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Warnaar's bijection and colored partition identities, I

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ABSTRACT

We provide a general and unified combinatorial framework for a number of colored partition identities, which include the five, recently proved analytically by B. Berndt, that correspond to the exceptional modular equations of prime degree due to H. Schröter, R. Russell and S. Ramanujan. Our approach generalizes that of S. Kim, who has given a bijective proof for two of these five identities, namely the ones modulo 7 (also known as the Farkas–Kra identity) and modulo 3. As a consequence of our method, we determine bijective proofs also for the two highly nontrivial identities modulo 5 and 11, thus leaving open combinatorially only the one modulo 23.

1. Introduction

Colored partition identities are a very active research area within the theory of integer partitions. In particular, they provide natural combinatorial interpretations for certain classes of objects coming from other mathematical fields, including equations that involve modular forms or theta functions. The simplest and perhaps best known identity of this family is the so-called "Farkas–Kra identity modulo 7" (see [5]), which states that there are as many integer partitions of $2N+1$ into distinct odd parts as there are integer partitions of $2N$ into distinct even parts, provided the multiples of 7 appear in two different copies. A combinatorial proof of this result had been asked for by H.M. Farkas and I. Kra, R. Stanley, B. Berndt and a number of other authors, and was recently given by S. Kim [8].

The Farkas–Kra identity is part of a set of five exceptional colored partition identities, sometimes referred to as "identities of the Schröter, Russell and Ramanujan type", which correspond to five, conjecturally unique, modular equations of prime degree, discovered independently by H. Schröter [15], R. Russell [12,13] and S. Ramanujan [3,11]. These modular equations, respectively of degree 3, 5, 7, 11

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and 23, as Berndt pointed out in [4], appear to be the only ones of such a simple type. See [4] for an interesting and detailed discussion of the history of these equations. In fact, in his paper, Berndt determined and proved analytically the five corresponding partition identities. As Berndt remarked, however (see also M.D. Hirschhorn [7]), these five identities remained “manifestly mysterious”, as they still lacked “simple bijective proofs”, which “would be of enormous interest”.

Soon afterwards, S. Kim [8], who employed in a clever fashion a bijection of S.O. Warnaar [17] and generalized one of his results, provided an entirely bijective proof of, among other facts, two of the above identities — the one modulo 7, as we have said, and that modulo 3.

A main goal of this paper is to respond to Berndt’s call for a unified combinatorial framework in which to look at the five identities of the Schröter, Russell and Ramanujan type. In fact, extending Kim’s idea, we prove an equivalence between a very broad family of colored partition identities, which include the above five, and suitable equations in $(\nu_1,\dots,\nu_t;d_1,\dots,d_t)$, where $t\geqslant 1$, the ν_i are partitions, and the d_i are integers whose sum is odd.

In particular, our approach allows us to prove bijectively two more of the identities of the Schröter, Russell and Ramanujan type, namely those corresponding to the modular equations of degree 5 and 11, whose specific proofs turn out to be highly nontrivial. Unfortunately, we have not been able to show bijectively the last identity, that modulo 23. We state its equivalent equation as Conjecture 3.14.

In a sequel to this paper [14], we will prove, again as a consequence of our method, a number of new (and challenging) colored partition identities.

2. The master bijection

Let us first briefly recall the main definitions from partition theory that we are going to use in this paper. For an introduction, a survey of the main techniques, or a discussion of the philosophy behind this fascinating field, see e.g. [1,2,10], Section I.1 of [9], and Section 1.8 of [16].

Given a nonnegative integer N, we say that the nonincreasing sequence $\lambda=(\lambda^{(1)},\dots,\lambda^{(s)})$ of nonnegative integers is a *partition* of N, and often write $|\lambda|=N$, if $\sum_{i=1}^{s}\lambda^{(i)}=N$. The $\lambda^{(i)}$ are called the *parts* of λ, and the number of parts of λ is its *length*, denoted by $l(\lambda)$. As usual, we define $p(N)$ to be the number of partitions of N into positive parts; thus $p(a)=0$ for $a<0$, and $p(0)=1$, since we adopt the standard convention that $\emptyset$ is the only partition of $N=0$.

Finally, let P be the set of all partitions into positive parts, D_0 the set of partitions into distinct nonnegative parts, and $D=P\cap D_0$ the set of partitions into distinct positive parts. For instance, $\lambda=(6,6,3)\in P$ has length $l(\lambda)=3$, and $\lambda=(7,6,3,0)\in D_0$ has length $l(\lambda)=4$.

We begin with the following crucial bijection due to S.O. Warnaar [17], who generalized an earlier bijection of E.M. Wright [18]. As usual, we set $\binom{d}{2}=d(d-1)/2$, for any $d\in\mathbb{Z}$.

Lemma 2.1. *(See [17].) There exists a bijection between the set of triples (α,β,d), where $\alpha\in D_0$, $\beta\in D$ and $d=l(\alpha)-l(\beta)$, and the set of pairs (ν,d), where $\nu\in P$ and $d\in\mathbb{Z}$, such that*

$$|\alpha|+|\beta|=|\nu|+\binom{d}{2}.$$

Proof. See [17], pp. 48–49, for a description of the bijection. □

The next theorem is the main general result of this paper. (We present it in a form that suffices for our purposes, even though it could easily be stated in more general terms.) It is an immediate corollary of the following lemma:

Lemma 2.2. *Fix integers $t\geqslant 1$, $C_1,\dots,C_t\geqslant 1$, and $0\leqslant A_i\leqslant C_i/2$ for all $i=1,\dots,t$. Let S be the set containing one copy of all positive integers congruent to $\pm A_i$ modulo C_i for each i, and let $D_S(N)$ be the number of partitions of N into distinct elements of S, where we require such partitions to have an odd number of parts if no A_i is equal to zero. Finally, set $r=|\{A_i=0\}|-1$, adopting the convention that $|X|=1$ if $X=\emptyset$. Then, for all $N\geqslant 1$,*

$$2^r\cdot D_S(N)$$

is the number of solutions $(\nu_1, \ldots, \nu_t; d_1, \ldots, d_t)$ *to the equation*

$$\sum_{i=1}^{t} C_i|\nu_i| + \sum_{i=1}^{t} C_i\binom{d_i}{2} + \sum_{i=1}^{t} A_i d_i = N, \tag{1}$$

where $\nu_i \in P$ *and* $d_i \in \mathbb{Z}$ *for all i, and* $\sum_{i=1}^{t} d_i$ *is odd.*

Proof. This proof will greatly generalize, but proceed for the most part in a similar way to, Kim's combinatorial proof of the Farkas–Kra identity modulo 7 (cf. [8], second proof of Theorem 2.1). A substantial difference is that we are going to push the bijectivity of this type of argument all the way through, so that Theorem 2.3 below will give us (ii) equivalent to (i), which is going to be the crucial tool in attacking the identities of the Schröter, Russell and Ramanujan type.

Fix $N \geqslant 1$. We start by assuming that all of the A_i are positive, and consider any partition π of N into distinct elements of S. We first split π into t pairs of partitions $(\lambda_1, \mu_1), \ldots, (\lambda_t, \mu_t)$, where, for any i, both λ_i and μ_i are in D, all parts of λ_i come from the copy of the integers of S that are congruent to $A_i \pmod{C_i}$, and all parts of μ_i come from the copy of the integers of S that are congruent to $-A_i \pmod{C_i}$.

Let us now construct a new partition π^* from π, which we split as $(\lambda_1^*, \mu_1^*), \ldots, (\lambda_t^*, \mu_t^*)$, where (entrywise) $\lambda_i^* = (\lambda_i - A_i)/C_i$ and $\mu_i^* = (\mu_i + A_i)/C_i$, for all i. Notice that, clearly, $\lambda_i^* \in D_0$ and $\mu_i^* \in D$, for all i.

Set

$$d_i = l(\lambda_i) - l(\mu_i) = l(\lambda_i^*) - l(\mu_i^*).$$

Note that $\sum_{i=1}^{t} d_i \equiv \sum_{i=1}^{t}(l(\lambda_i) + l(\mu_i)) = l(\pi) \pmod 2$; that is, $\sum_{i=1}^{t} d_i$ is odd if and only if π has an odd number of parts.

By Lemma 2.1, the triples $(\lambda_i^*, \mu_i^*, d_i)$ are in (Warnaar's) bijection with pairs (ν_i, d_i), where $\nu_i \in P$ and $|\lambda_i^*| + |\mu_i^*| = |\nu_i| + \binom{d_i}{2}$. Therefore, it is easy to see that

$$N = |\pi| = \sum_{i=1}^{t}\big(|\lambda_i| + |\mu_i|\big) = \sum_{i=1}^{t} C_i|\lambda_i^*| + \sum_{i=1}^{t} C_i|\mu_i^*| + \sum_{i=1}^{t} d_i A_i$$
$$= \sum_{i=1}^{t} C_i|\nu_i| + \sum_{i=1}^{t} C_i\binom{d_i}{2} + \sum_{i=1}^{t} d_i A_i.$$

Since all previous steps are reversible, this implies that the number of solutions to Eq. (1) is $D_S(N) = 2^0 \cdot D_S(N)$, as desired.

This completes the proof when all of the A_i are positive.

Suppose now that some $A_i = 0$. Let us assume, without loss of generality, that $A_1 = A_2 = \cdots = A_{r+1} = 0$ for some $r \geqslant 0$, and that all other $A_j \neq 0$. The proof of this case goes along the same lines, except that now the partitions λ_i^* are in D (not in D_0), for all $i \leqslant r+1$. Therefore, it is easy to see that, for $i \leqslant r+1$, the same partition λ_i corresponds to exactly two solutions to Eq. (1) — one given by operating with Warnaar's bijection with λ_i^*, and the other with λ_i^* to which a 0 is added at the end. Thus, each of our partitions π corresponds bijectively to 2^{r+1} solutions to (1), when $\sum_{i=1}^{t} d_i$ is arbitrary.

Now, it is immediate to see that, in every solution to (1), d_i can be replaced by $1 - d_i$, for any $i \leqslant r+1$. Since the parity of d_i and $1 - d_i$ is different, it follows that exactly $2^{r+1}/2 = 2^r$ of the solutions to (1) corresponding to the partition π yield an odd value for $\sum_{i=1}^{t} d_i$. This easily concludes the proof of the lemma. □

Theorem 2.3. *Consider the equation*

$$\sum_{i=1}^{t} C_i|\mu_i| + \sum_{i=1}^{t} C_i\binom{d_i}{2} + \sum_{i=1}^{t} A_i d_i = \sum_{i=1}^{t} C_i|\alpha_i| + \sum_{i=1}^{t} C_i\binom{e_i}{2} + \sum_{i=1}^{t} B_i e_i + m, \tag{2}$$

for given integers $t \geqslant 1$, $C_1, \ldots, C_t \geqslant 1$, $0 \leqslant A_i \leqslant C_i/2$ and $0 \leqslant B_i \leqslant C_i/2$ for all i, and $m \geqslant 0$. Let S be the set containing one copy of all positive integers congruent to $\pm A_i$ modulo C_i for each i, and T the set containing one copy of all positive integers congruent to $\pm B_i$ modulo C_i for each i. Let $D_S(N)$ (respectively, $D_T(N)$) be the number of partitions of N into distinct elements of S (respectively, T), where we require such partitions to have an odd number of parts if no A_i (respectively, no B_i) is equal to zero. Finally, set

$$p = \left|\{B_i = 0\}\right| - \left|\{A_i = 0\}\right|,$$

adopting the convention that $|X| = 1$ if $X = \emptyset$. Then the following are equivalent:

(i) *For any $N \geqslant N_0 \geqslant 1$, the number of tuples $(\mu_1, \ldots, \mu_t; d_1, \ldots, d_t)$ such that the left-hand side of (2) equals N, $\mu_i \in P$ and $d_i \in \mathbb{Z}$ for all i, and $\sum_{i=1}^t d_i$ is odd, is equal to the number of tuples $(\alpha_1, \ldots, \alpha_t; e_1, \ldots, e_t)$ such that the right-hand side of (2) equals N, $\alpha_i \in P$ and $e_i \in \mathbb{Z}$ for all i, and $\sum_{i=1}^t e_i$ is odd;*
(ii) *For any $N \geqslant N_0 \geqslant 1$,*

$$D_S(N) = 2^p \cdot D_T(N - m).$$

Proof. Straightforward from Lemma 2.2. □

3. The identities of the Schröter, Russell and Ramanujan type

The object of the rest of this paper is to show bijectively, using Theorem 2.3, four of the five partition identities proved by Berndt in [4], which correspond to the five exceptional modular equations of prime degree due to Schröter, Russell and Ramanujan, as we discussed in the introduction. They will be proved in Theorems 3.3, 3.6, 3.11, and 3.13. We have not been able to show the identity modulo 23; we will state an equation equivalent to it via Theorem 2.3 as Conjecture 3.14.

We start with the partition identities modulo 7 (i.e., the Farkas–Kra identity) and modulo 3. These are the two of the five for which a bijective proof is already known, thanks to the work of Kim [8] (we will just slightly modify Kim's bijection here so as to fit our setting).

Lemma 3.1. *Condition* (i) *of Theorem 2.3 holds for $N_0 = 1$, $t = 4$, $C_1 = \cdots = C_4 = 14$, $m = 1$, and*

$$(A_1, A_2, A_3, A_4) = (1, 3, 5, 7), \qquad (B_1, B_2, B_3, B_4) = (0, 2, 4, 6).$$

Proof. It is easy to verify that the result follows by associating the tuple

$$(\mu_1, \mu_2, \mu_3, \mu_4;\ d_1 = 2s + 1 - k - l + n,\ d_2 = k - n,\ d_3 = l - n,\ d_4 = n),$$

where n, l, k, and s are arbitrary integers, to the tuple

$$(\alpha_1 = \mu_1,\ \alpha_2 = \mu_2,\ \alpha_3 = \mu_3,\ \alpha_4 = \mu_4;\ e_1 = 2n + 1 - k - l + s,\ e_2 = k - s,$$
$$e_3 = l - s,\ e_4 = -s).$$

(This is exactly Kim's map of [8], Theorem 1.1, except that here we needed to set $e_4 = -s$ in place of $e_4 = s$.) □

Example 3.2. For any $N \geqslant 1$, Lemma 3.1 puts the tuples $(\mu_1, \ldots, \mu_4; d_1, \ldots, d_4) \in P^4 \times \mathbb{Z}^4$ such that the left-hand side of the following equation equals N and $d_1 + d_2 + d_3 + d_4$ is odd in bijection with the tuples $(\alpha_1, \ldots, \alpha_4; e_1, \ldots, e_4) \in P^4 \times \mathbb{Z}^4$ such that the right-hand side equals N and $e_1 + e_2 + e_3 + e_4$ is odd:

$$14\sum_{i=1}^{4}|\mu_i| + 14\sum_{i=1}^{4}\binom{d_i}{2} + 1d_1 + 3d_2 + 5d_3 + 7d_4$$
$$= 14\sum_{i=1}^{4}|\alpha_i| + 14\sum_{i=1}^{4}\binom{e_i}{2} + 0e_1 + 2e_2 + 4e_3 + 6e_4 + 1. \qquad (3)$$

Let $N = 15$. It is easy to check that there are exactly six such tuples. The left-hand side of (3) equals 15 for $(\mu_1, \mu_2, \mu_3, \mu_4; d_1, d_2, d_3, d_4)$ equal to:

$$\big((1), \emptyset, \emptyset, \emptyset; 1, 0, 0, 0\big), \quad \big(\emptyset, (1), \emptyset, \emptyset; 1, 0, 0, 0\big), \quad \big(\emptyset, \emptyset, (1), \emptyset; 1, 0, 0, 0\big),$$
$$\big(\emptyset, \emptyset, \emptyset, (1); 1, 0, 0, 0\big), \quad (\emptyset, \emptyset, \emptyset, \emptyset; 0, 1, 1, 1), \quad (\emptyset, \emptyset, \emptyset, \emptyset; 0, 1, 1, -1).$$

The bijection given in the proof of Lemma 3.1 maps the above solutions, respectively, to the following six tuples $(\alpha_1, \alpha_2, \alpha_3, \alpha_4; e_1, e_2, e_3, e_4)$ for which the right-hand side of Eq. (3) equals 15:

$$\big((1), \emptyset, \emptyset, \emptyset; 1, 0, 0, 0\big), \quad \big(\emptyset, (1), \emptyset, \emptyset; 1, 0, 0, 0\big), \quad \big(\emptyset, \emptyset, (1), \emptyset; 1, 0, 0, 0\big),$$
$$\big(\emptyset, \emptyset, \emptyset, (1); 1, 0, 0, 0\big), \quad (\emptyset, \emptyset, \emptyset, \emptyset; 0, 1, 1, -1), \quad (\emptyset, \emptyset, \emptyset, \emptyset; -1, 0, 0, 0).$$

Theorem 3.3. *(See [8].) Let S be the set containing one copy of the odd positive integers and one more copy of the odd positive multiples of 7, and T the set containing one copy of the even positive integers and one more copy of the even positive multiples of 7. Then, for any $N \geqslant 1$,*

$$D_S(N) = D_T(N - 1).$$

Proof. Straightforward from Theorem 2.3 and Lemma 3.1. □

Example 3.4. Let $N = 15$ in Theorem 3.3. By Example 3.2 and Lemma 2.2, we have

$$D_S(15) = D_T(14) = 6.$$

Indeed, it is easy to check that 15 can be partitioned in the following six ways into distinct odd positive integers, where the multiples of 7 appear in two copies, say $7n$ and $\overline{7n}$:

$$(15),\ (11, 3, 1),\ (9, 5, 1),\ (7, 5, 3),\ (\bar{7}, 5, 3),\ (7, \bar{7}, 1).$$

Similarly, 14 can be partitioned in the following six ways into distinct even positive integers, where the multiples of 7 appear in two copies:

$$(14),\ (\overline{14}),\ (12, 2),\ (10, 4),\ (8, 6),\ (8, 4, 2).$$

Lemma 3.5. *Condition* (i) *of Theorem 2.3 holds for $N_0 = 1$, $t = 4$, $C_1 = \cdots = C_4 = 6$, $m = 1$, and*

$$(A_1, A_2, A_3, A_4) = (1, 1, 3, 3), \qquad (B_1, B_2, B_3, B_4) = (0, 0, 2, 2).$$

Proof. The exact same bijection as in Lemma 3.1 easily gives the result. □

Theorem 3.6. *(See [8].) Let S be the set containing 2 copies of the odd positive integers and 2 more copies of the odd positive multiples of 3, and T the set containing 2 copies of the even positive integers and 2 more copies of the even positive multiples of 3. Then, for any $N \geqslant 1$,*

$$D_S(N) = 2D_T(N - 1).$$

Proof. Straightforward from Theorem 2.3 and Lemma 3.5. □

Notice that the two equations we are going to deal with next, which are equivalent, respectively, to the partition identities modulo 5 and 11 of the Schröter, Russell and Ramanujan type, will be:

$$\begin{aligned}
&2|\mu_1| + 2|\mu_2| + 10|\mu_3| + 10|\mu_4| + 2\binom{d_1}{2} + 2\binom{d_2}{2} + 10\binom{d_3}{2} + 10\binom{d_4}{2} \\
&\quad + d_1 + d_2 + 5d_3 + 5d_4 \\
&\ = 2|\alpha_1| + 2|\alpha_2| + 10|\alpha_3| + 10|\alpha_4| + 2\binom{e_1}{2} + 2\binom{e_2}{2} + 10\binom{e_3}{2} + 10\binom{e_4}{2} \\
&\quad + 0e_1 + 0e_2 + 0e_3 + 0e_4 + 3
\end{aligned}$$

and

$$2|\mu_1| + 22|\mu_2| + 2\binom{d_1}{2} + 22\binom{d_2}{2} + d_1 + 11d_2$$
$$= 2|\alpha_1| + 22|\alpha_2| + 2\binom{e_1}{2} + 22\binom{e_2}{2} + 0e_1 + 0e_2 + 3.$$

Therefore, one moment's thought gives that the type of argument that held for Lemmas 3.1 and 3.5, where the bijection between the solutions could simply be taken to be the identity on all the partitions μ_i, will not apply here, where $m = 3$. For instance, in the first of the two equations, the tuple

$$(\mu_1, \dots, \mu_4; d_1, \dots, d_4) = \big(\emptyset, \emptyset, \emptyset, (1); 1, 0, 0, 0\big),$$

which makes the left-hand side equal 11, must be mapped to a tuple $(\alpha_1, \dots, \alpha_4; e_1, \dots, e_4)$ such that the right-hand side equals 11, so we clearly need to have the partition $\alpha_4 = \emptyset$. An entirely similar argument holds for the second equation. This is why the two corresponding partition results will be far more difficult to treat bijectively than the previous ones.

The following lemma is a classical application of Euler's Pentagonal Number Theorem:

Lemma 3.7. *For any $n > 0$,*

$$\sum_{i \in \mathbb{Z}} (-1)^i p\left(n - \frac{i(3i-1)}{2}\right) = 0.$$

Proof. See e.g. [10], formula 5.1.2, or [16], Eq. (1.91). □

Lemma 3.8. *Fix arbitrary $C_1, \dots, C_t, A_1, \dots, A_t, B_1, \dots, B_t$, such that $0 \leqslant A_i \leqslant C_i/2$ and $0 \leqslant B_i \leqslant C_i/2$, for all $i = 1, \dots, t$. Let S_N be the set of all tuples of t partitions and t integers $(\mu_1, \dots, \mu_t; d_1, \dots, d_t)$ such that $\sum_{i=1}^t d_i$ is odd and*

$$\sum_{i=1}^t C_i|\mu_i| + \sum_{i=1}^t C_i\binom{d_i}{2} + \sum_{i=1}^t A_i d_i = N.$$

Similarly, let T_N be the set of all tuples of t partitions and t integers $(\alpha_1, \dots, \alpha_t; e_1, \dots, e_t)$ such that $\sum_{i=1}^t e_i$ is odd and

$$\sum_{i=1}^t C_i|\alpha_i| + \sum_{i=1}^t C_i\binom{e_i}{2} + \sum_{i=1}^t B_i e_i + m = N,$$

where m is an integer chosen so that the smallest value of N for which $T_N \neq \emptyset$ is also the second smallest value of N for which $S_N \neq \emptyset$. Define k to be the smallest value such that $S_k \neq \emptyset$. Further, let U_N be the union of the set of all tuples of t integers $(d_1, \dots, d_t)$ such that $\sum_{i=1}^t d_i$ is odd and

$$\sum_{i=1}^t C_i\binom{d_i}{2} + \sum_{i=1}^t A_i d_i = N,$$

with $|S_k|$ copies of the set of all tuples of t integers $(f_1, \dots, f_t)$ such that $\sum_{i=1}^t f_i$ is odd and

$$\sum_{i=1}^t C_i \frac{f_i(3f_i - 1)}{2} + k = N.$$

Finally, let V_N be the union of the set of all tuples of t integers $(e_1, \dots, e_t)$ such that $\sum_{i=1}^t e_i$ is odd and

$$\sum_{i=1}^t C_i \binom{e_i}{2} + \sum_{i=1}^t B_i e_i + m = N,$$

with $|S_k|$ copies of the set of all tuples of t integers $(f_1, \dots, f_t)$ such that $\sum_{i=1}^t f_i$ is even and

$$\sum_{i=1}^t C_i \frac{f_i(3f_i - 1)}{2} + k = N.$$

Then $|S_N| = |T_N|$ for all $N > k$ if and only if $|U_N| = |V_N|$ for all N.

Proof. For every N, let U_N^* be the set of all pairs consisting of a tuple of t partitions $(\mu_1, \dots, \mu_t)$ and an element of U_{N-x}, where $x = \sum_{i=1}^t C_i|\mu_i|$. Likewise, let V_N^* be the set of all pairs consisting of a tuple of t partitions $(\alpha_1, \dots, \alpha_t)$ and an element of V_{N-x}, where $x = \sum_{i=1}^t C_i|\alpha_i|$.

Obviously, if $|U_N| = |V_N|$ for all N, then $|U_N^*| = |V_N^*|$ for all N. Conversely, if $|U_N^*| = |V_N^*|$ and $|U_x| = |V_x|$ for all $x < N$, then all of the terms of $|U_N^*|$ and $|V_N^*|$ in which any of the partitions are nonempty cancel out, leaving $|U_N| = |V_N|$. Thus, $|U_N| = |V_N|$ for all N if and only if $|U_N^*| = |V_N^*|$ for all N. So we need only prove that $|S_N| = |T_N|$ for all $N > k$ if and only if $|U_N^*| = |V_N^*|$ for all N. By definition, $|S_N| = |T_N| = 0$ for all $N < k$, and $|T_k| = 0$ as well. Hence, for any $N < k$, $|U_N^*| = |V_N^*| = 0$, and it is easy to see that

$$\left|U_k^*\right| = |S_k| + 0 = 0 + |S_k| \cdot 1 = \left|V_k^*\right|.$$

Therefore, it suffices to show that $|U_N^*| - |S_N| = |V_N^*| - |T_N|$ for all $N > k$. This is equivalent to the existence of a bijection between the set of all $(\mu_1, \dots, \mu_t; f_1, \dots, f_t)$ such that $\sum_{i=1}^t f_i$ is odd and

$$\sum_{i=1}^t C_i|\mu_i| + \sum_{i=1}^t C_i f_i(3f_i - 1)/2 + k = N,$$

and the set of all $(\alpha_1, \dots, \alpha_t; f_1, \dots, f_t)$ such that $\sum_{i=1}^t f_i$ is even and

$$\sum_{i=1}^t C_i|\alpha_i| + \sum_{i=1}^t C_i f_i(3f_i - 1)/2 + k = N.$$

We can associate each element of either set with a tuple $(n_1, \dots, n_t)$, where, for any index i,

$$n_i = |\mu_i| + f_i(3f_i - 1)/2 \quad \text{or} \quad n_i = |\alpha_i| + f_i(3f_i - 1)/2,$$

as appropriate. For any given $(n_1, \dots, n_t)$, it is easy to see that the difference between the number of elements of the second set associated with $(n_1, \dots, n_t)$ and the number of elements of the first set associated with it, is

$$\prod_{i=1}^t \sum_{f_i \in \mathbb{Z}} (-1)^{f_i} p\left(n_i - \frac{f_i(3f_i - 1)}{2}\right).$$

Thus, by Lemma 3.7, unless $(n_1, \dots, n_t) = (0, \dots, 0)$, the last displayed formula is 0. But we have $\sum_{i=1}^t C_i n_i = N - k$, which implies that $n_i = 0$ for all i if and only if $N = k$. This proves the bijection between the two sets for any $N > k$, as desired. □

Remark 3.9. Notice that, given a bijection f between U_N and V_N, we can construct a bijection between S_N and T_N as follows. First, create a bijection f^* between U_N^* and V_N^* by having f^* leave their partition components unchanged and act as f on their integer components. Also, let g be a bijection between $U_N^* - S_N$ and $V_N^* - T_N$ (where these set differences are defined in the obvious way). Constructing a bijection between S_N and T_N is now a standard variation of the Garsia–Milne involution principle [6] (see e.g. [16], formula (2.33)). For instance, construct a graph where every element of U_N^* or V_N^* is a vertex, and there is an edge between any pair of elements that are in correspondence through f^* or g. Thus, each element of S_N or T_N has one edge in this graph, and each element of $U_N^* - S_N$ or $V_N^* - T_N$ has two, so every element of S_N is the other endpoint of a path ending at an element of T_N, and vice-versa. Therefore, the desired bijection between S_N and T_N maps each element of S_N or T_N to the other end of its path.

We are now ready for the bijective proofs of the two partition identities corresponding, respectively, to the modular equations of degree 5 and 11 of the Schröter, Russell and Ramanujan type (see Theorems 4.2 and 6.2 of [4]).

Lemma 3.10. *Condition* (i) *of Theorem* 2.3 *holds for* $N_0 = 3$, $t = 4$, $C_1 = C_2 = 2$, $C_3 = C_4 = 10$, $m = 3$, *and*

$$(A_1, A_2, A_3, A_4) = (1, 1, 5, 5), \qquad (B_1, B_2, B_3, B_4) = (0, 0, 0, 0).$$

Proof. It is easy to check that, in the notation of Lemma 3.8, we have $k = 1$ and $|S_k| = 4$. Thus, by Lemma 3.8, proving the lemma is equivalent to showing the existence of a bijection from the union of the set of all quadruples (d_1, d_2, d_3, d_4) with $d_1 + d_2 + d_3 + d_4$ odd and the set containing 4 copies of each quadruple (f_1, f_2, f_3, f_4) with $f_1 + f_2 + f_3 + f_4$ odd, to the union of the set of all quadruples (e_1, e_2, e_3, e_4) with $e_1 + e_2 + e_3 + e_4$ odd and the set containing 4 copies of each quadruple (f_1', f_2', f_3', f_4') with $f_1' + f_2' + f_3' + f_4'$ even, such that, for every pair of corresponding elements,

$$\begin{aligned} &2\binom{d_1}{2} + 2\binom{d_2}{2} + 10\binom{d_3}{2} + 10\binom{d_4}{2} + d_1 + d_2 + 5d_3 + 5d_4 \\ &\quad \text{or } f_1(3f_1 - 1) + f_2(3f_2 - 1) + 5f_3(3f_3 - 1) + 5f_4(3f_4 - 1) + 1 \\ &= 2\binom{e_1}{2} + 2\binom{e_2}{2} + 10\binom{e_3}{2} + 10\binom{e_4}{2} + 0e_1 + 0e_2 + 0e_3 + 0e_4 + 3 \\ &\quad \text{or } f_1'(3f_1' - 1) + f_2'(3f_2' - 1) + 5f_3'(3f_3' - 1) + 5f_4'(3f_4' - 1) + 1. \end{aligned}$$

Notice that, by replacing d_i with $1/2 - e_i$ in the above d-formula, for $i = 1, \dots, 4$, we obtain the e-formula. Thus, if we apply the map $d_i = 1/2 - e_i$, we can view the d-tuples and e-tuples as all being in the set $D = \{d \in \mathbb{Z}^4 \cup (\mathbb{Z} + 1/2)^4\colon\ d_1 + d_2 + d_3 + d_4 \in 2\mathbb{Z} + 1\}$. (A tuple $(d_1, d_2, d_3, d_4) \in D$ will in some sense be considered "of negative type" if the d_i are half-integers, since it will come from the opposite side of the bijection as the tuples in which the d_i are integers.) Furthermore, if we define a dot product so that

$$(d_1, d_2, d_3, d_4) \cdot (d_1', d_2', d_3', d_4') = d_1 d_1' + d_2 d_2' + 5d_3 d_3' + 5d_4 d_4',$$

then every point in this set corresponds to a quadruple with a value in the previous equation of its "length" squared.

Now, let

$$V_1 = (1, 1, 1, 1), \qquad V_2 = (1, -1, 1, -1), \qquad V_3 = (5, 5, -1, -1), \qquad V_4 = (5, -5, -1, 1).$$

It is easy to check that these vectors are pairwise orthogonal. For each $i = 1, \dots, 4$, set $M_i = \|V_i\|^2/12$. Thus, $M_1 = 1$, $M_2 = 1$, $M_3 = 5$, and $M_4 = 5$. Also, for arbitrary $d = (d_1, d_2, d_3, d_4)$, we clearly have that $d \cdot V_1$, $d \cdot V_2$, $d \cdot V_3$, and $d \cdot V_4$ are all odd integers, and $d \cdot V_3$ and $d \cdot V_4$ are divisible by 5. It follows that $d \cdot V_i$ is an odd multiple of M_i, for all d and i.

Now, for each $d \in D$ and i, let

$$r_i(d) = d - \frac{d \cdot V_i}{6M_i} V_i.$$

It is easy to check that $\|r_i(d)\| = \|d\|$, for all d and i. If $d \cdot V_i \equiv 0 \pmod{3M_i}$ then $\frac{d \cdot V_i}{6M_i}$ is a half-integer, so $r_i(d)$ is a vector that corresponds to an e-quadruple if d corresponds to a d-quadruple, and vice-versa. Hence, we can map every point in D that has a dot product with at least one of the V_i that is divisible by $3M_i$, to a point of the opposite type and the same value in the above d-formula, by sending it to $r_i(d)$, where i is the smallest integer such that $d \cdot V_i \equiv 0 \pmod{3M_i}$.

Note that $r_i(d) \cdot V_i = -d \cdot V_i$. Also, $r_i(d)$ has the same dot product with V_j as d does, for all $j \neq i$, because of the orthogonality of the vectors. This implies that $r_i(r_i(d)) \cdot V_j = d \cdot V_j$ for all j, and thus that $r_i(r_i(d)) = d$. Therefore, the above map is an involution. Hence, now we only need to consider the points in D whose dot products with V_i are not divisible by $3M_i$, for any i. Let $d \in D$ be any such point. For each i, let x_i be the nearest integer to $\frac{d \cdot V_i}{6M_i}$, $y_i = \frac{d \cdot V_i - 6M_i x_i}{M_i}$, and $z = d - \sum_{i=1}^{4} \frac{x_i}{2} V_i$.

For any i, $d \cdot V_i \equiv \pm M_i \pmod{6M_i}$, so $y_i = \pm 1$. Thus, by the Pythagorean Theorem we obtain:

$$\|d\|^2 = \sum_{i=1}^{4} \frac{(d \cdot V_i)^2}{\|V_i\|^2} = \sum_{i=1}^{4} \frac{(6M_i x_i + M_i y_i)^2}{12M_i} = 1 + \sum_{i=1}^{4} M_i x_i (3x_i + y_i).$$

We easily have that z must be either a quadruple of integers or a quadruple of half-integers, and $z \cdot V_i = y_i M_i = \pm M_i$, for each i. It is a simple exercise to verify that the only z that fit these criteria are: $(1, 0, 0, 0)$, $(-1, 0, 0, 0)$, $(0, 1, 0, 0)$, and $(0, -1, 0, 0)$. Therefore, we can choose a bijection between the 4 possible values of z and the 4 copies of each tuple $(f_1, \dots, f_4)$, and then map d to the copy of $(f_1, \dots, f_4) = (-x_1 y_1, \dots, -x_4 y_4)$ corresponding to z. It easily follows that

$$f_1(3f_1 - 1) + f_2(3f_2 - 1) + 5f_3(3f_3 - 1) + 5f_4(3f_4 - 1) + 1 = \|d\|^2.$$

Also, the y_i are determined by z, and for any given choice of z, since $x_i = -y_i f_i$, the only d that maps to a given tuple $(f_1, \dots, f_4)$ is $z - \sum_{i=1}^{4} \frac{y_i f_i}{2} V_i$. Furthermore, the entries of such d are all half-integers if $\sum_{i=1}^{4} f_i$ is odd, and integers if it is even. Thus, this map always takes elements of D corresponding to tuples of d's to tuples of f's with an even sum, and elements of D corresponding to tuples of e's to tuples of f's with an odd sum. This completes the proof of the lemma. □

Theorem 3.11. *Let S be the set containing 4 copies of the odd positive integers and 4 more copies of the odd positive multiples of 5, and T the set containing 4 copies of the even positive integers and 4 more copies of the even positive multiples of 5. Then, for any $N \geqslant 3$,*

$$D_S(N) = 8D_T(N-3).$$

Proof. Straightforward from Theorem 2.3 and Lemma 3.10. □

Lemma 3.12. *Condition* (i) *of Theorem 2.3 holds for $N_0 = 3$, $t = 2$, $C_1 = 2$, $C_2 = 22$, $m = 3$, and*

$$(A_1, A_2) = (1, 11), \qquad (B_1, B_2) = (0, 0).$$

Proof. It is easy to check that, in the notation of Lemma 3.8, we have $k = 1$ and $|S_k| = 2$. Thus, proving the lemma is equivalent to proving the existence of a bijection from the union of the set of all pairs (d_1, d_2) with $d_1 + d_2$ odd and the set containing 2 copies of each pair (f_1, f_2) with $f_1 + f_2$ odd, to the union of the set of all pairs (e_1, e_2) with $e_1 + e_2$ odd and the set containing 2 copies of each pair (f_1', f_2') with $f_1' + f_2'$ even, such that, for every pair of corresponding elements,

$$\begin{aligned} &2\binom{d_1}{2} + 22\binom{d_2}{2} + d_1 + 11d_2 \text{ or } f_1(3f_1 - 1) + 11f_2(3f_2 - 1) + 1 \\ &= 2\binom{e_1}{2} + 22\binom{e_2}{2} + 0e_1 + 0e_2 + 3 \text{ or } f_1'\bigl(3f_1' - 1\bigr) + 11f_2'\bigl(3f_2' - 1\bigr) + 1. \end{aligned}$$

Thus, similarly to what we did in Lemma 3.10, we can apply the map $d_1 = 1/2 - e_1$, $d_2 = e_2 - 1/2$, in order to view the d-pairs and e-pairs as both being in the same set $D = \{d \in \mathbb{Z}^2 \cup (\mathbb{Z} + 1/2)^2\colon\ d_1 + d_2 \in 2\mathbb{Z} + 1\}$. Notice that, for any $(d_1, d_2) \in D$, $d_1 + d_2$ is odd.

If $d_1 + d_2 \equiv 0 \pmod 3$, then the map

$$d_1' = d_1 - \frac{11(d_1 + d_2)}{6}, \qquad d_2' = d_2 - \frac{d_1 + d_2}{6}$$

yields a pair having the same value as (d_1, d_2), since $(d_1')^2 + 11(d_2')^2 = d_1^2 + 11d_2^2$. Furthermore, it is easy to see that (d_1', d_2') is an e-tuple (i.e., it has half-integer entries) if and only if (d_1, d_2) is a d-tuple (i.e., it has integer entries). Thus, this map cancels out all such elements.

If $d_1 + d_2 \not\equiv 0 \pmod 3$, then let x be the closest integer to $(d_1 + d_2)/6$, and let $d_1' = d_1 - 11x/2$ and $d_2' = d_2 - x/2$.

We have $d_1' + d_2' = \pm 1$, so there must exist an integer y such that $d_1' = y/2 \pm 1$ and $d_2' = -y/2$. This means that $d_1 = y/2 + 11x/2 \pm 1$ and $d_2 = -y/2 + x/2$. It easily follows that in this case $(d_1, d_2) \in D$ has a value of

$$d_1^2 + 11d_2^2 = 11x(3x \pm 1) + y(3y \pm 1) + 1,$$

and therefore we can map (d_1, d_2) to a copy of $(f_1 = \mp y, f_2 = \mp x)$.

Finally, it is a standard task to verify that (d_1, d_2) and $(-d_1, -d_2)$ get mapped to the same pair (f_1, f_2), and that, for any (d_1, d_2), $f_1 + f_2$ is even if (d_1, d_2) is a d-tuple and odd if it is an e-tuple. This completes the bijection and the proof of the lemma. □

Theorem 3.13. *Let S be the set containing* 2 *copies of the odd positive integers and* 2 *more copies of the odd multiples of* 11*, and T the set containing* 2 *copies of the even positive integers and* 2 *more copies of the even multiples of* 11*. Then, for any $N \geqslant 3$,*

$$D_S(N) = 2D_T(N-3).$$

Proof. Straightforward from Theorem 2.3 and Lemma 3.12. □

Finally, we state as a conjecture the "missing lemma" of this paper, whose bijective proof eludes us. By Theorem 2.3, such a proof will imply a bijective proof also for the last of the five identities of the Schröter, Russell and Ramanujan type (the one modulo 23, proved analytically in [4], Theorem 7.2), and will therefore complete our unified combinatorial approach to the five identities.

Conjecture 3.14. *Condition* (i) *of Theorem* 2.3 *holds for $N_0 = 3$, $t = 12$, $C_1 = \cdots = C_{12} = 46$, $m = 3$, and*

$$(A_1, \ldots, A_{12}) = (1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23),$$

$$(B_1, \ldots, B_{12}) = (0, 2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22).$$

Corollary to Conjecture 3.14. *Let S be the set containing one copy of the odd positive integers and one more copy of the odd positive multiples of* 23*, and T the set containing one copy of the even positive integers and one more copy of the even positive multiples of* 23*. Then, for any $N \geqslant 3$,*

$$D_S(N) = D_T(N-3).$$

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References

[1] G. Andrews, The Theory of Partitions, Encyclopedia Math. Appl., vol. II, Addison–Wesley, Reading, MA, London, Amsterdam, 1976.
[2] G. Andrews, K. Eriksson, Integer Partitions, Cambridge University Press, Cambridge, UK, 2004.
[3] B.C. Berndt, Ramanujan's Notebooks, Part III, Springer-Verlag, New York, 1991.
[4] B.C. Berndt, Partition-theoretic interpretations of certain modular equations of Schröter, Russell, and Ramanujan, Ann. Comb. 11 (2) (2007) 115–125.
[5] H.M. Farkas, I. Kra, Partitions and theta constant identities, in: The Mathematics of Leon Ehrenpreis, in: Contemp. Math., vol. 251, Amer. Math. Soc., Providence, RI, 2000, pp. 197–203.
[6] A.M. Garsia, S.C. Milne, A Rogers–Ramanujan bijection, J. Combin. Theory Ser. A 31 (1981) 289–339.
[7] M.D. Hirschhorn, The case of the mysterious sevens, Int. J. Number Theory 2 (2006) 213–216.
[8] S. Kim, Bijective proofs of partition identities arising from modular equations, J. Combin. Theory Ser. A 116 (3) (2009) 699–712.
[9] I.G. Macdonald, Symmetric Functions and Hall Polynomials, second ed., Oxford Math. Monogr., The Clarendon Press, Oxford University Press, 1995.
[10] I. Pak, Partition bijections, a survey, Ramanujan J. 12 (2006) 5–75.
[11] S. Ramanujan, Notebooks, vols. 1–2, Tata Institute of Fundamental Research, Bombay, India, 1957.
[12] R. Russell, On $\kappa\lambda - \kappa'\lambda'$ modular equations, Proc. Lond. Math. Soc. 19 (1887) 90–111.
[13] R. Russell, On modular equations, Proc. Lond. Math. Soc. 21 (1890) 351–395.
[14] C. Sandon, F. Zanello, Warnaar's bijection and colored partition identities, II, preprint.
[15] H. Schröter, Beiträge zur Theorie der elliptischen Funktionen, Acta Math. 5 (1884) 205–208.
[16] R. Stanley, Enumerative Combinatorics, vol. I, second ed., Cambridge University Press, Cambridge, UK, 2012.
[17] S.O. Warnaar, A generalization of the Farkas and Kra partition theorem for modulus 7, J. Combin. Theory Ser. A 110 (1) (2005) 43–52.
[18] E.M. Wright, An enumerative proof of an identity of Jacobi, J. Lond. Math. Soc. 40 (1965) 55–57.