \frac{4}{3}|E(G)|$.
We may now use the following easy result:
Proposition 3. A graph has a $D_{3}$ flow iff it has a nowhere-zero $\left(\mathbb{Z}_{2}\right)^{2}$ flow.

Proof. If $\phi=\left(\phi_{1}, \phi_{2}, \phi_{3}\right)$ is a $D_{3}$ flow, then $\phi^{\prime}=\left(\phi_{1}, \phi_{2}\right)$ is a nowhere-
zero $\left(\mathbb{Z}_{2}\right)^{2}$-flow. Conversely, if $\phi^{\prime}=\left(\phi_{1}, \phi_{2}\right)$ is a nowhere-zero $\left(\mathbb{Z}_{2}\right)^{2}$-flow, $\phi=\left(\phi_{1}, \phi_{2}, \phi_{1}+\phi_{2}\right)$ is a $D_{3}$-flow.

Now, applying Propositions 2 and 3 together with some known results on the existence of nowhere-zero $\left(\mathbb{Z}_{2}\right)^{2}$-flows, we obtain

Corollary 1. Every bridgeless planar graph $G$ has a cycle cover $\mathscr{C}$ with $l(\mathscr{C}) \leqslant \frac{4}{3}|E(G)|$.

Proof. Use the four color theorem [1;13, Proposition 3].
Corollary 2. Let $G$ be a cubic 3-edge-colorable graph. The length of a shortest cycle cover of $G$ is equal to $\frac{4}{3}|E(G)|$.

Proof. As already seen in Subsection 1.3, the length of a shortest cycle cover of $G$ is at least $\frac{4}{3}|E(G)|$. The equality follows from Propositions 2,3 and [13, Proposition 2].

Corollary 3. Every bridgeless graph $G$ without 3-cuts has a cycle cover $\mathscr{C}$ with $l(\mathscr{C}) \leqslant \frac{4}{3}|E(G)|$.

Proof. Use [13, Proposition 10].
(3) For $k=4$, we shall use the following observation:

Proposition 4. If a graph has a $D_{4}$ flow, it has a $D_{3}$ flow.
Proof. Let $\phi=\left(\phi_{1}, \phi_{2}, \phi_{3}, \phi_{4}\right.$,) be a $D_{4}$-flow. Then it is easy to check that $\phi^{\prime}=\left(\phi_{1}+\phi_{2}, \phi_{1}+\phi_{3}, \phi_{1}+\phi_{4}\right)$ is a $D_{3}$-flow. Hence this case reduces to the previous one.
(4) For $k=5$, we have nothing but a conjecture which has been proposed by several authors $[4,16]$, and which is stronger than the doublecover conjecture.

Conjecture. Every bridgeless graph has a $D_{5}$-flow.
By Proposition 1, this implies the following:
Conjecture. Every bridgeless graph $G$ has a cycle cover with $l(\mathscr{C}) \leqslant \frac{8}{5}|E(G)|$.
(5) For $k=6$, the existence of a $D_{6}$-flow in $G$ implies that $G$ has a cycle cover $\mathscr{C}$ with $l(\mathscr{C}) \leqslant \frac{5}{3}|E(G)|$.
We shall show (by different methods) that this last property holds for every bridgeless graph $G$.

## 3. The Main Result

### 3.1. Introduction

The following result is proved in [13].

8-Flow Theorem. Every bridgeless graph has a nowhere-zero $\left(\mathbb{Z}_{2}\right)^{3}$ flow.

Using this result only, one can prove that every bridgeless graph $G$ has a cycle cover $\mathscr{C}$ with $l(\mathscr{C}) \leqslant \frac{12}{7}|E(G)|$. We now present the proof of Theorem 1. It uses a refinement of an alternative proof for the 8 -flow Theorem (the idea is indicated in [13, Sect. V].
3.2. Lemma. Every bridgeless graph $G$ has $a \mathbb{Z}_{2}$-cycle $C$ such that $|C| \geqslant \frac{2}{3}|E(G)|$ and $C$ intersects every 3-cut of $G$.

Proof. It is clear that to prove the lemma, it is enough to prove it for loopless 2-edge-connected graphs. Let $G$ be such a graph, and let $v \in V(G)$. $A$ splitting of $G$ at $v$ is the graph $G^{\prime}$ obtained by replacing $v$ by two distinct vertices $v^{\prime}$ and $v^{\prime \prime}$, each edge of $G$ with end vertices $v, x(x \in V(G)-\{v\})$ being replaced by an edge with end vertices $v^{\prime}, x$ or $v^{\prime \prime}, x$ in such a way that $v^{\prime}$ has degree 2 in $G^{\prime}$. A splitting of $G$ is any graph obtained from $G$ by a succession of vertex-splittings. It follows from a result of Fleischner [8] (see also Mader [14]) that $G$ has a 2-edge-connected splitting $G^{\prime}$ which has no vertices of degree greater than three. Note that identifying edges of $G^{\prime}$ with edges of $G$ in the obvious way, every $\mathbb{Z}_{2}$-cycle of $G^{\prime}$ is a $\mathbb{Z}_{2}$-cycle of $G$, and every 3 -cut of $G$ is a 3 -cut of $G^{\prime}$. We conclude that:
(1) To prove the lemma it is enough to prove it for loopless 2-edgeconnected graphs with no vertex of degree greater than three.

Let $G$ be such a graph. If $G$ has no vertices of degree 3 , the result is clear. Otherwise there exists a cubic 2 -edge-connected graph $H$ such that $G$ can be obtained from $H$ by replacing each edge $e$ of $H$ by a simple path $P_{e}$ of length $f(e) \geqslant 1$. For $F \subseteq E(H)$ we shall denote by $f(F)$ the sum $\sum_{e \in F} f(e)$. It follows from a result of Edmonds [6] that there exists an integer $k \geqslant 1$ and a family ( $M_{1}, \ldots, M_{3 k}$ ) of $3 k$ perfect matchings of $H$ (not necessarily distinct) such that every edge of $H$ appears in exactly $k$ of the $M_{i}$ 's.

Let $K$ be a 3-cut of $H$. For every perfect matching $M$ of $H, E(H)-M$ is a 2-factor of $H$ and hence a $\mathbb{Z}_{2}$-cycle. Hence $|K \cap(E(H)-M)|$ is even, so that $|K \cap M|$ equals 1 or 3 . Now each one of the 3 edges of $K$ appears in exactly $k$ of the $M_{i}$ 's $(i=1, \ldots, 3 k)$, so that $\sum_{i=1}^{3 k}\left|K \cap M_{i}\right|=3 k$. It follows that $\forall i \in\{1, \ldots, 3 k\},\left|K \cap M_{i}\right|=1$.

Finally we note that $\sum_{i=1}^{3 k} f\left(M_{i}\right)=k f(E(H))$. Hence there exists
$i \in\{1, \ldots, 3 k\}$ with $f\left(M_{i}\right) \leqslant \frac{1}{3} f(E(H))$. Then $F=E(H)-M_{i}$ is a $\mathbb{Z}_{2}$-cycle of $H$ which intersects every 3 -cut of $H$ and such that $f(F) \geqslant \frac{2}{3} f(E(H))$. Let $C$ be the subset of edges of $G$ equal to $\bigcup_{e \in F} P_{e}$. Clearly $C$ is a $\mathbb{Z}_{2}$-cycle of $G$ and $|C|=f(F) \geqslant \frac{2}{3} f(E(H))=\frac{2}{3}|E(G)|$. Moreover, no 3-cut of $G$ contains two edges of a single path $P_{e}, e \in E(H)$ (the remaining edge of the 3-cut would be a bridge). Hence every 3 cut of $G$ is obtained by considering some 3-cut $\left\{e_{1}, e_{2}, e_{3}\right\}$ of $H$ and choosing exactly one edge from each of $P_{e_{1}}, P_{e_{2}}, P_{e_{3}}$. It follows that $C$ intersects every 3 -cut of $G$. This completes the proof.

### 3.3. A Consequence of the Lemma

Proposition 5. Every connected bridgeless graph $G$ has a postman tour of length at most $\frac{4}{3}|E(G)|$.

Proof. Let $C$ be a $\mathbb{Z}_{2}$-cycle of $G$ with $|C| \geqslant \frac{2}{3}|E(G)|$. Replace every edge of $E(G)-C$ by two parallel edges. This yields an Eulerian graph $H$ with $|E(H)| \leqslant \frac{4}{3}|E(G)|$.

Remark. Propositions 5 and 1 together give another proof of Corollary 1 which does not rely on the four color theorem.

### 3.4. Proof of Theorem 1

Let $G$ be a bridgeless graph with $|E(G)|=m$. By the lemma, there exists a $\mathbb{Z}_{2}$-flow $\phi_{1}$ of $G$ such that $\left|\sigma\left(\phi_{1}\right)\right| \geqslant 2 m / 3$ and $\sigma\left(\phi_{1}\right)$ intersects every 3 -cut of $G$. For each edge $e$ of $\sigma\left(\phi_{1}\right)$, add to $G$ an edge $e^{\prime}$ parallel to $e$ (i.e., with the same pair of ends). We obtain a new bridgeless graph $G^{\prime}$ which contains $G$ as a subgraph. Moreover it is clear that $G^{\prime}$ has no 3-cuts. By Proposition 10 of [13], $G^{\prime}$ has a nowhere-zero $\left(\mathbb{Z}_{2}\right)^{2}$-flow $\phi^{\prime}=\left(\phi_{2}^{\prime}, \phi_{3}^{\prime}\right)$.

For $e \in E(G)$ and $i \in\{2,3\}$ let $\phi_{i}(e)=\phi_{i}^{\prime}(e)$ if $e \notin \sigma\left(\phi_{1}\right)$ and $\phi_{i}(e)=\phi_{i}^{\prime}(e)+$ $\phi_{i}^{\prime}\left(e^{\prime}\right)$ if $e \in \sigma\left(\phi_{1}\right)$. This defines two $\mathbb{Z}_{2}$-flows $\phi_{2}, \phi_{3}$ of $G$. Since $\phi^{\prime}=\left(\phi_{2}^{\prime}, \phi_{3}^{\prime}\right)$ is nowhere-zero, the $\left(\mathbb{Z}_{2}\right)^{2}$-flow $\left(\phi_{2}, \phi_{3}\right)$ of $G$ takes nonzero values on $E(G)-\sigma\left(\phi_{1}\right)$. It follows that $\left(\phi_{1}, \phi_{2}, \phi_{3}\right)$ is a nowhere-zero $\left(\mathbb{Z}_{2}\right)^{3}$-flow of $G$.

Consider the vector space $[G F(2)]^{3}$ (over $G F(2)$ ) of the 3-tuples $x=\left(\alpha_{1}, \alpha_{2}, \alpha_{3}\right)\left(\alpha_{i} \in G F(2), i=1,2,3\right)$. To every element $x=\left(\alpha_{1}, \alpha_{2}, \alpha_{3}\right)$ of this space we associate the flow $\phi_{x}=\Sigma_{\alpha_{i}=1} \phi_{i}$. In particular,

$$
\phi_{(1,0,0)}=\phi_{1}, \quad \phi_{(0,1,0)}=\phi_{2}, \quad \text { and } \quad \phi_{(0, \mathbf{0}, 1)}=\phi_{3} .
$$

It is easy to show that for every basis $\left\{x_{1}, x_{2}, x_{3}\right\}$ of $[G F(2)]^{3},\left(\phi_{x_{1}}, \phi_{x_{2}}, \phi_{x_{3}}\right)$ is a nowhere-zero $\left(\mathbb{Z}_{2}\right)^{3}$-flow of $G$. Denote by $X$ the set $[G F(2)]^{3}-\{(0,0,0)\}$ and by $X^{\prime}$ the set $X-\{(1,0,0)\}$. One can easily check that each edge appears in exactly 4 of the $\sigma\left(\phi_{x}\right)(x \in X)$, and hence $\Sigma_{x \in X}\left|\sigma\left(\phi_{x}\right)\right|=4 m$. Then $\sum_{x \in X^{\prime}} \sigma\left(\phi_{x}\right)\left|=4 m-\left|\sigma\left(\phi_{1}\right)\right| \leqslant 4 m-\frac{2}{3} m=\frac{10}{3} m\right.$. Let $\mathscr{P}$ be the set of bases of
$[G F(2)]^{3}$ which do not contain the vector $(1,0,0)$. Every vector of $X^{\prime}$ appears in exactly 8 elements of $\mathscr{B}$. Hence $\Sigma_{B \in \mathscr{B}}\left(\Sigma_{x \in B}\left|\sigma\left(\phi_{x}\right)\right|\right)=$ $8 \Sigma_{x \in X^{\prime}}\left|\sigma\left(\phi_{x}\right)\right| \leqslant 80 \mathrm{~m} / 3$. Since $|\mathscr{B}|=16$, there exists $B \in \mathscr{B}$ with

$$
\sum_{x \in B}\left|\sigma\left(\phi_{x}\right)\right| \leqslant \frac{1}{16} \frac{80 m}{3}=\frac{5 m}{3} .
$$

Then the supports of the $\mathbb{Z}_{2}$-flows $\phi_{x}$ for $x \in B$ will give a cycle cover $\mathscr{C}$ with $l(\mathscr{C}) \leqslant 5 \mathrm{~m} / 3$. This completes the proof.

### 3.5. 4-Covers

We observe that using the seven $\mathbb{Z}_{2}$-cycles $\sigma\left(\phi_{x}\right)\left(x \in[G F(2)]^{3}-\right.$ $\{(0,0,0)\})$ defined in the above proof it is possible to obtain a cycle cover $\mathscr{C}$ such that every edge appears in exactly 4 cycles of $\mathscr{C}$. Calling such a cycle cover a cycle 4 -cover, we have

Proposition 6. Every bridgeless graph has a cycle 4-cover.

## 4. Proof of Theorem 2

Let $G$ be a bridgeless graph. We may assume $G$ is connected. Let $H$ be a subset of $E(G)$ such that the graph $(V(G), H)$ is 2-edge-connected and minimal with this property. It is easy to show, using $[5,15]$, that $|H| \leqslant$ $2|V(G)|-2$. By Theorem $1,(V(G), H)$ has a cycle cover $\mathscr{C}_{1}$ with $l\left(\mathscr{C}_{1}\right) \leqslant \frac{5}{3}$ $|H|$. Let $F=E(G)-H$, and consider a spanning tree $T$ contained in $H$. For every $e$ in $F$, there is a unique $\mathbb{Z}_{2}$-flow $\phi_{e}$ such that $e \in \sigma\left(\phi_{e}\right) \subseteq T \cup\{e\}$. Let $\phi=\Sigma_{e \in F} \phi_{e}$. Then clearly $F \subseteq \sigma(\phi) \subseteq T \cup F$. Let $\mathscr{C}_{2}$ be a cycle decomposition of $\sigma(\phi)$. Now $\mathscr{C}_{1} \cup \mathscr{C}_{2}$ is a cycle cover $\mathscr{C}$ of $G$, with

$$
l(\mathscr{C})=l\left(\mathscr{C}_{1}\right)+l\left(\mathscr{C}_{2}\right)=l\left(\mathscr{C}_{1}\right)+|\sigma(\phi)| \leqslant \frac{5}{3}|H|+|T \cup F| .
$$

Since $\quad|T \cup F|=|T|+|F|=|V(G)|-1+|E(G)|-|H| \quad$ we have $l(\mathscr{C}) \leqslant$ $|E(G)|+|V(G)|-1+\frac{2}{3}|H| \leqslant|E(G)|+\frac{7}{3}(|V(G)|-1)$. This completes the proof.

## 5. Vertex Cycle Covers

Given a graph $G$, a vertex cycle cover of $G$ is a set of cycles $\mathscr{C}$ of $G$ such that each vertex of $G$ belongs to at least one cycle of $\mathscr{C}$.

Proposition 7. Let $G$ be a graph such that each vertex of $G$ lies in a cycle. Then $G$ has a vertex cycle cover $\mathscr{C}$ such that $l(\mathscr{C}) \leqslant \frac{10}{3}(|V(G)|-1)$.

Proof. We may assume that $G$ is 2 -edge-connected. Let $H$ be a critically 2-edge-connected spanning subgraph of $G$, so that $|E(H)| \leqslant 2|V(G)|-2$. By Theorem $1, H$ has a cycle cover $\mathscr{B}$ such that $l(\mathscr{C}) \leqslant \frac{5}{3}|E(H)|$. Clearly $\mathscr{B}$ is a vertex cycle cover of $G$ and $l(\mathscr{C}) \leqslant \frac{10}{3}(|V(G)|-1)$.

## 6. Covering of the Vertices of a Strong Digraph with Circuits

In this section, circuit means "directed circuit." Let $f(2 p)=p^{2}+p$ and $f(2 p+1)=(p+1)^{2}$.

Proposition 8. For any strong digraph $D$ with $n$ vertices, there exists a vertex circuit cover $\mathscr{C}$ such that $l(\mathscr{B}) \leqslant f(n)$.

Proof. Let $k$ be the length of the longest circuit of $D$ and let $C_{0}$ be such a longest circuit. We can cover the vertices of $D$ with $C_{0}$ and for each vertex not in $C_{0}$ with a circuit of length at most $k$. Therefore we can cover with $(n-k+1)$ circuits of length at most $k$. This yields a vertex circuit cover $\mathscr{C}$ with $l(\mathscr{C}) \leqslant k(n-k+1)$. But it is known that $\max _{k} k(n-k+1)=f(n)$.

The result is best possible in the sense that there exists a strong digraph $D$ of order $n$ such that for any covering family $\mathscr{C}, l \mathscr{C}) \geqslant f(n)$. Consider the digraph $D$ consisting of a directed circuit of length $k=|n / 2|$ in which we have replaced one vertex by a stable set of ( $n-k+1$ ) vertices (see Fig. 1). Each vertex $y_{i}$ belongs to the unique circuit $C_{i}=\left(x_{1}, y_{i}, x_{2}, \ldots, x_{k-1}\right)$. Therefore to cover all the $y_{i}$ we need to use all the circuits $C_{i}$. But $\Sigma l\left(C_{i}\right)=$ $k(n-k+1)=\lceil n / 2\rceil(\lfloor n / 2\rfloor+1)=f(n)$.


Figure I.

## 7. Open Problems

7.1.

In view of Theorem 1, the main problem is to find the infimum $\rho$ of the set of numbers $r$ with the property that every bridgeless graph $G$ has a cycle cover $\mathscr{C}$ with $l(\mathscr{C}) \leqslant r|E(G)|$. All we know is that $\frac{7}{5} \leqslant \rho \leqslant \frac{5}{3}$. The lower bound $\frac{7}{5}$ is given by the Petersen graph (see subsection 1.3). In fact, by combining several Petersen graphs together as in Fig. 2, we obtain an infinite family of graphs $G$ whose shortest cycle cover $\mathscr{C}$ satisfies $l(\mathscr{C})=\frac{7}{5}|E(G)|$. We note further that both the Blanuša snarks on 18 vertices, the flower snark on 20 vertices, and both the Loupekhine snarks on 22 vertices, have cycle covers of length $\frac{4}{3}|E(G)|$.

## 7.2.

A problem related to Theorem 2 is proposed by Itai and Rodeh [12, Open Problem (i)]. Does every bridgcless graph $G$ have a cycle cover $\mathscr{C}$ with $l(\mathscr{C}) \leqslant|E(G)|+|V(G)|-1$ ? They prove this for graphs with two edge disjoint spanning trees. By Theorem 1, the result is true for graphs $G$ with $|E(G)| \leqslant \frac{3}{2}(|V(G)|-1)$ (e. g., subdivisions of cubic graphs containing at least three vertices of degree 2 ).

By Proposition 1 and using the obvious property that a shortest postman tour of $G$ has length at most $|E(G)|+|V(G)|-1$ it follows that the result is also true for planar graphs. On the other hand, it can be checked that if $G$ is the complete bipartite graph $K_{n, 3}$, the length of a shortest cycle cover is $|E(G)|+|V(G)|-3$.

## 7.3.

Finally we propose the following conjecture
Every 2 -connected graph $G$ has a vertex cycle cover of length at most $2|V(G)|-2$.

Note that this conjecture would be best possible because of the completc bipartite graph $K_{n, 2}$ ( $n$ odd).


Figure 2.

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