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On the discrete spectrum of non-selfadjoint operators

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Abstract

We prove quantitative bounds on the eigenvalues of non-selfadjoint unbounded operators obtained from selfadjoint operators by a perturbation that is relatively-Schatten. These bounds are applied to obtain new results on the distribution of eigenvalues of Schrödinger operators with complex potentials.

Keywords: Non-selfadjoint operators; Eigenvalues; Schrödinger operators; Complex potentials

1. Introduction and results

This paper is devoted to the study of the set of isolated eigenvalues of non-selfadjoint unbounded operators acting on a Hilbert space $\mathcal{H}$ (throughout this paper all Hilbert spaces are assumed to be complex and separable). More precisely, we are interested in operators of the form

$$H = H_0 + M, \qquad \mathrm{dom}(H) = \mathrm{dom}(H_0),$$

where H_0 is selfadjoint with spectrum $\sigma(H_0) = [0, \infty)$, and M is a relatively-compact perturbation of H_0, i.e. $\mathrm{dom}(H_0) \subset \mathrm{dom}(M)$ and $M[\lambda - H_0]^{-1}$ is compact for some (hence all)

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$\lambda \in \mathbb{C} \setminus [0, \infty)$. Under these assumptions, the spectrum of H is included in a half-plane and we assume that the same is true for its numerical range $N(H)$, i.e. there exists an $\omega_0 \geqslant 0$ such that

$$\sigma(H) \subset \overline{N(H)} \subset \{\lambda \in \mathbb{C}\colon \Re(\lambda) \geqslant -\omega_0\} =: \mathbb{H}_{\omega_0}.$$

By Weyl's theorem, the essential spectra of H and H_0 coincide, so that $\sigma_{\mathrm{ess}}(H) = [0, \infty)$. However, the perturbation M may give rise to a discrete set of eigenvalues, whose only possible limiting points are on the interval $[0, \infty)$ or at infinity. This set of eigenvalues is called the discrete spectrum of H and will be denoted by $\sigma_d(H)$.

It is the aim of this work to obtain further information on $\sigma_d(H)$, of a quantitative nature, by imposing additional restrictions on the perturbation M. We shall assume, first of all, that for $p > 0$, M is relatively p-Schatten, that is, for some (hence all) $\lambda \in \mathbb{C} \setminus [0, \infty)$

$$M[\lambda - H_0]^{-1} \in \mathbf{S}_p, \tag{1}$$

where $\mathbf{S}_p$ is the Schatten class of order p (see Section 2.1 for the relevant definitions). Moreover, we assume that the p-Schatten norm of $M[\lambda - H_0]^{-1}$ satisfies a certain bound (see (2) below), which in particular restricts its growth as λ approaches the spectrum of H_0, and as $|\lambda|$ goes to infinity. It will be shown that these assumptions can be used to derive quantitative information on the set of eigenvalues, in particular on how *fast* sequences in $\sigma_d(H)$ must converge to $[0, \infty)$. Our main abstract result is presented and discussed in Section 1.1 below.

In Section 1.2 we apply our abstract theorem to Schrödinger operators with a complex potential. In this way we demonstrate both that the hypotheses of our theorem are natural (in the sense that they can be verified in concrete cases), and that it yields new results for a problem which has previously been studied by other methods.

Our abstract result will be proved by constructing a holomorphic function whose zeros are the eigenvalues of H, and using complex analysis to obtain information on these zeros. Variants of this approach were used previously, e.g. in [2,6]. One of our main tools will be a result of Borichev, Golinskii and Kupin [2] providing bounds on the zeros of a holomorphic function in the unit disk, in terms of its growth near the boundary (see also [9]). We note that theorems of this type are a classical theme in complex function theory, see e.g. [8], but most results in this direction, unlike those of [2], are not suitable for dealing with the kind of holomorphic functions that arise here, which grow exponentially near the boundary. In [2] the complex analysis result was used to obtain inequalities for eigenvalues of Jacobi operators.

1.1. Eigenvalue inequalities for general operators

The following theorem is our main result. It will be proved in Section 3.

Theorem 1. *Let H_0 be a selfadjoint operator on a Hilbert space $\mathcal{H}$, with $\sigma(H_0) = [0, \infty)$, $H = H_0 + M$, where M satisfies* (1) *for some $p > 0$, and $N(H) \subset \mathbb{H}_{\omega_0}$. Assume that for $\mu \in \mathbb{C}$ with $\Im(\mu) > 0$,*

$$\big\| M\big[\mu^2 - H_0\big]^{-1}\big\|_{\mathbf{S}_p}^p \leqslant K_0 \frac{|\mu + i|^\delta}{|\Im(\mu)|^\alpha |\mu|^\nu}, \tag{2}$$

where $\alpha, \delta, \nu, K_0 \geqslant 0$. Let $0 < \tau < 1$, and define

$$\begin{aligned}
\rho &= \delta + 2(p-\alpha) - \nu, \\
\eta_1 &= \frac{1}{2}(\alpha + 1 + \tau), \\
\eta_2 &= \frac{1}{2}(\nu - 1 + \tau)_+, \\
\eta_3 &= \frac{1}{2}(\alpha + \nu - \delta) - \tau,
\end{aligned} \tag{3}$$

where $x_+ = \max(x, 0)$. *Then the following holds,*

$$\sum_{\lambda \in \sigma_d(H)} \frac{\operatorname{dist}(\lambda, [0,\infty))^{2\eta_1}}{|\lambda|^{\eta_1 - \eta_2}(|\lambda| + 1)^{\eta_1 + \eta_2 - \eta_3}} \leqslant C K_0, \tag{4}$$

where $C = C(p, \alpha, \delta, \nu, \tau)(1 + \omega_0)^{\eta_1 + \eta_2 + \frac{1}{2}(\alpha + \rho)}$.

In the summation over the eigenvalues in (4) and elsewhere in this article, each eigenvalue is summed according to its algebraic multiplicity. The constants used in the inequalities throughout this article will be regarded as generic, i.e. the value of a constant may change from line to line. However, we will always carefully indicate the parameters that a constant depends on.

Noting that, for any $\varepsilon > 0$ and $|\lambda| \geqslant \varepsilon$

$$\frac{1}{|\lambda| + 1} = \frac{1}{|\lambda|} \frac{1}{1 + |\lambda|^{-1}} \geqslant \frac{1}{|\lambda|} \frac{1}{1 + \varepsilon^{-1}},$$

we obtain the following corollary of Theorem 1.

Corollary 1. *Under the assumptions of Theorem* 1 *we have, for all* $\varepsilon > 0$,

$$\sum_{\lambda \in \sigma_d(H),\, |\lambda| \geqslant \varepsilon} \frac{\operatorname{dist}(\lambda, [0,\infty))^{2\eta_1}}{|\lambda|^{2\eta_1 - \eta_3}} \leqslant C K_0 \left(1 + \frac{1}{\varepsilon}\right)^{\eta_1 + \eta_2 - \eta_3} \tag{5}$$

where $C = C(p, \alpha, \delta, \nu, \tau)(1 + \omega_0)^{\eta_1 + \eta_2 + \frac{1}{2}(\alpha + \rho)}$.

To demonstrate that these inequalities contain a lot of information about the discrete spectrum of H, we would like to discuss some of their immediate consequences.

The finiteness of the sum on the LHS of (4) has consequences regarding sequences $\{\lambda_k\}$ of isolated eigenvalues converging to some $\lambda^* \in \sigma_{\mathrm{ess}}(H)$. Taking a subsequence, we can suppose that one of the following options holds:

(i) $\lambda^* = 0$ and $\Re(\lambda_k) \leqslant 0$ for all k.
(ii) $\lambda^* = 0$ and $\Re(\lambda_k) > 0$ for all k.
(iii) $\lambda^* \in (0, \infty)$.

In case (i), since $\operatorname{dist}(\lambda_k, [0,\infty)) = |\lambda_k|$, (4) implies

$$\sum_{k=1}^{\infty} |\lambda_k|^{\eta_1+\eta_2} < \infty,$$

which means that any such sequence must converge to 0 at a sufficiently fast rate. In case (ii), (4) implies

$$\sum_{k=1}^{\infty} \frac{|\Im(\lambda_k)|^{2\eta_1}}{|\lambda_k|^{\eta_1-\eta_2}} < \infty,$$

and in case (iii) we obtain

$$\sum_{k=1}^{\infty} \left|\Im(\lambda_k)\right|^{2\eta_1} < \infty,$$

so that the sequence must converge to the real line sufficiently fast.

Theorem 1 also provides information about divergent sequences of eigenvalues. For example, if $\{\lambda_k\}$ is a sequence of eigenvalues which stays bounded away from $[0,\infty)$, that is

$$\operatorname{dist}\bigl(\lambda_k, [0,\infty)\bigr) \geqslant \delta, \tag{6}$$

for some $\delta > 0$ and all k, then (5) implies that

$$\sum_{k=1}^{\infty} \frac{1}{|\lambda_k|^{2\eta_1-\eta_3}} < \infty. \tag{7}$$

In case $2\eta_1 > \eta_3$, (7) implies that $|\lambda_k|$ must go to infinity sufficiently fast and in case $2\eta_1 \leqslant \eta_3$, (7) implies by contradiction that the number of eigenvalues outside any δ-neighbourhood of $[0,\infty)$ must be finite.

For all these results on the asymptotic behaviour of sequences of eigenvalues, it would be of interest to know whether they are sharp, that is, if possible, to construct examples of operators that have precisely the types of asymptotic behaviour indicated above, but no better.

1.2. Applications to Schrödinger operators

We consider Schrödinger operators on $L^2(\mathbb{R}^d)$ that is,

$$H_0 = -\Delta, \qquad H = H_0 + M_V,$$

where M_V is the operator of multiplication with a complex-valued potential V.

Although the study of such operators has recently attracted increasing attention (see the papers by Bruneau and Ouhabaz [3], Abramov et al. [1], Davies [4], Frank et al. [10] and the monograph by Davies [5]), relatively little is known in comparison to the case of real-valued potentials. What is known indicates some essential differences in the behaviour of the discrete spectrum in the real and complex cases. For example, while in the real case the condition $|V(x)| = O((1+|x|)^{-2-\epsilon})$

is sufficient to guarantee that the number of eigenvalues is finite, Pavlov [12] (in dimension one) has constructed complex potentials with $|V(x)| = O(e^{-c|x|^\alpha})$, where $\alpha < \frac{1}{2}$, for which there exists an infinite sequence of eigenvalues, converging to points in $(0, \infty)$.

For $f \in L^2$ we set $\langle f, f\rangle = \|f\|_{L^2}^2$. The following theorem will be proved in Section 4.

Theorem 2. *Let $H = H_0 + M_V$ acting on $L^2(\mathbb{R}^d)$, where $d \geqslant 2$. Suppose that $V \in L^p(\mathbb{R}^d)$, where $p > \frac{d}{2}$ and $p \geqslant 2$, and let ω_0 be such that*

$$\langle H_0 f, f\rangle + \big\langle \Re(V) f, f\big\rangle \geqslant -\omega_0 \langle f, f\rangle, \quad f \in \mathrm{dom}(H_0).$$

Then, for any $\tau \in (0,1)$, the following holds: if $p - \frac{d}{2} \geqslant 1 - \tau$, then

$$\sum_{\lambda \in \sigma_d(H)} \frac{\mathrm{dist}(\lambda, [0,\infty))^{p+\tau}}{|\lambda|^{\frac{d}{4}+\frac{1}{2}}(|\lambda|+1)^{\frac{d}{4}-\frac{1}{2}+2\tau}} \leqslant C_1 \int_{\mathbb{R}^d} \big|V(y)\big|^p\, dy, \tag{8}$$

where $C_1 = C(d,p,\tau)(1+\omega_0)^{(\frac{d}{4}+p-\frac{1}{2}+\tau)}$, and if $p - \frac{d}{2} < 1 - \tau$, then

$$\sum_{\lambda \in \sigma_d(H)} \frac{\mathrm{dist}(\lambda, [0,\infty))^{p+\tau}}{|\lambda|^{\frac{1}{2}(p+\tau)}(|\lambda|+1)^{\frac{1}{2}(d-p+3\tau)}} \leqslant C_2 \int_{\mathbb{R}^d} \big|V(y)\big|^p\, dy, \tag{9}$$

where $C_2 = C(d,p,\tau)(1+\omega_0)^{\frac{1}{2}(d+p+\tau)}$.

Using the same estimate as in the derivation of Corollary 1, the previous theorem implies the following result.

Corollary 2. *Given the assumptions of Theorem 2, for any $\tau \in (0,1)$ and $\varepsilon > 0$,*

$$\sum_{\lambda \in \sigma_d(H),\, |\lambda| \geqslant \varepsilon} \frac{\mathrm{dist}(\lambda, [0,\infty))^{p+\tau}}{|\lambda|^{\frac{d}{2}+2\tau}} \leqslant C\left(1+\frac{1}{\varepsilon}\right)^{\gamma} \int_{\mathbb{R}^d} \big|V(y)\big|^p\, dy, \tag{10}$$

where

$$\gamma = \frac{1}{2}\left(d - p + 3\tau + \left(p - \frac{d}{2} - 1 + \tau\right)_+\right)$$

and

$$C = C(d,p,\tau)(1+\omega_0)^{\frac{1}{2}(d+p+\tau+(p-\frac{d}{2}-1+\tau)_+)}.$$

Let us mention that we can derive similar results for dimension $d = 1$, but since some details of the proof are slightly different, it is not presented here.

It is interesting to compare the above estimates with the following known result for eigenvalues *outside* of a sector $\{\lambda\colon |\Im(\lambda)| < \chi\Re(\lambda)\}$, where $\chi > 0$.

Theorem 3. *(See Frank et al. [10].) Let $H = H_0 + M_V$ acting on $L^2(\mathbb{R}^d)$, where $d \geqslant 1$. Suppose that $V \in L^{\frac{d}{2}+\kappa}(\mathbb{R}^d)$ with $\kappa \geqslant 1$. Then, for any $\chi > 0$,*

$$\sum_{\lambda\in\sigma_d(H),\,|\Im(\lambda)|\geqslant\chi\Re(\lambda)} |\lambda|^\kappa \leqslant C(d,\kappa)\left(1+\frac{2}{\chi}\right)^{\frac{d}{2}+\kappa}\int_{\mathbb{R}^d}\left|V(y)\right|^{\frac{d}{2}+\kappa} dy. \tag{11}$$

These generalized Lieb–Thirring inequalities were proved by reduction to a selfadjoint problem, and employing the selfadjoint Lieb–Thirring inequalities (a similar approach has been used in [3]). The authors of [10] conjecture that the restriction $\kappa \geqslant 1$ is superfluous, and that (11) might be true for κ fulfilling the same restrictions as in the selfadjoint case, that is $\kappa \geqslant 0$ when $d \geqslant 3$, $\kappa > 0$ when $d = 2$, and $\kappa \geqslant \frac{1}{2}$ when $d = 1$.

Since the sum in (11) excludes a sector containing the positive real axis, Theorem 3 does not provide explicit information on sequences of eigenvalues converging to some point in $(0,\infty)$, as is provided in Theorem 2. However, the following corollary of Theorem 3, which will be proved in Section 4, gives a bound on a sum over *all* eigenvalues. This corollary is of interest in itself, and also allows a more direct comparison with Theorem 2.

Corollary 3. *Let $H = H_0 + M_V$ acting on $L^2(\mathbb{R}^d)$, where $d \geqslant 1$. Suppose that $V \in L^p(\mathbb{R}^d)$ with $p - \frac{d}{2} \geqslant 1$. Then, for any $0 < \tau < 1$,*

$$\sum_{\lambda\in\sigma_d(H)} \frac{\operatorname{dist}(\lambda,[0,\infty))^{p+\tau}}{|\lambda|^{\frac{d}{2}+\tau}} \leqslant C(d,p,\tau)\int_{\mathbb{R}^d}\left|V(y)\right|^p dy. \tag{12}$$

The similarity between the estimates (8), (9) and (12) is apparent. In particular, as can be seen by comparing Corollary 3 with Corollary 2, for eigenvalues accumulating on $(0,\infty)$, Theorem 2 and Corollary 3 provide exactly the same estimates. However, for eigenvalues accumulating at 0, the estimates provided in (12) are stronger than the corresponding estimates provided in (8) and (9). On the other hand, whereas (12) has been proved only for $p - \frac{d}{2} \geqslant 1$, the inequalities (8) and (9) remain true as long as $p - \frac{d}{2} > (2 - \frac{d}{2})_+$, e.g. for any $p > \frac{d}{2}$ when $d \geqslant 4$. Thus our method allows to prove a somewhat weaker inequality, which is valid for a wider range of potentials. Whether this trade-off is an essential feature of the problem (indicating a different behaviour of the discrete spectrum at 0 and ∞, respectively), or whether it is just an artifact of the methods used in the proofs of [10] and in our proof of Theorem 2 is an open question, related to the conjecture made in [10] which was mentioned above.

2. Preliminaries

In this section we gather some results about determinants and holomorphic functions needed for the proof of Theorem 1. Also, some useful inequalities are derived.

2.1. Schatten classes and determinants

For a Hilbert space $\mathcal{H}$ let $\mathbf{C}(\mathcal{H})$ and $\mathbf{B}(\mathcal{H})$ denote the classes of closed and of bounded linear operators on $\mathcal{H}$, respectively. We denote the ideal of all compact operators on $\mathcal{H}$ by $\mathbf{S}_\infty$ and the ideal of all Schatten class operators by $\mathbf{S}_p$, $p > 0$, i.e. a compact operator $C \in \mathbf{S}_p$ if

$$\|C\|_{\mathbf{S}_p}^p = \sum_{n=1}^{\infty} \mu_n(C)^p < \infty$$

where $\mu_n(C)$ denotes the n-th singular value of C. For $C \in \mathbf{S}_n$, $n \in \mathbb{N}$, one can define the (regularized) determinant

$$\det_n(I - C) = \prod_{\lambda \in \sigma(C)} \left[(1-\lambda) \exp\left(\sum_{j=1}^{n-1} \frac{\lambda^j}{j} \right) \right],$$

having the following properties (see e.g. Dunford and Schwartz [7], Gohberg and Krein [11] or Simon [15]):

1. $I - C$ is invertible if and only if $\det_n(I - C) \neq 0$.
2. $\det_n(I) = 1$.
3. $\det_n(I - AB) = \det_n(I - BA)$ for $A, B \in \mathbf{B}(\mathcal{H})$ with $AB, BA \in \mathbf{S}_n$.
4. If $C(\lambda) \in \mathbf{S}_n$ depends holomorphically on $\lambda \in \Omega$, where $\Omega \subset \mathbb{C}$ is open, then $\det_n(I - C(\lambda))$ is holomorphic on Ω.
5. If $C \in \mathbf{S}_p$ for some $p > 0$, then $C \in \mathbf{S}_{\lceil p \rceil}$, where

   $$\lceil p \rceil = \min\{n \in \mathbb{N}\colon\ n \geqslant p\},$$

   and the following inequality holds,

   $$\left|\det_{\lceil p \rceil}(I - C)\right| \leqslant \exp\left(\Gamma_p \|C\|_{\mathbf{S}_p}^p\right), \tag{13}$$

   where Γ_p is some positive constant, see [7, p. 1106]. We remark that $\Gamma_p = \frac{1}{p}$ for $p \leqslant 1$, $\Gamma_2 = \frac{1}{2}$ and $\Gamma_p \leqslant e(2 + \log p)$ in general, see Simon [14].

For $A, B \in \mathbf{B}(\mathcal{H})$ with $B - A \in \mathbf{S}_p$, the $\lceil p \rceil$-regularized perturbation determinant of B with respect to A is a well-defined holomorphic function on $\rho(A) = \mathbb{C} \setminus \sigma(A)$, given by

$$d(\lambda) = \det_{\lceil p \rceil}\left(I - (\lambda - A)^{-1}(B - A)\right).$$

Furthermore, $\lambda_0 \in \rho(A)$ is an eigenvalue of B of algebraic multiplicity k_0 if and only if λ_0 is a zero of $d(\cdot)$ of the same multiplicity.

2.2. A theorem of Borichev, Golinskii and Kupin

The following result, proved in [2], gives a bound on the zeros of a holomorphic function in the unit disk in terms of its growth near the boundary. An important feature of this theorem is that it enables to take into account the existence of 'special' points (ξ_j) on the boundary of the unit disk, where the rate of growth is higher than at generic points.

Theorem 4. *Let h be a holomorphic function in the unit disk $\mathbb{U}$ with $h(0) = 1$. Assume that h satisfies a bound of the form*

$$\log\bigl|h(z)\bigr| \leqslant K \frac{1}{(1-|z|)^\alpha} \prod_{j=1}^N \frac{1}{|z-\xi_j|^{\beta_j}},$$

where $|\xi_j| = 1$ $(1 \leqslant j \leqslant N)$, and the exponents α, β_j are nonnegative. Let $\tau > 0$. Then the zeros of h satisfy the inequality

$$\sum_{h(z)=0} \bigl(1-|z|\bigr)^{\alpha+1+\tau} \prod_{j=1}^N |z-\xi_j|^{(\beta_j-1+\tau)_+} \leqslant C\bigl(\alpha, \{\beta_j\}, \{\xi_j\}, \tau\bigr)K.$$

2.3. Some inequalities

We will need some elementary inequalities, which we collect here for convenient reference.

Lemma 1. *For $\mu \in \mathbb{C}$*

$$|\mu|\bigl|\Im(\mu)\bigr| \leqslant \operatorname{dist}\bigl(\mu^2, [0,\infty)\bigr) \leqslant 2|\mu|\bigl|\Im(\mu)\bigr|.$$

Proof. If $\Re(\mu^2) > 0$ then $|\Re(\mu)| > |\Im(\mu)|$ and we have

$$\operatorname{dist}\bigl(\mu^2, [0,\infty)\bigr) = \bigl|\Im\bigl(\mu^2\bigr)\bigr| = 2\bigl|\Re(\mu)\bigr|\bigl|\Im(\mu)\bigr| \leqslant 2|\mu|\bigl|\Im(\mu)\bigr|,$$
$$\operatorname{dist}\bigl(\mu^2, [0,\infty)\bigr) = \bigl|\Im\bigl(\mu^2\bigr)\bigr| = 2\bigl|\Re(\mu)\bigr|\bigl|\Im(\mu)\bigr| \geqslant \sqrt{2}|\mu|\bigl|\Im(\mu)\bigr|.$$

If $\Re(\mu^2) \leqslant 0$ then $|\Re(\mu)| \leqslant |\Im(\mu)|$ and we have

$$\operatorname{dist}\bigl(\mu^2, [0,\infty)\bigr) = |\mu|^2 = \Re(\mu)^2 + \Im(\mu)^2 \leqslant 2\Im(\mu)^2 \leqslant 2\bigl|\Im(\mu)\bigr||\mu|,$$
$$\operatorname{dist}\bigl(\mu^2, [0,\infty)\bigr) = |\mu|^2 \geqslant |\mu|\bigl|\Im(\mu)\bigr|.$$

Taking the worst-case scenarios we get the result. $\square$

For $a > 0$, we define the conformal map $\phi_a : \mathbb{U} \to \mathbb{C} \setminus [0,\infty)$ by

$$\phi_a(z) = -a^2\left(\frac{z+1}{z-1}\right)^2. \tag{14}$$

Note that $\phi_a(0) = -a^2$.

Lemma 2. *For $a > 0$ and $\lambda \in \mathbb{C} \setminus [0,\infty)$, the following holds*

$$\frac{a}{2}\frac{\operatorname{dist}(\lambda,[0,\infty))}{|\lambda|^{\frac{1}{2}}(|\lambda|+a^2)} \leqslant 1-\left|\phi_a^{-1}(\lambda)\right| \leqslant 4a\frac{\operatorname{dist}(\lambda,[0,\infty))}{|\lambda|^{\frac{1}{2}}(|\lambda|+a^2)},$$
$$\frac{\sqrt{2}a}{(|\lambda|+a^2)^{\frac{1}{2}}} \leqslant \left|\phi_a^{-1}(\lambda)-1\right| \leqslant \frac{2a}{(|\lambda|+a^2)^{\frac{1}{2}}},$$
$$\frac{\sqrt{2}|\lambda|^{\frac{1}{2}}}{(|\lambda|+a^2)^{\frac{1}{2}}} \leqslant \left|\phi_a^{-1}(\lambda)+1\right| \leqslant \frac{2|\lambda|^{\frac{1}{2}}}{(|\lambda|+a^2)^{\frac{1}{2}}}.$$

Proof. This is a standard computation, see e.g. Corollary 1.4 in [13]. □

3. Proof of the eigenvalue inequalities for general operators

The following lemma, which is of independent interest, is the main ingredient in the proof of Theorem 1, provided in Section 3.2.

Lemma 3. *Let H_0 be selfadjoint with $\sigma(H_0)=[0,\infty)$, $H=H_0+M$, where M satisfies* (1) *for some $p>0$, and $N(H)\subset \mathbb{H}_{\omega_0}$. For $a>0$ with $a^2>\omega_0$ and $\mu\in\mathbb{C}$ with $\Im(\mu)>0$ assume that*

$$\left\|\left[a^2+H\right]^{-1}M\left[\mu^2-H_0\right]^{-1}\right\|_{\mathbf{S}_p}^p \leqslant K_1\frac{|\mu+ia|^\delta}{|\Im(\mu)|^\alpha|\mu|^\nu}, \tag{15}$$

where $\alpha,\delta,\nu\geqslant 0$ and $K_1>0$. For $0<\tau<1$ let ρ, η_1, η_2 be defined as in (3)*, and let*

$$\eta_0=\frac{1}{2}(\rho-1+\tau)_+. \tag{16}$$

Then

$$\sum_{\lambda\in\sigma_d(H)}\frac{\operatorname{dist}(\lambda,[0,\infty))^{2\eta_1}}{|\lambda|^{\eta_1-\eta_2}(a^2+|\lambda|)^{\eta_0+2\eta_1+\eta_2}} \leqslant C(p,\alpha,\delta,\nu,\tau)\frac{K_1}{a^{2\eta_0+2\eta_1-\alpha-\rho}}. \tag{17}$$

3.1. Proof of Lemma 3

We start with the construction of a holomorphic function $f:\mathbb{C}\setminus[0,\infty)\to\mathbb{C}$ whose zeros coincide with the eigenvalues of H in $\mathbb{C}\setminus[0,\infty)$.

3.1.1. The function $f(\lambda)$

To begin with, we note that the resolvent-identity

$$\left[a^2+H_0\right]^{-1}-\left[a^2+H\right]^{-1}=\left[a^2+H\right]^{-1}M\left[a^2+H_0\right]^{-1}$$

implies that

$$
\begin{aligned}
&I-\big(\lambda+a^2\big)\big[a^2+H\big]^{-1}\\
&\quad=I-\big(\lambda+a^2\big)\big[a^2+H_0\big]^{-1}+\big(\lambda+a^2\big)\big[a^2+H\big]^{-1}M\big[a^2+H_0\big]^{-1}.
\end{aligned}
\tag{18}
$$

Assuming that $\lambda\notin[0,\infty)$ and multiplying both sides of (18) from the right by $[I-(\lambda+a^2)[a^2+H_0]^{-1}]^{-1}$ we obtain

$$
\begin{aligned}
&\big[I-\big(\lambda+a^2\big)\big[a^2+H\big]^{-1}\big]\big[I-\big(\lambda+a^2\big)\big[a^2+H_0\big]^{-1}\big]^{-1}\\
&\quad=I+\big(\lambda+a^2\big)\big[a^2+H\big]^{-1}M\big[a^2+H_0\big]^{-1}\big[I-\big(\lambda+a^2\big)\big[a^2+H_0\big]^{-1}\big]^{-1}\\
&\quad=I-\big(\lambda+a^2\big)\big[a^2+H\big]^{-1}M[\lambda-H_0]^{-1}.
\end{aligned}
\tag{19}
$$

Note that the LHS of (19) is invertible if and only if $I-(\lambda+a^2)[a^2+H]^{-1}$ is invertible, which is the case if and only if $\lambda\notin\sigma_d(H)$. Therefore, defining

$$
F(\lambda)=\big(\lambda+a^2\big)\big[a^2+H\big]^{-1}M[\lambda-H_0]^{-1},
\tag{20}
$$

it follows that

$$
\lambda\in\sigma_d(H)\quad\Leftrightarrow\quad I-F(\lambda)\text{ is not invertible.}
\tag{21}
$$

$F(\lambda)$ is an operator-valued function defined on $\mathbb{C}\setminus[0,\infty)$, and by assumption (1) we have $F(\lambda)\in\mathbf{S}_p$. Hence, (21) can be rewritten as

$$
\lambda\in\sigma_d(H)\quad\Leftrightarrow\quad \det{}_{\lceil p\rceil}\big(I-F(\lambda)\big)=0.
$$

Defining $f(\lambda)=\det_{\lceil p\rceil}(I-F(\lambda))$, we obtain that f is holomorphic on $\mathbb{C}\setminus[0,\infty)$ and

$$
\sigma_d(H)=\big\{\lambda\in\mathbb{C}\setminus[0,\infty)\ \big|\ f(\lambda)=0\big\}.
\tag{22}
$$

Moreover, $F(-a^2)=0$ implies that

$$
f\big(-a^2\big)=1.
\tag{23}
$$

It should be noted that

$$
F(\lambda)=\big[\big(\lambda+a^2\big)^{-1}-\big[a^2+H_0\big]^{-1}\big]^{-1}\big(\big[a^2+H\big]^{-1}-\big[a^2+H_0\big]^{-1}\big),
$$

providing the alternative representation

$$
f(\lambda)=\det{}_{\lceil p\rceil}\big(I-\big[\big(\lambda+a^2\big)^{-1}-\big[a^2+H_0\big]^{-1}\big]^{-1}\big(\big[a^2+H\big]^{-1}-\big[a^2+H_0\big]^{-1}\big)\big).
$$

This shows that f is the $\lceil p\rceil$-regularized perturbation determinant of $[a^2+H]^{-1}$ with respect to $[a^2+H_0]^{-1}$ as defined in Section 2.1. Together with the spectral mapping theorem this implies that the order of λ_0 as a zero of f coincides with its algebraic multiplicity as an eigenvalue of H.

We conclude this subsection with the following bound on $f(\lambda)$.

Lemma 4. *Assume* (15). *Then for all* $\mu \in \mathbb{C}$ *with* $\Im(\mu) > 0$

$$\log\left|f\left(\mu^2\right)\right| \leqslant \Gamma_p K_1 \frac{|\mu - ia|^p |\mu + ia|^{\delta+p}}{|\Im(\mu)|^\alpha |\mu|^\nu}. \tag{24}$$

Proof. Using (20) and (15), we have

$$\left\|F\left(\mu^2\right)\right\|_{\mathbf{S}_p}^p \leqslant \left|\mu^2 + a^2\right|^p \left\|\left[a^2 + H\right]^{-1} M\left[\mu^2 - H_0\right]^{-1}\right\|_{\mathbf{S}_p}^p \leqslant K_1 \frac{|\mu^2 + a^2|^p |\mu + ia|^\delta}{|\Im(\mu)|^\alpha |\mu|^\nu},$$

and the result follows by (13). $\square$

In the sequel, we want to study the zeros of $f(\lambda)$. Since our tool will be a theorem on zeros of a holomorphic function in the unit disk $\mathbb{U}$, we have to transform the problem from $\mathbb{C} \setminus [0, \infty)$ to $\mathbb{U}$.

3.1.2. The function $h(z)$

Recall the conformal map $\phi_a : \mathbb{U} \to \mathbb{C} \setminus [0, \infty)$ given by (14), and define $h : \mathbb{U} \to \mathbb{C}$ by

$$h(z) = f\left(\phi_a(z)\right).$$

Then h is holomorphic in the unit disk, and (22) implies that

$$\sigma_d(H) = \left\{\phi_a(z) \,\middle|\, z \in \mathbb{U},\ h(z) = 0\right\}. \tag{25}$$

By (23) we have

$$h(0) = 1.$$

The bound on f provided by Lemma 4 is now translated into a bound on h.

Lemma 5. *Assume* (15). *Then for all* $z \in \mathbb{U}$

$$\log\left|h(z)\right| \leqslant C(p, \delta) K_1 a^{\alpha+\rho} \frac{|z|^p}{(1 - |z|)^\alpha |z + 1|^\nu |z - 1|^\rho},$$

where ρ *was defined in* (3).

Proof. Set $\mu = ia\frac{1+z}{1-z}$ and note that $\Im(\mu) > 0$. Then by Lemma 4

$$\log\left|h(z)\right| = \log\left|f\left(\mu^2\right)\right| \leqslant \Gamma_p K_1 \frac{|\mu - ia|^p |\mu + ia|^{\delta+p}}{|\Im(\mu)|^\alpha |\mu|^\nu}. \tag{26}$$

Since

$$|\mu + ia| = \frac{2a}{|z - 1|}, \qquad |\mu - ia| = \frac{2a|z|}{|z - 1|} \quad \text{and} \quad \frac{1}{|\Im(\mu)|} \leqslant \frac{|1 - z|^2}{a(1 - |z|)},$$

we obtain

$$\frac{|\mu - ia|^p |\mu + ia|^{\delta+p}}{|\Im(\mu)|^\alpha |\mu|^\nu} \leqslant 2^{\delta+2p} a^{\delta+2p-\alpha-\nu} \frac{|z|^p |z-1|^{2\alpha+\nu-\delta-2p}}{(1-|z|)^\alpha |z+1|^\nu},$$

which together with (26) concludes the proof. □

We are now in a position to apply Theorem 4, the result by Borichev, Golinskii and Kupin. Since $\rho = \delta + 2(p-\alpha) - \nu$ can be negative, Lemma 5 implies that

$$\log\bigl|h(z)\bigr| \leqslant C(p,\alpha,\delta,\nu) K_1 a^{\alpha+\rho} \frac{|z|^p}{(1-|z|)^\alpha |z-1|^{\rho_+} |z+1|^\nu}.$$

Applying Theorem 4 with $N = 2$, $\xi_1 = 1$, $\xi_2 = -1$, $\beta_1 = \rho_+$, $\beta_2 = \nu$ and $K = C(p,\alpha,\delta,\nu) K_1 a^{\alpha+\rho}$ we obtain, for $0 < \tau < 1$,

$$\sum_{h(z)=0,\, z\in\mathbb{U}} \bigl(1-|z|\bigr)^{\alpha+1+\tau} |z-1|^{(\rho-1+\tau)_+} |z+1|^{(\nu-1+\tau)_+} \leqslant C(p,\alpha,\delta,\nu,\tau) K_1 a^{\alpha+\rho}. \tag{27}$$

Here it was used that $(\rho_+ - 1 + \tau)_+ = (\rho - 1 + \tau)_+$ for $0 < \tau < 1$.

Recalling the definition of η_0, η_1 and η_2 (see (3) and (16)), inequality (27) can be rewritten as follows

$$\sum_{h(z)=0,\, z\in\mathbb{U}} \bigl(1-|z|\bigr)^{2\eta_1} |z+1|^{2\eta_2} |z-1|^{2\eta_0} \leqslant C(p,\alpha,\delta,\nu,\tau) K_1 a^{\alpha+\rho}. \tag{28}$$

3.1.3. Back to the eigenvalues

It remains to retranslate the bound obtained in (28) into a bound on the eigenvalues of H. Using the inequalities derived in Section 2.3 this is straightforward. From (25) and (28) we obtain

$$\sum_{\lambda\in\sigma_d(H)} \bigl(1-\bigl|\phi_a^{-1}(\lambda)\bigr|\bigr)^{2\eta_1} \bigl|\phi_a^{-1}(\lambda)+1\bigr|^{2\eta_2} \bigl|\phi_a^{-1}(\lambda)-1\bigr|^{2\eta_0} \leqslant C(p,\alpha,\delta,\nu,\tau) K_1 a^{\alpha+\rho} \tag{29}$$

and, using Lemma 2, the sum on the left-hand side of (29) can be bounded from below by

$$C(p,\alpha,\delta,\nu,\tau) a^{2\eta_0+2\eta_1} \sum_{\lambda\in\sigma_d(H)} \frac{\operatorname{dist}(\lambda,[0,\infty))^{2\eta_1}}{|\lambda|^{\eta_1-\eta_2}} \frac{1}{(|\lambda|+a^2)^{\eta_0+2\eta_1+\eta_2}}. \tag{30}$$

(29) and (30) imply (17). We have thus completed the proof of Lemma 3.

3.2. Proof of Theorem 1

We will see that Theorem 1 is a direct consequence of Lemma 3.

Let $a > 0$ with $a^2 > \omega_0$. Since

$$\bigl\|\bigl[a^2+H\bigr]^{-1}\bigr\| \leqslant \frac{1}{\operatorname{dist}(-a^2, \overline{N(H)})} \leqslant \frac{1}{a^2-\omega_0},$$

and for $\Im(\mu) > 0$

$$|\mu + i| \leqslant \left(1 + \frac{1}{a}\right)|\mu + ia| \leqslant \sqrt{2\left(1 + \frac{1}{a^2}\right)}|\mu + ia|,$$

we obtain from assumption (2) that

$$\left\| \left[a^2 + H\right]^{-1} M \left[\mu^2 - H_0\right]^{-1} \right\|_{\mathbf{S}_p}^p \leqslant \frac{2^{\frac{\delta}{2}} K_0 (1 + \frac{1}{a^2})^{\frac{\delta}{2}}}{(a^2 - \omega_0)^p} \frac{|\mu + ia|^\delta}{|\Im(\mu)|^\alpha |\mu|^\nu}.$$

Hence, an application of Lemma 3 shows that

$$\sum_{\lambda \in \sigma_d(H)} \frac{\operatorname{dist}(\lambda, [0, \infty))^{2\eta_1}}{|\lambda|^{\eta_1 - \eta_2} (a^2 + |\lambda|)^{\eta_0 + 2\eta_1 + \eta_2}} \leqslant L \frac{(a^2 + 1)^{\frac{\delta}{2}}}{(a^2 - \omega_0)^p a^{2\eta_0 + 2\eta_1 - \alpha - \rho + \delta}},$$

where $L = C(p, \alpha, \delta, \nu, \tau) K_0$. To simplify the notation we set

$$\begin{aligned} b &= a^2, \\ \varphi_1 &= \eta_0 + \eta_1 - \frac{\alpha + \rho - \delta}{2} + p - 1 - \tau, \\ \varphi_2 &= \eta_0 + 2\eta_1 + \eta_2. \end{aligned}$$

Then, the last inequality is equivalent to

$$\sum_{\lambda \in \sigma_d(H)} \frac{\operatorname{dist}(\lambda, [0, \infty))^{2\eta_1} b^{\varphi_1}}{|\lambda|^{\eta_1 - \eta_2} (b + |\lambda|)^{\varphi_2} (b + 1)^{\frac{\delta}{2}}} \leqslant L \frac{b^{p-1-\tau}}{(b - \omega_0)^p}. \tag{31}$$

Note that (31) holds for any $b > \omega_0$, so we may integrate both sides of (31) with respect to $b \in (\omega_0 + 1, \infty)$. For the RHS, we obtain

$$\int_{\omega_0+1}^{\infty} db \, \frac{b^{p-1-\tau}}{(b - \omega_0)^p} \leqslant \tau^{-1} (1 + \omega_0)^{p-\tau}. \tag{32}$$

Integrating the LHS of (31), interchanging sum and integral, it follows that

$$\begin{aligned} &\int_{\omega_0+1}^{\infty} db \left[\sum_{\lambda \in \sigma_d(H)} \frac{\operatorname{dist}(\lambda, [0, \infty))^{2\eta_1} b^{\varphi_1}}{|\lambda|^{\eta_1 - \eta_2} (b + |\lambda|)^{\varphi_2} (b + 1)^{\frac{\delta}{2}}} \right] \\ &\quad = \sum_{\lambda \in \sigma_d(H)} \frac{\operatorname{dist}(\lambda, [0, \infty))^{2\eta_1}}{|\lambda|^{\eta_1 - \eta_2}} \int_{\omega_0+1}^{\infty} db \, \frac{b^{\varphi_1}}{(b + |\lambda|)^{\varphi_2} (b + 1)^{\frac{\delta}{2}}}. \end{aligned} \tag{33}$$

The finiteness of the above integral follows from (32), and we can bound it from below as follows,

$$\int_{\omega_0+1}^{\infty} db\, \frac{b^{\varphi_1}}{(b+|\lambda|)^{\varphi_2}(b+1)^{\frac{\delta}{2}}} \geqslant \frac{C(p,\alpha,\delta,\nu,\tau)}{(\omega_0+1)^{\eta_1+\eta_2-\eta_3}(|\lambda|+1)^{\eta_1+\eta_2-\eta_3}}. \tag{34}$$

Note that we used the easily verified fact that $\eta_1+\eta_2-\eta_3>0$, see definition (3) above. (31) to (34) imply that

$$\sum_{\lambda\in\sigma_d(H)} \frac{\operatorname{dist}(\lambda,[0,\infty))^{2\eta_1}}{|\lambda|^{\eta_1-\eta_2}(|\lambda|+1)^{\eta_1+\eta_2-\eta_3}} \leqslant C(p,\alpha,\delta,\nu,\tau)L(1+\omega_0)^{p-\tau+\eta_1+\eta_2-\eta_3}.$$

Noting that $p-\tau-\eta_3=\frac{\alpha+\rho}{2}$ concludes the proof of Theorem 1.

4. Proof of the inequalities for Schrödinger operators

4.1. Schatten norm bounds

We intend to prove Theorem 2 by an application of Theorem 1. To this end, some information on the Schatten norms of $M_V[\mu^2-H_0]^{-1}$ is needed. This will be dealt with in the following two lemmas.

Lemma 6. *Let $V\in L^p(\mathbb{R}^d)$, where $d\geqslant 2$, $p\geqslant 2$ and $p>\frac{d}{2}$. Then, for $\lambda\in\mathbb{C}\setminus[0,\infty)$ with $\Re(\lambda)>0$,*

$$\big\|M_V[\lambda-H_0]^{-1}\big\|_{\mathbf{S}_p}^p \leqslant C(p,d)\|V\|_{L^p}^p\left[\frac{|\Re(\lambda)|^{\frac{d-2}{2}}}{|\Im(\lambda)|^{p-1}}+\frac{1}{|\Im(\lambda)|^{p-\frac{d}{2}}}\right]$$

and for $\lambda\in\mathbb{C}$ with $\Re(\lambda)\leqslant 0$,

$$\big\|M_V[\lambda-H_0]^{-1}\big\|_{\mathbf{S}_p}^p \leqslant C(p,d)\|V\|_{L^p}^p\frac{1}{|\lambda|^{p-\frac{d}{2}}}.$$

Proof. Theorem 4.1 in Simon [15] implies, for $p\geqslant 2$,

$$\big\|M_V[\lambda-H_0]^{-1}\big\|_{\mathbf{S}_p}^p \leqslant (2\pi)^{-\frac{d}{p}}\big\|\big(\lambda-|\cdot|^2\big)^{-1}\big\|_{L^p}^p\|V\|_{L^p}^p.$$

We will show that, for $\lambda\in\mathbb{C}\setminus[0,\infty)$ with $\Re(\lambda)>0$

$$\big\|\big(\lambda-|\cdot|^2\big)^{-1}\big\|_{L^p}^p \leqslant C(p,d)\left[\frac{|\Re(\lambda)|^{\frac{d-2}{2}}}{|\Im(\lambda)|^{p-1}}+\frac{1}{|\Im(\lambda)|^{p-\frac{d}{2}}}\right]. \tag{35}$$

Set $\lambda=\lambda_0+i\lambda_1$, and assume first that $\lambda_0>0$. Since $\|(\lambda-|\cdot|^2)^{-1}\|_{L^p}=\|(\bar{\lambda}-|\cdot|^2)^{-1}\|_{L^p}$, it is sufficient to treat the case $\lambda_1>0$. Making the change of variable $r=\sqrt{\lambda_0-\lambda_1 s}$ we can express $\|(\lambda-|\cdot|^2)^{-1}\|_{L^p}^p$ as

$$C(d)\lambda_1^{1-p}\Bigg[\int_0^\infty \frac{(\lambda_0+\lambda_1 s)^{\frac{d-2}{2}}}{(s^2+1)^{\frac{p}{2}}}\,ds + \int_0^{\frac{\lambda_0}{\lambda_1}} \frac{(\lambda_0-\lambda_1 s)^{\frac{d-2}{2}}}{(s^2+1)^{\frac{p}{2}}}\,ds\Bigg]. \tag{36}$$

For the first integral in (36), we have, using $(\lambda_0+\lambda_1 s)^{\frac{d-2}{2}} \leqslant (2\lambda_0)^{\frac{d-2}{2}} + (2\lambda_1 s)^{\frac{d-2}{2}}$,

$$\int_0^\infty \frac{(\lambda_0+\lambda_1 s)^{\frac{d-2}{2}}}{(s^2+1)^{\frac{p}{2}}}\,ds \leqslant C(d,p)\big[\lambda_0^{\frac{d-2}{2}} + \lambda_1^{\frac{d-2}{2}}\big]. \tag{37}$$

Similarly, for the second integral in (36) we obtain

$$\int_0^{\frac{\lambda_0}{\lambda_1}} \frac{(\lambda_0-\lambda_1 s)^{\frac{d-2}{2}}}{(s^2+1)^{\frac{p}{2}}}\,ds \leqslant \lambda_0^{\frac{d-2}{2}} \int_0^\infty \frac{1}{(s^2+1)^{\frac{p}{2}}}\,ds = C(p)\lambda_0^{\frac{d-2}{2}}. \tag{38}$$

(36)–(38) imply the validity of (35) in case that $\lambda_0 > 0$. A similar argument shows the validity of (35) in case that $\lambda_0 \leqslant 0$. $\square$

Lemma 7. *Let $V \in L^p(\mathbb{R}^d)$, where $d \geqslant 2$, $p \geqslant 2$ and $p > \frac{d}{2}$. Then, for $\mu \in \mathbb{C}$ with $\Im(\mu) > 0$,*

$$\big\|M_V\big[\mu^2 - H_0\big]^{-1}\big\|_{\mathbf{S}_p}^p \leqslant C(p,d)\|V\|_{L^p}^p \frac{|\mu+i|^\delta}{|\Im(\mu)|^\alpha|\mu|^\nu} \tag{39}$$

where

$$\nu = p - \frac{d}{2}, \qquad \delta = \frac{d}{2} - 1, \qquad \alpha = p - 1.$$

Proof. Let us consider first the case $0 < \Im(\mu) < |\Re(\mu)|$. Since $\Re(\mu^2) = \Re(\mu)^2 - \Im(\mu)^2 > 0$ and $\Im(\mu^2) = 2\Re(\mu)\Im(\mu)$, Lemma 6 implies

$$\big\|M_V\big[\mu^2 - H_0\big]^{-1}\big\|_{\mathbf{S}_p}^p \leqslant C(p,d)\|V\|_{L^p}^p \bigg[\frac{|\Re(\mu)^2 - \Im(\mu)^2|^{\frac{d-2}{2}}}{|2\Re(\mu)\Im(\mu)|^{p-1}} + \frac{1}{|2\Re(\mu)\Im(\mu)|^{p-\frac{d}{2}}}\bigg].$$

Hence, to show (39), it is sufficient to show that

$$\frac{|\Re(\mu)^2 - \Im(\mu)^2|^{\frac{d-2}{2}}|\mu|^{p-\frac{d}{2}}}{|2\Re(\mu)|^{p-1}|\mu+i|^{\frac{d}{2}-1}} \quad \text{and} \quad \frac{|\mu|^{p-\frac{d}{2}}\Im(\mu)^{\frac{d}{2}-1}}{|2\Re(\mu)|^{p-\frac{d}{2}}|\mu+i|^{\frac{d}{2}-1}} \tag{40}$$

are bounded from above by a suitable constant $C(p,d)$. In the following, we will provide such a bound for the first quotient in (40) (a similar computation for the second quotient will be omitted). Since $|\Re(\mu)| > \Im(\mu) > 0$ one deduces that $|\mu| \leqslant \sqrt{2}|\Re(\mu)|$ and $|\Re(\mu)^2 - \Im(\mu)^2| \leqslant 2|\Re(\mu)|^2$. Thus, we obtain

$$\frac{|\Re(\mu)^2-\Im(\mu)^2|^{\frac{d-2}{2}}|\mu|^{p-\frac{d}{2}}}{|2\Re(\mu)|^{p-1}|\mu+i|^{\frac{d}{2}-1}} \leqslant \frac{1}{2^{\frac{p}{2}-\frac{d}{4}}}\left(\frac{|\Re(\mu)|}{|\mu+i|}\right)^{\frac{d}{2}-1} \leqslant \frac{1}{2^{\frac{p}{2}-\frac{d}{4}}}.$$

The proof of (39) in case that $\Im(\mu) > 0$ and $|\Re(\mu)| \leqslant \Im(\mu)$ follows the same lines as above and will therefore be omitted. □

4.2. Proof of Theorem 2

Lemma 7 implies that for $p \geqslant 2$ and $p > \frac{d}{2}$

$$\left\| M_V\left[\mu^2 - H_0\right]^{-1}\right\|_{\mathbf{S}_p}^p \leqslant C(p,d)\|V\|_{L^p}^p \frac{|\mu+i|^\delta}{|\Im(\mu)|^\alpha|\mu|^\nu},$$

where $\nu = p - \frac{d}{2}$, $\delta = \frac{d}{2} - 1$ and $\alpha = p - 1$. With the notation of Theorem 1 we have for $0 < \tau < 1$,

$$\begin{aligned}
\rho &= \delta + 2(p-\alpha) - \nu = d - p + 1,\\
\eta_1 &= \frac{1}{2}(\alpha + 1 + \tau) = \frac{1}{2}(p+\tau),\\
\eta_2 &= \frac{1}{2}(\nu - 1 + \tau)_+ = \frac{1}{2}\left(p - \frac{d}{2} - 1 + \tau\right)_+,\\
\eta_3 &= \frac{\alpha + \nu - \delta}{2} - \tau = p - \frac{d}{2} - \tau
\end{aligned}$$

and an application of Theorem 1 shows that

$$\sum_{\lambda\in\sigma_d(H)} \frac{\operatorname{dist}(\lambda,[0,\infty))^{p+\tau}}{|\lambda|^{\frac{1}{2}(p+\tau-(p-\frac{d}{2}-1+\tau)_+)}(|\lambda|+1)^{\frac{1}{2}(d-p+3\tau+(p-\frac{d}{2}-1+\tau)_+)}} \leqslant C\|V\|_{L^p}^p,$$

where $C = C(d,p,\tau)(1+\omega_0)^{\frac{1}{2}(p+d+\tau+(p-\frac{d}{2}-1+\tau)_+)}$. Simplifying the above expression in the cases $p - \frac{d}{2} \geqslant 1 - \tau$ and $p - \frac{d}{2} < 1 - \tau$ we get (8), (9).

4.3. Proof of Corollary 3

Restricting the generalized Lieb–Thirring inequality (11) to the set $\Re(\lambda) > 0$, we obtain

$$\sum_{\lambda\in\sigma_d(H),\,|\Im(\lambda)|\geqslant\chi\Re(\lambda)>0} |\lambda|^\kappa \leqslant C(d,\kappa)\left(1+\frac{2}{\chi}\right)^{\frac{d}{2}+\kappa}\int_{\mathbb{R}^d}\left|V(y)\right|^{\frac{d}{2}+\kappa}dy. \tag{41}$$

We multiply both sides of (41) with $\chi^{\frac{d}{2}+\kappa-1+\tau}$, where $0 < \tau < 1$, and integrate over $\chi \in (0,1)$. Interchanging sum and integral, one obtains for the LHS

$$
\begin{aligned}
&\int_0^1 d\chi\, \chi^{\frac{d}{2}+\kappa-1+\tau} \sum_{\lambda\in\sigma_d(H),\, |\Im(\lambda)|\geqslant \chi\Re(\lambda)>0} |\lambda|^{\kappa} \\
&\quad = \sum_{\lambda\in\sigma_d(H),\, \Re(\lambda)>0} |\lambda|^{\kappa} \int_0^{\min(\frac{|\Im(\lambda)|}{\Re(\lambda)},1)} d\chi\, \chi^{\frac{d}{2}+\kappa-1+\tau} \\
&\quad = C(d,\kappa,\tau) \sum_{\lambda\in\sigma_d(H),\, \Re(\lambda)>0} |\lambda|^{\kappa} \min\left(1, \left(\frac{|\Im(\lambda)|}{\Re(\lambda)}\right)^{\frac{d}{2}+\kappa+\tau}\right) \\
&\quad \geqslant C(d,\kappa,\tau) \sum_{\lambda\in\sigma_d(H),\, |\Im(\lambda)|\leqslant\Re(\lambda)} |\lambda|^{\kappa} \left(\frac{|\Im(\lambda)|}{\Re(\lambda)}\right)^{\frac{d}{2}+\kappa+\tau} \\
&\quad \geqslant C(d,\kappa,\tau) \sum_{\lambda\in\sigma_d(H),\, |\Im(\lambda)|\leqslant\Re(\lambda)} \frac{\operatorname{dist}(\lambda,[0,\infty))^{\frac{d}{2}+\kappa+\tau}}{|\lambda|^{\frac{d}{2}+\tau}}.
\end{aligned}
$$

Similarly, we obtain for the RHS of (41)

$$
\int_0^1 d\chi \left(1+\frac{2}{\chi}\right)^{\frac{d}{2}+\kappa} \chi^{\frac{d}{2}+\kappa-1+\tau} \int_{\mathbb{R}^d} \bigl|V(y)\bigr|^{\frac{d}{2}+\kappa} dy \leqslant C(d,\kappa,\tau) \int_{\mathbb{R}^d} \bigl|V(y)\bigr|^{\frac{d}{2}+\kappa} dy.
$$

This shows that

$$
\sum_{\lambda\in\sigma_d(H),\, |\Im(\lambda)|\leqslant\Re(\lambda)} \frac{\operatorname{dist}(\lambda,[0,\infty))^{\frac{d}{2}+\kappa+\tau}}{|\lambda|^{\frac{d}{2}+\tau}} \leqslant C(d,\kappa,\tau) \int_{\mathbb{R}^d} \bigl|V(y)\bigr|^{\frac{d}{2}+\kappa} dy.
$$

Using Theorem 3 with $\chi = 1$ gives that the same inequality is true summing over all eigenvalues λ with $|\Im(\lambda)| \geqslant \Re(\lambda)$. Setting $p = \kappa + \frac{d}{2}$ completes the proof of Corollary 3.

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