Basic characteristics and development of yield criteria for geomaterials

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\textbf{Abstract:} The yield criteria of geomaterials play a crucial role in studying and designing the strength of materials and structures. The basic characteristics of yield criteria for geomaterials need to be studied under the framework of continuum mechanics. These characteristics include the effects of strength difference (SD) of materials in tension and compression, normal stress, intermediate principal stress, intermediate principal shear stress, hydrostatic stress, twin-shear stresses, and the convexity of yield surface. Most of the proposed yield criteria possess only one or some of these basic characteristics. For example, the Tresca yield criterion considers only single-shear stress effect, and ignores the effect of SD, normal stress, intermediate principal stress, intermediate principal shear stress, hydrostatic stress, and twin-shear stresses. The Mohr-Coulomb yield criterion accounts for the effect of SD, normal stress, single-shear stress and hydrostatic stress, but disregards the effect of intermediate principal stress, intermediate principal shear stress, and twin-shear stresses. The basic characteristics remain to be fully addressed in the development of yield criterion. In this paper, we propose a new yield criterion with three features, that is, newly developed, better than existing criteria and ready for application. It is shown that the proposed criterion performs better than the existing ones and is ready for application. The development of mechanical models for various yield criteria and the applications of the unified strength theory to engineering are also summarized. According to a new tetragonal mechanical model, a tension-cut condition is added to the unified strength theory. The unified strength theory is extended to the tension-tension region.

\textbf{Key words:} yield criteria; failure criteria; unified strength theory; tension cut-off; orthogonal octahedral element; geomaterial; beauty of a strength theory

1 Introduction

Geomaterials are the engineering materials used in civil engineering. Although commonly used, geomaterials are very complex. It is very difficult to describe their behaviours, and even more complicated to develop constitutive models for such materials. Engineers frequently assume that geomaterials are homogeneous and isotropic media. However, most geomaterials are neither homogeneous nor isotropic. Despite these factors, some simple mechanical behaviours of geomaterials can still be observed in both uniaxial and polyaxial stress states. A triaxial test was carried out on crushed stone by Yangtze River Scientific Research Institute \cite{1}. The triaxial testing setup and a three-dimensional (3D) stress state of a sample of crushed rock are shown in Figs.1(a) and (b), respectively. The relations between $\sigma_1$ and $\sigma_3$, $\sigma_1 - \sigma_3$, and $\sigma_1 + \sigma_3$, and $\sigma_1 - \sigma_3$ and $\sigma_3$ are shown in Figs.2, 3, and 4, respectively. It is shown that although the crushed rock particles are randomly distributed, the simple linear response of the sample can still be observed.

![Triaxial test apparatus and a 3D stress state of crushed stone.](image)

Fig.1 Triaxial test apparatus and a 3D stress state of crushed stone.

![Relation between $\sigma_1$ and $\sigma_3$.](image)

Fig.2 Relation between $\sigma_1$ and $\sigma_3$.  

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A three-dimensional, atom-based model with the plastic yield criterion of metallic glass is developed by Schuh et al. [2, 3]. The simulation results are given in Fig.5, showing that the material strength clearly varies with different stresses (Fig.5(b)). It is also noted that the strength locus lies between the two bounds of convex region.

The relations between shear stress $\tau$ and normal stress $\sigma$ of various fractured rocks are shown in Fig.6 [4], in which $d_r$ is connectivity rate of discontinuity.

The relations between shear stress and normal stress for glass balls, Toyoura sand-soil and crushed sand are shown in Fig.7 [5, 6]. Although the strengths of different materials are different, the linear relation is observed. Similar results were obtained by other researchers.

Strength theories deal with the yield or failure of materials in the frame of continuum mechanics. Hundreds of yield criteria have been proposed or studied in the past [7–9]. However, the study of the basic characteristics of yield criteria for geomaterials under complex stress is still necessary.

## 2 Basic characteristics of failure criteria for geomaterials under complex stress

According to experimental data [1–6, 10–28], the basic characteristics of failure criteria for geomaterials are summarized as follows.

### 2.1 Effect of SD

The strength of geomaterials is greater under compression than that under tension. The compressive strength $\sigma_c$ is greater than tensile strength $\sigma_t$. The difference between the compressive and the tensile strengths is called the
effect of SD. Therefore, the single-parameter failure criteria such as the Tresca criterion and Huber-von Mises criterion with $\alpha = \sigma_1 / \sigma_2 = 1$ are not suitable for geomaterials. The two-parameter failure criteria have to be used for geomaterials to include the effect of SD.

2.2 Effect of normal stress

It is worthy to note that the strength of a geomaterial usually depends on shear stresses. Hence many efforts are devoted to the research on the shear strengths of geomaterials and the relation between shear and normal stresses. The values of shear stress $\tau$ plotted against normal stress $\sigma$ obtained by Jaeger and Cook [13] for marble (A) and Trachyte (B) are shown in Fig.8, where $\mu$ is the coefficients of friction [13]. Relation between shear stress and normal stress of geomaterial is shown in Fig.9 [5].

![Fig.8 Relation between shear strength and normal stress obtained by Jaeger and Cook [13].](image)

![Fig.9 Relation between shear stress and normal stress for a rock-soil [5].](image)

Byerlee (1978) collected a large number of experimental results of this type and divided them into three situations: low pressure, corresponding to the stress encountered in most civil engineering applications; regular pressure, with in-situ stress magnitude about 100 MPa at 300 m in depth in underground engineering; and high pressure, for high stress situations at greater depth in earthcrust in geophysical applications. He plotted the normal stress-shear stress behaviours for these three situations, as shown in Fig.10 [14].

![Fig.10 Relations between shear stress and normal stress [14].](image)

The general behaviour between shear strength $\tau$ and normal stress $\sigma$ was considered as linear and was expressed as [5, 6, 13]

$$\tau = c + \sigma \tan \varphi$$

(1)

The effects of normal stress on shear strength can be extended to effects on other shear stresses. According to the experimental data of Tang [26], the relation between twin-shear stresses and normal stresses at the shear sections $\tau_{13} + \tau_{12} = f(\sigma_{13} + \sigma_{12})$ can be obtained, as shown in Fig.11, where $\tau_{tw} = \tau_{13} + \tau_{12}$, $\sigma_{tw} = \sigma_{13} + \sigma_{12}$.
Fig. 11 Relation between twin-shear strength and relevant normal stresses [26].

2.3 Effect of hydrostatic stresses

Hydrostatic stresses, or mean stress $\sigma_m = (\sigma_1 + \sigma_2 + \sigma_3)/3$, has a great influence on the strength of geomaterials. Many studies have been devoted to the effect of hydrostatic stress [5–14]. The relation between the limit stress circle and the confining pressure is shown in Fig. 12 [5, 6]. The effect of the hydrostatic stress is similar to that of normal stress.

The effect of the hydrostatic stress can be extended to twin-shear stresses behaviour by the experimental results shown in Fig. 13 [29, 30], where $T_{tw} = \tau_{13} + \tau_{12}$ (see Fig. 13(a)) and $T_{tw} = \tau_{13} + \tau_{12}$ ($\sigma_2 \geq \sigma_1/1 + \alpha$, see Fig. 13(b)).

2.4 Effect of the intermediate principal stress

The effect of the intermediate principal stress in rock is a very important issue in both theory and practice.

The results of experiments obtained by Mogi [23] at Tokyo University are shown in Fig. 14. It can be seen that the strength of rock increases quickly as the intermediate principal stress increases. However, if the intermediate principal stress reaches a certain value, the strength of rock decreases gradually.

Fig. 12 Test results by using a large triaxial apparatus [5, 6].

Fig. 13 Relations between twin-shear strength and hydrostatic pressure [29, 30].

Fig. 14 Relation between strength and $\sigma_2$ [23].

The effects of $\sigma_2$ for sand [15] and for marble [16] are shown in Figs. 15 and 16, respectively.

The effect of intermediate principal stress can be extended to the effect of the intermediate principal shear stress. The effect of the intermediate principal shear...
stress has not been discussed before. Some experiments, however, have shown this effect. The effect of the intermediate principal shear stress for clay and sand-rock of Silesian were obtained by Ergun [18], Ramamurthy et al. [19] and Kwasniewski et al. [20], as shown in Figs.17, 18 and 19 (\(\tau_{23} = (\sigma_2 - \sigma_3)/2\), \(\tau_{12} = (\sigma_1 - \sigma_2)/2\), \(\tau_{13} = (\sigma_1 - \sigma_3)/2\)).

2.5 Bounds of convex yield criteria
Various yield criteria and failure criteria for geomaterials have been proposed in the past. They are situated between some bounds if the convexity requirements for yield criteria are considered. The lower bound is provided by the single-shear strength theory (the Mohr-Coulomb strength theory) and the single-shear yield criterion (the Tresca yield criterion), as shown in Fig.20. The upper bound is given by the generalized twin-shear strength theory and the twin-shear yield criterion or the maximum deviatoric stress criterion, or the shape change criterion. Other failure criteria are situated between the two bounds.

Large amounts of yield criteria were presented, which are summarized in Refs. [7, 8, 28–56].

The yield loci on deviatoric plane (\(\sigma_1', \sigma_2', \sigma_3'\) threefold symmetry axes) of these criteria can vary from the curvilinear loci to circle, or from circle to curvilinear, as shown in Figs.21 to 31.

Fig.16 Effect of \(\sigma_1\) for marble [16].

Fig.17 Effect of \(\tau_{23}\) [18].

Fig.18 Effect of \(\tau_{23}\) [19].

Fig.19 Effect of \(\tau_{12}\) of sandstone [20].

Fig.20 Two bounds and region of convex yield loci.
Fig. 23 Shi-Yang model [49]. Fig. 24 Twin-shear smooth model [52].

Figures 25 and 26 are the Matsuoka-Nakai criterion and Lade-Duncan criterion [38], respectively. A 3D generalized criterion was proposed by Yoshimine in 2004, as shown in Fig. 27 [56]. A linear combination of the Huber-von Mises criterion and Matsuoka-Nakai criterion was proposed in 2004 [55]. The yield loci is similar to the Yoshimine generalized criterion, as shown in Figs. 27 and 28.

Fig. 25 Matsuoka-Nakai model [37]. Fig. 26 Lade-Duncan model [38].

Fig. 27 Yoshimine model [56]. Fig. 28 Yao-Lu model [55].

Based on the Ehlers criterion [46], an adjusting model was proposed by Wunderlich et al. [35], as shown in Fig. 29. The shape of yield loci in deviatoric plane can be adjusted by some parameters. Maiolino yield locus [48] is shown in Fig. 30. Bardet proposed a Lode angle-dependent criterion for isotropic SD materials, which is the combination of the Matsuoka-Nakai criterion [37] and the Lade-Duncan criterion [38] (LMN model, Fig. 31). This criterion is referred to as the LMN criterion by Bardet, after the first letter of Lade, Matsuoka, and Nakai [51]. Krenk criterion [47] is shown in Fig. 32.

Fig. 31 LMN model [51]. Fig. 32 Krenk family [47].

Most curvilinear criteria, however, cannot vary from curvilinear loci to the upper bound. As shown in Fig. 33, a curvilinear unified criterion is discussed by Hu and Yu, which covers only the one third of all convex regions [54]. Most regions of this kind of criteria are non-convex, as shown in Fig. 33(b).

Fig. 33 Curvilinear unified criterion [54].

The variations of the Yoshimine generalized failure criterion [56] are shown in Figs. 34 and 35, in which \( \alpha = (s_1 / s_3)^2 \), \( \beta = (s_2 / s_3)^2 \), \( s_1 \), \( s_2 \) and \( s_3 \) are directional cosine components, \( R \) is friction coefficient, \( k \) and \( C \) are material parameters. They can match many experimental results, which will be shown in next section.

Fig. 34 Generalized 3D failure criterion [56]. Fig. 35 Special cases of generalized 3D criterion [56].
3 Experimental results of geomaterials under complex stress

The development of the strength theory is one of the basic issues in geomechanics and geotechnical engineering. At present, the conventional triaxial test is one of the elementary tests in geomechanics. Based on the conventional triaxial test, however, the difference between different yield criteria cannot be identified. Increasingly, true triaxial, bi-axial and plane-strain tests or torsion-axial load tests have been performed. Most of these experimental results are not consistent with the Mohr-Coulomb strength theory, and the intermediate principal stress can not be taken into account in the single-shear theory.

The early researches on the failure criteria for soils under true triaxial stress states or plane strain states were pursued by Shibata and Karube (1965) at Kyoto University, Wood and Roth (1972) at Cambridge University, Ko and Scott (1967) at Colorado State University, Brown and Casbarian (1965), Sutherland and Mesdary (1969) at the University of Glasgow, Bishop (1971) and Green (1972) at Imperial College. Several summaries were given in Refs. [7, 8, 50].

It is noted that the experimental data available lie almost between the two bounds. Some results are illustrated in Figs.36 to 46.

Fig.36 Limit loci for Ottawa fine sand [21, 56].

(a) Loose sand (b) Dense sand

(c) Limit loci of a medium fine sand \( (n = 0.37) \).

(b) Limit loci of a medium fine sand \( (n = 0.39) \).

Fig.37 Experimental results of sand with different porosities [15].

(c) Limit loci of a medium fine sand \( (n = 0.41) \).

(d) Limit loci of a medium fine sand \( (n = 0.43) \).

Fig.38 Limit loci of clay (Shibata-Karube, 1965).

Fig.39 Limit loci of clay.

Fig.40 Limit loci of volcanic rock [23].
The experimental results of loess obtained by Yoshimine at Tokyo Metropolitan University in Japan are shown in Fig.43, which are close to the unified strength theory with $b=1$. It is interesting that these results can also be matched by the piecewise linear unified strength theory with different values of parameter $b$, as shown in Fig.46. Comparison of Yoshimine criterion with test data is shown in Figs.44 and 45.

The applications of yield criteria with emphasis on the unified strength theory are presented in Refs. [57–98]. Recently, a series of researches on yield criteria are given by Kolupaev et al. [99–105]. A general model [99, 100] was proposed as follows:

$$
\sigma_{eq}^{2} (\sum b_i I_i + a_1 b_1 I_1 + a_2 b_2 I_2 + a_3 b_3 I_3 + c_1 d_1 + c_2 d_2 + c_3 d_3 + 1/3d_4 + 2c_1/3) = b_2 + a_1 + a_2 + a_3 + 1/3d_4 + 2c_1/3
$$

Equation (2) contains the model:

$$
A_1 I_1^2 + D_3 I_1 I_2 + I_3 = 0
$$

which includes the Matsuoka-Nakai criterion when $A_1=0$, and the Lade-Ducan criterion when $D_3=0$.

The subject of strength of geomaterials under complex stress states is complicated for both theoretical and experimental researches. The experimental verification of strength theories is of paramount importance. If a failure criterion, a material model or a strength theory is proposed, it needs to be verified by independent experiments and theoretical analysis. The experimental verification of yield criteria for geomaterials shows...
that the yield loci are situated between the two convex bounds.

4 Developments of failure criteria for geomaterials

New ideas, new models, new methods, new equations, new criteria and new theories are the results of research and development. The originality, advantages over the existing criteria and their availability are the three elements of research objectives for a new yield criterion.

4.1 Models of yield criteria for geomaterials

Mechanical and mathematical models play important roles in establishing a new theory and understanding a presented theory. A mechanical model is an abstraction, a formation of an idea or ideas that may involve the subject with special configurations. Mathematical model may involve relations between continuous functions of space, time and other variations. Establishing a mechanical model is a basic step for developing a yield criterion for geomaterials. Some mechanical models for establishing yield criteria of geomaterials are summarized in Table 1. When the stress state is determined, various models taken from the element under the action of the same stress are equivalent.

<table>
<thead>
<tr>
<th>No.</th>
<th>Models</th>
<th>Provider</th>
<th>Introduced criterion</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Cubic element, or principal stress element, a commonly used model</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>Regular isoclinal octahedron model proposed by Ros-Eichinger and Nadai for introducing the Huber-von Mises criterion in 1926 and 1931</td>
<td></td>
<td>Huber-von Mises criterion, Drucker-Prager criterion</td>
</tr>
<tr>
<td>3</td>
<td>Spherical model proposed by Novozhilov for introducing the Huber-von Mises criterion in 1952</td>
<td></td>
<td>Huber-von Mises criterion</td>
</tr>
<tr>
<td>4</td>
<td>Single-shear model proposed by Yu to explain and introduce the Tresca and Mohr-Coulomb criteria and it is referred as the single-shear criteria in 1988</td>
<td></td>
<td>Tresca and Mohr-Coulomb criteria. It is clear that the intermediate principal stress ( \sigma_2 ) is not taken into account in single-shear theory</td>
</tr>
<tr>
<td>5</td>
<td>Multi-shear model, first used by Yu in 1961 for proposing the twin-shear criterion [44]</td>
<td></td>
<td>Twin-shear criterion, the three-shear criterion (Mises criterion) can be also introduced by using this model</td>
</tr>
<tr>
<td>6</td>
<td>Dodecahedron model: developed by Yu for proposing the generalized twin-shear criterion in 1985 [40]. The unified strength theory can be also introduced by using this model</td>
<td></td>
<td>Generalized twin-shear criterion, the three-shear criterion (Huber-von Mises criterion) can be also introduced by using this model</td>
</tr>
<tr>
<td>7</td>
<td>Orthogonal octahedral twin-shear model proposed by Yu to introduce the two equations for the twin-shear strength criterion and the unified strength theory in 1985 and 1991, respectively</td>
<td></td>
<td>Twin-shear criterion, Unified strength theory</td>
</tr>
<tr>
<td>8</td>
<td>Pentahedron twin-shear model for unified strength theory proposed by Yu in 2006. A cut-off equation is added to the unified strength theory</td>
<td></td>
<td>The three equations of the unified strength theory can be introduced by using this new model</td>
</tr>
<tr>
<td>9</td>
<td>26 polyhedron model proposed by Yu for introducing the general criterion in 2007. The single-shear, twin-shear and the three-shear criteria are unified [42]</td>
<td></td>
<td>Single-shear, twin-shear, three-shear and their combined equations for various criteria</td>
</tr>
</tbody>
</table>
Model 1 is a cubic element, and is widely used in mechanics and engineering. Models 2 to 4 are proposed for well-known yield criterion. Models 5 to 9 are new models for introducing the new criteria and new strength theory. The three principal stresses state (model 1) is converted to the three shear stresses and their three normal stresses imposed on the shear sections (models 5 to 8).

4.2 Mathematical models for yield criteria of geomaterials

The model 5, a dodecahedron model, was first used to introduce a new twin-shear criterion in 1961 [43, 44]. The model 6, a dodecahedron model, was first used to introduce a new generalized twin-shear criterion in 1985 [40]. Since there are only two independent principal shear stresses, and the maximum shear stress equals the sum of other two, i.e. \( \sigma_{13} = \tau_{13} + \tau_{12} \), the three shear stresses state can be converted into the twin-shear stress state \((\tau_{13}, \tau_{12}, \sigma_{13}, \sigma_{12})\) or \((\tau_{13}, \tau_{23}, \sigma_{13}, \sigma_{23})\). This stress state corresponds to the twin-shear model [40, 43, 44].

4.3 Development of new yield criteria for geomaterials

The mathematical modeling of a linear yield criterion and a nonlinear criterion based on mechanical model 9 in Table 1 can be given as

\[
a_1 \tau_{13} + a_2 \tau_{12} + a_3 \tau_{23} + a_4 \sigma_{13} + a_5 \sigma_{12} + a_6 \sigma_{23} + a_7 \tau_{\text{oct}} + a_8 \sigma_{\text{oct}} + a_9 \sigma_2 + a_{10} \sigma_3 = c
\]

(5)

\[
a_1 \tau_{13} + a_2 \tau_{12} + a_3 \tau_{23} + a_4 \sigma_{13} + a_5 \sigma_{12} + a_6 \sigma_{23} + a_7 \tau_{\text{oct}} + a_8 \sigma_{\text{oct}} + a_9 \tau_{\text{oct}}^2 + a_{10} \tau_{\text{oct}}^3 + a_1 \sigma_1 + a_2 \sigma_2 + a_3 \sigma_3 = c
\]

(6)

It is noted that the nonlinear criterion can match the experimental results, but the linear criterion cannot, as shown in Fig.47(a). The nonlinear criterion, however, is difficult to be used for analytical solutions of problems of geomaterials. The piecewise linear criterion may be a better choice for the analytical solution for structures. It also matches the experimental results well, as shown in Fig.47(b).

5 Applications of new yield criteria for geomaterials

The application of a new constitutive relation is of great importance for the development of geomechanics and geological engineering. A series of results can be obtained by using the Yu unified strength theory (Yu UST or UST). It is convenient to use the UST by analytical and numerical methods. The UST has been implemented into several elastoplastic programs and applied to engineering problems [28–30, 57–62, 66–98]. The singularities at the corners of single-shear theory, twin-shear theory and the unified strength theory have been processed conveniently by using a unified numerical procedure. A unified elastoplastic program (UEPP) has been established, which has been applied to some engineering problems [10, 89–92, 106–108].

The twin-shear strength theory has been implemented into some finite element programs for solving the hydropower structure [66], mechanical structure [73], soil mechanics problems [29, 30, 50], composite materials [71, 72], and structural analysis of dam [70].

The elasto-visco-plastic finite element analysis of self-enhanced thick cylinder by using the twin-shear strength theory was given by Liu et al. [73]. The twin-shear yield criterion and the twin-shear strength theory have been implemented into three commercial finite element codes by Quint Co. [75–77]. The twin-shear strength theory was implemented into a finite element code and applied to analyze the stability of the high slopes of Three Gorges shiplock by Yangtze River Scientific Research Institute [1].

An application of the generalized model of Yu to plastics is given by Altenbach and Kolupaev [102]. The applications of the unified strength theory to unreinforced polymers are described.

A 3D finite element numerical modeling for a large underground caves and the stability, analysis of excavation rock mass of the Tai’an Pumped Storage Hydraulic Plant in Zhejiang Province in China were performed by Sun et al. [79, 80].

Recently, analyses of textural stress and rock failure of diversion tunnels by using the twin-shear strength theory are given by Yang and Zhang [85]. The twin-shear theory is also used to study the sudden-crack phenomenon and to simulate the response of surrounding rock in diversion tunnel [86]. The adaptive arithmetic of arch dam cracking analysis by using the twin-shear strength theory is given by Yang et al. [87]. The difficulty of singularity has been overcome, and it is easy to be used.

The unified yield criterion and the unified strength...
theory have been implemented and applied to some plasticity and engineering problems [89–98].

UST is also implemented into the general commercial codes, such as ABAQUS and AutODYN by Fan and Qiang in Singapore [60], Zhang et al. at Griffith University, Australian researchers in the punch of concrete and dynamic problems [59]. Normal high-velocity impact concrete slabs were simulated by using the unified strength theory [60]. The unified strength theory was implemented into nonlinear FEM by Zhou [96] for numerical analysis of reinforced concrete under dynamic load.

UST is used to study the topology optimization of evolutionary structure by Li et al. [68]. The abstract of the paper shows that: “Based on the traditional evolutionary structural optimization method and considering wide applications of the unified strength theory for all kinds of engineering structures, this paper presents a bi-directional evolutionary structural topology optimization method based on the unified strength criterion. It can be used not only for isotropic materials, but also for many kinds of anisotropic materials. Finally, some numerical examples are given, and the results show that this method has been widely used in topology optimization design for structures of fragile materials, the anisotropic material processing and die design fields.”

The three-parameter unified strength theory was implemented into finite element program and used to study the structural reliability by Wang et al. [81, 82] at Sichuan University. It is also used for nonlinear finite element analysis of reinforced concrete plate and shell at Nanyang Technological University, Singapore [84]. The three-parameter and five-parameter unified strength theories are used for bearing capability of concrete-filled steel tube component considering effect of intermediate principal stress and plastic seismic damage of concrete structure [78].

UST is also implemented in the finite difference computation. A new effective three-dimensional finite difference method (FDM) computer program, FLAC3D (fast Lagrange analysis of continua in 3-dimension) was presented. The stability analysis of the high slopes of the Three Gorges shiplock using FLAC3D was given. It is a pity, however, that only two failure criteria, the Mohr-Coulomb criterion and the Drucker-Prager criterion, were implemented into the original FLAC3D code. UST and its unified elastoplastic constitutive model are implemented into FLAC3D by Zhang et al. [93, 94]. The abstract of the paper shows that: “The unified strength theory is a new theory system which can almost describe the strength characters of most geomaterials and has been applied widely. And FLAC3D is an excellent geotechnical program. If the former can be integrated in the later, many complex problems in engineering will be settled well. So according to this problem, the numerical scheme of elastoplastic unified constitutive model in FLAC3D was studied. And the numerical format of the elastoplastic constitutive model based on the unified strength theory was derived.” The merits of the unified strength theory combined with the FLAC3D program can be clearly shown in geotechnical engineering [94].

UST is also implemented into the FLAC3D at University of Science and Technology Beijing for stability analysis of large scale underground caverns [69], and the stability and protections of underground caves group of Huanren power plant are calculated. The excavation, the spread of plastic region around the cave group and the distribution, and the displacement change situation are obtained by Li. The effects of irregular surface, in-situ stress field’s distribution and different constitutive relations on stability have been studied [69].

The seismic stability analysis of rockfill dams based on unified strength theory is conducted by Lin and Liu in 2008 [67]. The results show that the strength theory has significant influences on seismic stability analysis.

UST has also been used with the limit equilibrium method by Yangtze River Scientific Research Institute for the seismic stability and permanent displacement of joint rock slope [97]. The analytical formulae and practical example in expressway of critical filling height of roadbed based on the unified strength theory are given by Tong and Guo in 2007 [88]. The comparison shows that the result obtained the unified strength theory with the parameter $b=1$ closes to the site test result [88].

The bearing capacity of a footing can be given according to Fig.48. The convenient solution is a special case of the unified solution. The unified solution of bearing capacity of a trapezoid structure with unified strength theory parameter $b$ is obtained as illustrated in Fig.49.

![Fig.48 Beari ng capacity of footing foundations.](image)
6 New model for the unified strength theory

It is clear that there are three principal shear stresses, $\tau_{13}$, $\tau_{12}$, and $\tau_{23}$, in the three-dimensional principal stress space defined by three axes of $\sigma_1$, $\sigma_2$, and $\sigma_3$. However, only two principal shear stresses are independent variables among $\tau_{13}$, $\tau_{12}$, $\tau_{23}$ because the maximum principal shear stress equals the sum of the other two, i.e., $\tau_{13} = \tau_{12} + \tau_{23}$.

Since there are only two independent principal shear stresses, the shear stress state can also be converted into the twin-shear stress state ($\tau_{13}$, $\tau_{12}$, $\sigma_{13}$, $\sigma_{12}$) or ($\tau_{13}$, $\tau_{23}$, $\sigma_{13}$, $\sigma_{23}$). This stress state corresponds to the twin-shear model proposed by Yu in 1961 and 1985. The eight sections of element which two groups of shear stresses act on consist of the orthogonal octahedral elements, the twin-shear mechanical model can be obtained as shown in Fig.51.

By removing the half of the orthogonal octahedral model, a new pentahedron element can be obtained, as shown in Fig.52. The relation between the twin-shear stress and the principal stress $\sigma_1$ or $\sigma_3$ can be deduced from this element. Based on the orthogonal octahedral element and pentahedron element, the unified strength theory can be developed.

The twin-shear orthogonal octahedral model is different from the regular octahedral model. The orthogonal octahedral model consists of two groups of four sections perpendicular to each other and subjected to the maximum shear stress $\tau_{13}$ and the intermediate principal shear stress $\tau_{12}$ or $\tau_{23}$.
The mathematical modeling of the unified strength theory is given as
\[
F = \tau_{12} + b \tau_{12} + \beta (\sigma_{13} + b \sigma_{12}) = c
\]
(7a)
\[
F' = \tau_{13} + b \tau_{13} + \beta (\sigma_{11} + b \sigma_{13}) = c
\]
(7b)
\[
F'' = \sigma_1 = \sigma_i \quad (\sigma_i > \sigma_j > \sigma_i > 0)
\]
(7c)
where \( \beta \) and \( c \) are material parameters, \( b \) is failure criterion parameter.

The unified strength theory takes into account (1) the effect of SD, (2) the hydrostatic stress effect, (3) the normal stress effect, (4) the effect of the intermediate principal stress, and (5) the effect of the intermediate principal shear stress.

The magnitudes of \( \beta \) and \( c \) can be determined by experimental results of uniaxial tension strength \( \sigma_1 \), uniaxial compression strength \( \sigma_i \), or \( \alpha = \sigma_1 / \sigma_i \):
\[
\beta = \frac{\sigma_i - \sigma^*_i}{\sigma^*_i} = \frac{1 - \alpha}{1 + \alpha}, \quad c = \frac{2 \sigma^*_i \sigma_1}{\sigma^*_i + \sigma_1} = \frac{2}{1 + \alpha} \sigma_1
\]
(8)

The mathematical expression can be derived from the mathematical modeling and the uniaxial tensile and uniaxial compressive conditions as follows:
\[
F = \sigma_1 - \frac{\alpha}{1 + b} (b \sigma_2 + \sigma_3) = \sigma_i \quad \left( \sigma_2 \leq \frac{\sigma^*_2 + \alpha \sigma^*_3}{1 + \alpha} \right)
\]
(9a)
\[
F' = \frac{1}{1 + b} (\sigma_1 + b \sigma_2) - \alpha \sigma_1 = \sigma_i \quad \left( \sigma_2 \geq \frac{\sigma^*_2 + \alpha \sigma^*_3}{1 + \alpha} \right)
\]
(9b)
\[
F'' = \sigma_1 = \sigma_i \quad (\sigma_i > \sigma_j > \sigma_i > 0)
\]
(9c)

Equation (9c) is used only for the stress state of three tensile stresses. It is similar to the Mohr-Coulomb theory with tension cut-off suggested by Paul in 1961. UST with the tension cut-off can be used for geomaterials. The widely used Mohr-Coulomb criterion for geomaterials is a specific form of the unified strength theory (Eqs.(9a) and (9c), and \( b=0 \)).

For non-SD material, when the uniaxial tensile and compressive strengths are identical, i.e. \( \alpha = 1 \), Eqs.(9a) and (9b) are simplified as follows:
\[
F = \sigma_1 - \frac{1}{1 + b} (b \sigma_2 + \sigma_3) = \sigma_i \quad \left( \sigma_2 \leq \frac{\sigma^*_2 + \sigma^*_3}{2} \right)
\]
(10a)
\[
F' = \frac{1}{1 + b} (\sigma_1 + b \sigma_2) - \sigma_1 = \sigma_i \quad \left( \sigma_2 \geq \frac{\sigma^*_2 + \sigma^*_3}{1 + \alpha} \right)
\]
(10b)

Thus, the unified strength theory is applicable to both SD and non-SD materials.

7 Yield surfaces and yield loci of the unified strength theory

A series of yield surfaces of the unified strength theory are illustrated in the following. Figure 53 is the yield surface of the unified strength theory with different values of parameter \( b \) drawn by Zhang [95]. Figures 54(a), (b) and (d) are the inner yield surface (yield surface of the unified strength theory with \( b = 0 \) or the yield surface of the Mohr-Coulomb strength theory), the median yield surface of the unified strength theory with \( b = 1/2 \), and the outer yield surface (yield surface of the unified strength theory with \( b = 1 \) or the yield surface of the twin-shear strength theory) [95]. Figure 54(c) is the yield surface of the Matsuoka-Nakai criterion which is closed to the UST with \( b = 3/4 \).

![Fig.53 Serial yield surfaces of UST.](image)

![Fig.54 Special cases of UST.](image)

The yield surface of the unified strength theory with a tension cut-off in stress space was given by Zhang et al. [94], as shown in Fig.55.

![Fig.55 Yield surfaces of UST with a tension cut-off [94].](image)
The projections of the UST on the deviatoric plane are illustrated in Fig.56(a) (for SD materials, \( \alpha \neq 1 \)). A series of yield loci of unified yield criterion at the deviatoric plane for non-SD materials (\( \alpha = 1 \)) is shown in Fig.56(b). Unified strength theory gives a series of yield and strength criteria and establishes the relation among them. The cut-off line is shown in Fig.57.

\[ k = \sqrt{3} \frac{1 + b}{1 + b + \alpha} \]  

(11)

The lower bound is provided by the single-shear strength theory (the Mohr-Coulomb strength theory, or UST with \( b = 0 \)). The upper bound is given by the generalized twin-shear strength theory (Yu et al. 1985), or UST (Yu 1991) with \( b = 1 \). The median bound is a new series of yield criteria deduced from UST with \( b = 1/2 \). Other series of new yield criteria can also be deduced from UST with \( b = 1/4 \) or \( b = 3/4 \), as shown in Fig.56.

8 Characteristics of the unified strength theory

The characteristics of the UST can be summarized as follows:

1. UST is an assembly of a series of yield criteria adopted for non-SD and SD materials. The criteria cover all the region between the lower bound (single-shear theory) and upper bound (twin-shear theory).

2. UST combines various yield criteria through the parameter \( b \). The single-shear theory, the twin-shear theory and some other strength criteria are special cases or approximations of the unified strength theory as shown in Fig.54. A series of new criteria can also be obtained.

3. All the yield criteria of UST are piecewise linear. The application of the unified strength theory gives not only a single solution, but also a series of solutions. It is referred to as the unified solution for many materials and structures.

4. UST with different values of parameter \( b \) can match experimental results for various materials. The physical and geometrical meanings of parameter \( b \), however, have to be studied.

5. UST can be used to establish the unified elastoplastic constitutive equations. It can be implemented into a finite element code in a unified manner. It is also convenient for elastic limit design, elastoplastic and plastic limit analyses of structures because of its piece-wise linear form [106, 107].

9 Conclusions

The failure criterion of geomaterials is an important
element for research and design of the strength of materials and structures. The basic characteristics of a yield criterion for geomaterials were studied. The effects of SD, normal stress and its extension, intermediate principal stress and its extension, hydrostatic stress and its extension, twin-shear stresses and the convexity of yield surface were considered as the basic characteristics of yield criteria for geomaterials.

The main advances in strength theories for SD and non-SD materials from single-shear theory (lower bound) to the three-shear theory (median criteria) and to the twin-shear theory (upper bound), as well as from single criterion to unified criterion are briefly summarized in Table 2.

The basic characteristics are available for the innovation of new yield criterion. New (introduced for the first time), better than existing criteria and the availability are the three elements of innovation for yield criterion. The development of mechanical models for various yield criteria was also summarized. According to a new tetragonal mechanical model, a tension cut-off condition was added to the unified strength theory. The unified strength theory was extended to the tension-tension region, it is very important for geomaterials.

The unified strength theory is summarized in Ref. [106]. The applications of the unified strength theory to slip line field problems for both non-SD and SD materials were summarized by Yu [107]. The applications of the unified strength theory to limit, shake-down and dynamic plastic analyses of structures by analytical method were summarized by Yu [108]. The applications of the unified strength theory to several engineering problems by numerical method are summarized in a monograph entitled “Computational

<table>
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Plasticity”, which will be published soon.

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