**Abstract**

The purpose of this paper is to study the Lawson compactness of function spaces for L-domains. A basic notion of property \( RW \) for core compact spaces is introduced, which is proved to have a close relation to the Lawson compactness of function spaces for continuous L-domains as following:

1. Every Lawson compact continuous dcpo has property \( RW \) (via the Scott topology) and for each continuous L-domain, Lawson compactness is equivalent to property \( RW \);
2. Let \( P \) be a continuous dcpo with a least element. Then \([X \to P]\) is compact continuous for every core compact space \( X \) with property \( RW \) iff \( P \) is compact continuous L-domain;
3. Let \( X \) be a core compact space. Then \([X \to P]\) is compact for every compact continuous L-domain \( P \) iff \( X \) has property \( RW \).

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1. **Introduction**

Let \( X \) be a topological space and \( D \) a dcpo with the Scott topology, then the set \([X \to D]\) consisting of continuous morphisms from \( X \) to \( D \) with the pointwise order is
again a dcpo. This function set is one of the most basic structures, i.e., the function space in classical domain theory, and has been studied by many authors (see Lambrinos and Papadopoulos [8], Schwarz and Weck [15], Lawson [10], Liu and Liang [12], Erker and Keimel [2], Kou and Luo [6,7], etc.). One of the interesting questions arising from this important structure is to characterize those pairs \((X, P)\) such that \(X\) is a core compact space, \(P\) is a continuous dcpo, and \([X \to P]\) is a continuous dcpo for which the Lawson topology is compact (see [13, Problem 544]). It is proved in [12] that for a continuous dcpo \(D\) with bottom, \([X \to D]\) is continuous for every core compact space \(X\) if and only if \(D\) is an L-domain. Recently, Liang and Keimel in [11] defined an interesting new topological property \(W\) and showed that if \(D\) is a compact continuous L-domain and \(X\) is a core compact space with property \(W\) then \([X \to D]\) is a compact continuous dcpo. However, a number of continuous dcos with nice properties such as FS-domains need not have property \(W\) via the Scott topology. Moreover, even if \([X \to D]\) is compact for any compact continuous L-domain, the core compact space \(X\) also need not have property \(W\).

In this paper, we will go on considering the above question. We do not intend to give a complete solution to this problem but restrict ourselves also to continuous L-domains. We introduce a basic notion of property \(RW\) for core compact spaces. It is proved to be strictly weaker than property \(W\). The following Theorem shows that property \(RW\) of a core compact space \(X\) should be the most appropriate property such that \([X \to L]\) is compact for a compact continuous L-domain \(L\).

**Theorem.** Property \(RW\) and Lawson compactness have the following relations:

1. Every compact continuous domain has property \(RW\) and for continuous L-domains, Lawson compactness is equivalent to property \(RW\);
2. Let \(L\) be a continuous dcpo with bottom. Then \([X \to L]\) is compact for all core compact spaces \(X\) with property \(RW\) iff \(L\) is compact continuous L-domain;
3. Let \(X\) be a core compact space. Then \([X \to L]\) is compact for all compact continuous L-domains \(L\) iff \(X\) has property \(RW\).

We quickly recall some basic notions concerning continuous domains and function spaces (see, for example, Gierz et al. [3], Lawson [9], Abramsky and Jung [1] and Mislove [14]). A subset \(D\) of a partially ordered set \(P\) is directed if given \(x, y \in D\) there exists \(z \in D\) such that \(x, y \leq z\). A directed complete partially ordered set or dcpo is a partially ordered set \((P, \leq)\) such that every directed subset of \(P\) has a least upper bound (denoted by \(\lor\) in \(P\). For \(x, y \in P\), we write \(x \ll y\) if for every directed set \(D \subseteq P\) with \(y \leq \lor D\), there exists \(d \in D\) with \(x \leq d\). We set \(\downarrow y = \{x \in P: x \ll y\}\) and \(\uparrow y = \{x \in P: y \ll x\}\). A dcpo is called a continuous dcpo if for each \(y \in P\), \(\downarrow y\) is a directed set and \(y = \lor \downarrow y\). If a continuous dcpo is a complete lattice, then it is called a continuous lattice. Throughout this paper, \(X\) always denotes a topological space and \(\Omega(X)\) its open set lattice. A topological space \(X\) is core compact if \(\Omega(X)\) is a continuous lattice. A dcpo \(P\), as a topological space, is always equipped with the Scott topology \(\sigma(P)\). The Lawson topology \(\lambda(P)\) on a dcpo \(P\) is that of taking the family of \(\{U \setminus \uparrow x: U \in \sigma(P), x \in P\}\) as a subbase for the open sets. We call \(P\) Lawson compact or compact if \(P\) is a compact space with the Lawson topology. All functions in this paper are
Scott continuous maps and we write $[X \to P]$ for the set of continuous functions from $X$ to $P$.

A dcpo $P$ is an L-domain if $P$ has a least element $\perp$ and for each $x \in P$, the principle ideal $\downarrow x = \{ y \in P : y \leq x \}$ is a complete lattice. In this case, we write $\bigvee_x$ for the supremum operation in $\downarrow x$. The most important and beautiful property for L-domains is that all continuous L-domains with Scott continuous maps form one of the maximal Cartesian closed full subcategories of the category of continuous domains with least elements (see Jung [4,5]).

Next, we define step functions. For $A \subseteq X$, $a \in P$ (with a bottom $\perp$), define a map $A \downarrow a : X \to P$ by

$$A \downarrow a(x) = \begin{cases} a, & \text{if } x \in A, \\ \perp, & \text{if } x \notin A. \end{cases}$$

**Lemma 1.1** (see [12]). Let $X$ be a core compact space and $P$ a continuous L-domain, $a, b \in P$, $V \in \Omega(X)$, and $f \in [X \to P]$. Then $[X \to P]$ is a continuous L-domain satisfying:

1. $A \downarrow a \in [X \to P]$ if $A \in \Omega(X)$;
2. $V \downarrow b \ll f^{-1}(\uparrow a) \downarrow a$ if $b \ll a$ and $V \ll f^{-1}(\uparrow a)$;
3. $f = \bigvee \{ \bigvee_i V_i \downarrow b_i : i = 1, 2, \ldots, n \}$: $V_i \downarrow b_i \in \text{step}(f)$, $i = 1, 2, \ldots, n$, $n \geq 1$,

where

$$\text{step}(f) = \{ V \downarrow b : \exists a \in P, b \ll a, V \in \Omega(X), V \ll f^{-1}(\uparrow a) \}.$$

2. **Property RW**

The following definition is quoted from [11].

**Definition 2.1.** Let $(X, \Omega(X))$ be a topological space and let $V \in \Omega(X) \setminus \{ \emptyset \}$.

1. A family of open sets $\{ U_i : i \in I \}$ is called a decomposition of $V$ if
   (a) $\bigcup_{i \in I} U_i = V$;
   (b) $U_i \cap U_j = \emptyset$ whenever $i \neq j$ ($i, j \in I$);
   (c) $U_i \neq \emptyset$ for each $i \in I$.
2. A decomposition $\{ U_i : i \in I \}$ of $V$ is said to be maximal if for each decomposition $\{ V_j : j \in J \}$ of $V$ and for each $j \in J$, there is $i \in I$ such that $U_i \subseteq V_j$.
3. $X$ is said to have property $W$ if each nonempty open set has a finite maximal decomposition. If $X$ is also core compact, then we call it a $W$-space.

Using property $W$, Liang and Keimel in [11] gave an interesting characterization for the Lawson compactness of function spaces as following:

**Theorem 2.2.** Let $D$ be a compact continuous L-domain and $X$ a core compact space with property $W$. Then $[X \to D]$ is a compact continuous L-domain.
Notice that all compact continuous L-domains are FS-domains, so when we use an FS-domain instead of the above space \(X\) the function space is also compact. A natural question to ask is whether FS-domains have property \(W\). Let us see an example. Set 
\[X = N \cup \{\perp\} \] 
ordered as: \(\perp \subseteq n \text{ for all } n \in N\) and \(n \subseteq m\) iff \(n = m\). Then \((X, \subseteq)\) is bounded complete algebraic domain. Hence, \(X\) is an FS-domain and \([X \to D]\) is compact for all compact continuous L-domains \(D\), but the Scott open set \(N \subseteq X\) has no finite maximal decomposition, i.e., \(X\) does not have property \(W\). In the following, we will introduce a new topological property to overcome the shortcoming of property \(W\).

**Definition 2.3.** Let \(U, V \in \Omega(X)\) with \(V \subseteq U\). A decomposition \(\{V_i: i \in I\}\) of \(V\) is said to be relatively maximal to \(U\), denoted by \(\sum_V U\), if for each decomposition \(\{U_j: j \in J\}\) of \(U\) and for each \(j \in J\), \(U_j \cap V \neq \emptyset\) implies that 
\[U_j \cap V = \bigcup \{V_i: i \in I, U_j \cap V_i \neq \emptyset\}.\]

**Definition 2.4.** A topological space \(X\) is said to have property \(RW\) if for finitely many pairs \(V_1 \ll U_1, V_2 \ll U_2, \ldots, V_k \ll U_k\) in \(\Omega(X)\) with \(\bigcap_{i=1}^k V_i \neq \emptyset\), \(\bigcap_{i=1}^k V_i\) has a finite decomposition which is relatively maximal to \(\bigcap_{i=1}^k U_i\). A core compact space is called a \(RW\)-space if it has property \(RW\).

Here, the notation "\(RW\)" means that it is relative to property \(W\).

**Proposition 2.5.** Every \(W\)-space is a \(RW\)-space.

**Proof.** Let \(V_i, U_i \in \Omega(X)\) for \(i = 1, 2, \ldots, n\) such that \(V_i \ll U_i\) and \(\bigcap_{i=1}^n V_i \neq \emptyset\). Then \(\bigcap_{i=1}^n V_i\) has a finite maximal decomposition \(\{A_i: i' \in I'\}\). Let \(\{B_j: j \in J\}\) be a decomposition of \(\bigcap_{i=1}^n U_i\), and let 
\[V = \{A_{i'} \cap B_j: i' \in I', j \in J, A_{i'} \cap B_j \neq \emptyset\}.\]
Then \(V\) is a decomposition of \(\bigcap_{i=1}^n V_i\). From Definition 2.1, \(A_{i'} \cap B_j \in \Omega(X)\) implies \(A_{i'} \cap B_j \neq \emptyset\). Therefore, \(B_j \cap \bigcap_{i=1}^n V_i = \bigcup\{A_{i'}: i' \in I', B_j \cap A_{i'} \neq \emptyset\}\) if \(B_j \cap \bigcap_{i=1}^n V_i \neq \emptyset\). Hence by Definition 2.4, \(X\) is a \(RW\)-space. \(\square\)

In fact, the above example also shows that property \(RW\) is strictly weaker than property \(W\).

Recall that a space is locally connected if there exists a basis of open connected sets.

**Lemma 2.6.** Let \(X\) be a locally connected space and let \(V \ll U\) in the lattice \(\Omega(X)\) of open sets. Then \(V\) has a finite decomposition relatively maximal to \(U\).

**Proof.** Since \(X\) is locally connected, \(U\) has a maximal decomposition \(\{C_i\}_{i \in I}\), namely the collection of connected components (all of which are open by local connectivity). Then from \(V \ll U\), there exist finitely many \(C_1, C_2, \ldots, C_n\) covering \(V\) and \(C_i \cap V \neq \emptyset\) for each \(i \ll n\). Let \(V = \{C_i \cap V: i = 1, \ldots, n\}\), then \(V\) is easily seen to be relatively maximal to \(U\). \(\square\)
Note that since each Scott open filter is connected and all of them form a base for the Scott topology of a continuous domain, then every continuous domain is locally connected. Hence from Lemma 2.6, we have the following corollary.

**Corollary 2.7.** Let \( D \) be a continuous domain and \( U, V \subseteq D \) be two Scott open set with \( V \ll U \) in \( \sigma(D) \). Then \( V \) has a finite decomposition relatively maximal to \( U \).

Now we investigate the relation between Lawson compactness and property RW of continuous domains. A dcpo \( P \) is said to have property \( m \) if for any finite set \( F \subseteq P \), \( \bigcap_{a \in F} \uparrow a = \uparrow \mathrm{mub}\{a : a \in F\} \), where \( \mathrm{mub}\{a : a \in F\} \) is the set of all minimal upper bounds of \( F \) in \( P \). The following Lemma can be found in [1,4].

**Lemma 2.8.** For a continuous dcpo \( P \), the following are equivalent:

1. \( P \) is Lawson compact;
2. \( P \) is Scott quasi-compact and for all Scott open sets \( O, U, V \subseteq D \), \( O \ll U \) and \( O \ll V \) implies \( O \ll U \cap V \);
3. \( P \) is Scott quasi-compact and has property \( m \) and for all pairs \( a_1 \ll a, b_1 \ll b \) in \( P \), \( \uparrow a \cap \uparrow b \) is contained in a finite union of sets of the form \( \uparrow \uparrow c, c \in \mathrm{mub}\{a_1, b_1\} \).

**Theorem 2.9.** Every Lawson compact continuous domain \( L \) has property RW (with the Scott topology). Moreover, if \( L \) is an L-domain, then Lawson compactness and property RW are equivalent.

**Proof.** The first statement follows directly from Definition 2.4, Corollary 2.7 and Lemma 2.8(2). Let \( L \) be an L-domain with property RW and \( a_1 \ll a, b_1 \ll b \) in \( L \). Then there exist \( a_2, b_2 \in L \) such that \( a_1 \ll a_2 \ll a \) and \( b_1 \ll b_2 \ll b \), i.e., \( \uparrow a_2 \ll \uparrow a_1, \uparrow b_2 \ll \uparrow b_1 \); thus \( \uparrow a \cap \uparrow b \subseteq \uparrow a_2 \cap \uparrow b_2 \subseteq \uparrow a_1 \cap \uparrow b_1 \). By property RW of \( L \), \( \uparrow a_2 \cap \uparrow b_2 \) has a relatively maximal finite decomposition to \( \uparrow a_1 \cap \uparrow b_1 \), written

\[
\sum_{\uparrow a_1 \cap \uparrow b_1} \uparrow a_2 \cap \uparrow b_2 = \{ V_i : i = 1, 2, \ldots, k \}.
\]

Notice that since \( L \) is an L-domain, we have

\[\uparrow a_1 \cap \uparrow b_1 = \bigcup \{ \uparrow c : c \in \mathrm{mub}\{a_1, b_1\} \},\]

which is a decomposition of \( \uparrow a_1 \cap \uparrow b_1 \). By property RW again, for each \( i \), there exists a unique \( c \in \mathrm{mub}\{a_1, b_1\} \) such that \( V_i \subseteq \uparrow c \). Since \( \uparrow a \cap \uparrow b \subseteq \uparrow a_2 \cap \uparrow b_2 \), \( \uparrow a \cap \uparrow b \) is contained in a finite union of sets of the form \( \uparrow c, c \in \mathrm{mub}\{a_1, b_1\} \). So by Lemma 2.8 \( L \) is Lawson compact. \( \square \)

Since FS-domains, finite continuous domains and bifinite domains are Lawson compact (for details see Jung [4,5]), all of them have property RW. Generally, Lawson compactness is strictly stronger than property RW for a continuous dcpo. Let \( P = N \cup \{ a, b, \bot \} \), ordered as:

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\[a \ll b \ll \bot, \quad \bot \ll \bot.\]
(1) \( \forall m, n \in \mathbb{N}, m \sqsubseteq n \text{ iff } m \geq n \);
(2) \( \forall m \in \mathbb{N}, a, b \subseteq m \);
(3) \( \forall x \in P, \bot \sqsubseteq x \);
(4) \( a \nsubseteq b, b \nsubseteq a \).

Then \((P, \sqsubseteq)\) is a continuous dcpo. One can easily see that \(P\) is not Lawson compact, but has property \(RW\).

In fact, the above example can be extended as a property.

**Proposition 2.10.** If \(X\) is a core compact space such that \(U \cap V \neq \emptyset\) for all \(U, V \in \Omega(X) \setminus \{\emptyset\}\), then \(X\) is a RW-space.

**Proof.** It follows directly from Definition 2.4. \(\square\)

### 3. Compact continuous function spaces

In this section, we will consider the relations between property \(RW\) and compactness of function spaces.

Recall that for a core compact space \(X\), a continuous L-domain \(L\) and a Scott continuous function \(f : X \to L\),

\[
\text{step}(f) = \{ V \downarrow b : \exists a \in P, b \ll a, V \in \Omega(X), V \ll f^{-1}(\uparrow a) \}.
\]

**Lemma 3.1.** Let \(X\) be a RW-space and \(L\) a compact continuous L-domain. For any \(f_1, f_2 \in [X \to L]\), \(V_i \downarrow b_i \in \text{step}(f_k)\) (\(i = 1, 2, \ldots, n_1\) if \(k = 1\) and \(i = n_1 + 1, n_1 + 2, \ldots, n_1 + n_2\) if \(k = 2\)), then

\[
A = \left\{ \bigvee \{ V_i \downarrow b_i : i = 1, 2, \ldots, n_1 + n_2 \} : g \in \text{mub}\{f_1, f_2\} \right\}
\]

is finite.

**Proof.** Let \(I_1 = \{1, 2, \ldots, n_1\}\), \(I_2 = \{n_1 + 1, n_1 + 2, \ldots, n_1 + n_2\}\), and \(I = I_1 \cup I_2\). Then for each \(i \in I\), there exists \(a_i \in L\) such that \(b_i \ll a_i\) and \(V_i \ll f_k^{-1}(\uparrow a_i) (i \in I_k, k = 1, 2)\).

In order to give the proof, we need some notation. For a subset \(F\) of \(I\), let \(|F|\) denote the cardinal number of \(F\) and write

\[
\Phi = \left\{ F \subseteq I : \bigcap_{i \in F} V_i \neq \emptyset, |F| \geq 2 \right\};
\]

\[
M_I = \max\{k \in I : \exists F \in \Phi, |F| = k\};
\]

\[
\Phi_i = \left\{ F \in \Phi : |F| = i \right\} (i = 2, 3, \ldots, M_I).
\]

For each \(F \in \Phi\), set

\[
F_b = \{ c \in \text{mub}\{b_i : i \in F\} : \exists d \in \text{mub}\{a_i : i \in F\}, c \leq d \}.
\]
Notice that since $L$ is a compact continuous $L$-domain, then by Lemma 2.7(3), $F_b$ is finite for each $F \in \Phi$. Let

$$F = \{ \bot \} \cup \{ b_i: \ i \in I \} \cup \bigcup \{ F_b: \ F \in \Phi \},$$

then $F$ is finite. For each $g \in \text{mub}\{ f_1, f_2 \}$, set

$$G(g) = \bigvee \{ V_i \cap b_i: \ i \in I_k, \ k = 1, 2 \}.$$

Since $L$ is an $L$-domain, the image $G(g)(L)$ is contained in $F$ (see the proof of Lemma 2.4 in [11]).

Now for each $F \in \Phi$, let $U_1 = \bigcap_{i \in F \cap I_1} f_1^{-1}(\uparrow a_i)$ and $U_2 = \bigcap_{i \in F \cap I_2} f_2^{-1}(\uparrow a_i)$, and let $V(F)$ be the finite decomposition of $\bigcap_{i \in F} V_i$ relatively maximal to $U_1 \cap U_2$. Set

$$B = \left\{ V \left( \bigcup_{F \in \Phi_k} \bigcap_{i \in F} V_i: \ V \in V(F), \ F \in \Phi_k, \ k = 2, 3, \ldots, M \right) \right\}$$

$$\cup \left\{ V_i \setminus \bigcup_{F \in \Phi_2} \bigcap_{i \in F} V_i: \ i \in I \right\},$$

then $B$ is a finite family. Let $\text{Fun}(B)$ be a subset of $[X \rightarrow L]$ such that

$$f \in \text{Fun}(B) \iff \forall B \in B, \ \exists e_B \in F \text{ such that } f(B) = \{ e_B \} \text{ and } f(X \setminus \bigcup B) = \{ \bot \}.$$ 

then $\text{Fun}(B)$ is finite. We claim that the set $\{ G(g): \ g \in \text{mub}\{ f_1, f_2 \} \}$ is contained in $\text{Fun}(B)$. For each $g \in \text{mub}\{ f_1, f_2 \}$ and each $F \in \Phi$, we have

$$g(U_1 \cap U_2) \subseteq \bigcap_{i \in F} \uparrow a_i.$$ 

Since $L$ is an $L$-domain, we have

$$\bigcap_{i \in F} \uparrow a_i = \bigcup \{ \uparrow x: \ x \in \text{mub}\{ a_i: \ i \in F \} \}$$

and the right family is a decomposition of $\bigcap_{i \in F} \uparrow a_i$. Now let

$$U = \{ g^{-1}(\uparrow x) \cap U_1 \cap U_2: \ x \in \text{mub}\{ a_i: \ i \in F \} \},$$

then $U$ is a decomposition of $U_1 \cap U_2$. Hence, from the RW property of $X$ and the definition of $V(F)$, for each $x \in \text{mub}\{ a_i: \ i \in F \}$, if $g^{-1}(\uparrow x) \cap U_1 \cap U_2 \cap \bigcap_{i \in F} V_i \neq \emptyset$, then

$$g^{-1}(\uparrow x) \cap U_1 \cap U_2 \cap \bigcap_{i \in F} V_i = \{ V \in V(F): \ V \cap g^{-1}(\uparrow x) \cap U_1 \cap U_2 \neq \emptyset \}.$$ 

This shows that for each $V \in V(F)$, there exists a unique $x \in \text{mub}\{ a_i: \ i \in F \}$ such that $V \subseteq g^{-1}(\uparrow x)$. Notice that since $L$ is an $L$-domain, there is a unique $c \in \text{mub}\{ b_i: \ i \in F \}$ such that $c \leq x$. Hence,

$$G(g)\left( V \left( \bigcup_{F \in \Phi_k} \bigcap_{i \in F} V_i \right) \right) = \{ c \}.$$
and for each $i \in I$,
\[ G(g) \left( V_i \setminus \bigcup_{F \in \Phi} \bigcap_{i \in F} V_i \right) = \{b_i\}. \]

This shows that the values of the function $G(g)$ on $\bigcup B$ are completely determined by the elements of $B$, i.e., $G(g) \in \text{Fun}(B)$. This completes the proof. \(\square\)

**Theorem 3.2.** Let $X$ be a RW-space and $L$ a compact continuous L-domain. Then $[X \to L]$ is a compact continuous L-domain.

**Proof.** Obviously $[X \to L]$ is a continuous L-domain. Suppose $f_i, g_i \in [X \to L]$ with $f_i \ll g_i$ ($i = 1, 2$). Then by Lemma 1.1, there are finitely many $V_i \downarrow b_i \in \text{step}(g_k)$ ($i \in I_1$ if $k = 1$ and $i \in I_2$ if $k = 2$) such that
\[ f_k \ll \bigvee_{g_k} \{V_i \downarrow b_i : i \in I_k\} \ll g_k \quad (k = 1, 2). \]

By Lemma 3.1, the set
\[ A = \left\{ \bigvee_{g} \{V_i \downarrow b_i : i \in I_1 \cup I_2\} : g \in \text{mub}\{g_1, g_2\} \right\} \]
is finite. Since $\text{mub}\{g_1, g_2\} \subseteq \bigcup_{h \in A} \uparrow h$, there exists a finite set $F \subseteq \text{mub}\{f_1, f_2\}$ such that $\text{mub}\{g_1, g_2\} \subseteq \bigcup_{h \in F} \uparrow h$. By Lemma 2.7, $[X \to L]$ is compact. \(\square\)

Since compact continuous domains have property RW (Theorem 2.8), we have

**Corollary 3.3.** Let $D, E$ be compact continuous domains. If $E$ is an L-domain, then $[D \to E]$ is a compact continuous domain.

**Theorem 3.4** [11]. For a continuous dcpo $P$ with least element, the following are equivalent:

1. $[X \to P]$ is a continuous dcpo for all core compact space $X$;
2. $[X \to P]$ is a continuous dcpo for all core compact space $X$ with property $W$;
3. $P$ is an L-domain.

Since $W$-spaces are RW-space (Proposition 2.5), it follows from Theorems 3.2 and 3.4 we have

**Theorem 3.5.** Let $P$ be a continuous dcpo with a least element. Then the following are equivalent:

1. $[X \to P]$ is a compact continuous dcpo for all core compact spaces $X$ with property RW;
2. $P$ is a compact continuous L-domain.
Property RW also can be characterized by the compactness of function spaces as follows:

**Theorem 3.6.** Let $X$ be a core compact space. Then $X$ is a RW-space if and only if $[X \to L]$ is compact for every compact continuous L-domain $L$.

**Proof.** By Theorem 3.5, we only need to show the “if” part. Let $V_i, U_i \in \Omega(X)$ with $V_i \ll U_i$ ($i = 1, 2, \ldots, k$ and $k \geq 2$) and $\bigcap_{i \in I} V_i \neq \emptyset$. First, we use the index set $I$ to construct a compact continuous L-domain. Set $I = \{1, 2, \ldots, k\}$, then the power set $2^I$ of $I$ is finite. Let $L = \{c_1, c_2\} \cup 2^I \backslash \{I\}$, ordered as following:

- $\forall F \in 2^I \backslash \{I\}, F \subseteq c_1, c_2$,
- $\forall F, G \in 2^I \backslash \{I\}, F \subseteq G$ if $F \subseteq G$, and
- $c_1 \nsubseteq c_2, c_2 \nsubseteq c_1$.

One can easily see that $(L, \subseteq)$ is a compact algebraic L-domain and every element of $L$ is compact. For convenience, set $\{i\} = i$ ($i = 1, 2, \ldots, k$) and $\emptyset = \perp$. For each $x \in X$, let $I_x = \{i \in I : x \in V_i\}$. By the structure of $L$, one can see that all functions in $\mub\{f_i : i \in I\}$ take the same value $I_x$ if $x \in X \backslash \bigcap_{i \in I} V_i$. It is similar for the functions in $\mub\{g_i : i \in I\}$. Hence,

$$\forall f \in \mub\{f_i : i \in I\}, \quad f^{-1}(c_1) \cup f^{-1}(c_2) = \bigcap_{i \in I} V_i, \quad f^{-1}(c_1) \cap f^{-1}(c_2) = \emptyset,$$

$$\forall g \in \mub\{g_i : i \in I\}, \quad g^{-1}(c_1) \cup g^{-1}(c_2) = \bigcap_{i \in I} U_i, \quad g^{-1}(c_1) \cap g^{-1}(c_2) = \emptyset.$$

For each $g \in \mub\{g_i : i \in I\}$, set

$$h_g = \bigvee_{f} \{f_i : i \in I\}.$$

Then the set

$$A = \{h_g : g \in \mub\{g_i : i \in I\}\}$$

is finite and $A \subseteq \mub\{f_i : i \in I\}$ as $[X \to L]$ is a compact continuous L-domain. Let $A = \{h_j : j \in J, \quad |J| < \aleph_0\}$ with $h_i \neq h_j$ whenever $i \neq j$, and let

$$V = \left\{\bigcap_{i \in J} h_j^{-1}(c_k) : k = 1, 2\right\}.$$

Then by (1) and (2), $V$ is a finite decomposition of $\bigcap_{i \in I} V_i$ (without losing generality, we may assume that every member of $V$ is non-empty). We claim that $V$ is relatively maximal.
to $\bigcap_{i \in I} U_i$. Suppose that $U = \{U_{\alpha}; \alpha \in \nabla\}$ is decomposition of $\bigcap_{i \in I} U_i$. For any $U_{\alpha} \in U$ with $U_{\alpha} \cap \bigcap_{i \in I} V_i \neq \emptyset$, we define a map $g_{\alpha} : X \to L$ as following:

$$g_{\alpha}(x) = \begin{cases} 
    c_1, & \text{if } x \in U_{\alpha}, \\
    c_2, & \text{if } x \in \left(\bigcup U\right) \setminus U_{\alpha}, \\
    I_x, & \text{if } x \in X \setminus \left(\bigcup U\right),
\end{cases}$$

where $I_x = \{i \in I : x \in U_i\}$. One can easily see that $g_{\alpha}$ is Scott continuous and $g_{\alpha} \in \text{mub}\{g_i : i \in I\}$. Then there is $h_j \in A$ such that $h_j \leq g_{\alpha}$. Hence, $U_{\alpha} = g_{\alpha}^{-1}(c_1) \supseteq h_j^{-1}(c_1) \neq \emptyset$. By Definition 2.3, $V$ is a finite decomposition relatively maximal of $\bigcap_{i \in I} V_i$ to $\bigcap_{i \in I} U_i$, and hence the theorem is proved. $\square$

From the above results, property $RW$ of a core compact space $X$ should be the most appropriate property such that $[X \to L]$ is compact for a compact continuous $L$-domain $L$. As proved in Theorem 2.8, every compact continuous dcpo has property $RW$, but whether a coherent space has property $RW$ is unknown. Suppose that $L$ is a continuous $L$-domain and the Isbell topology on $[X \to L]$ agrees the Scott topology for all $RW$-spaces. Is $L$ compact? We leave these questions to the interested readers for further consideration. $\square$

References


