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A simple upper bound for the hamiltonian index of a graph

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Abstract

Let G be a connected graph other than a path and $\mathrm{ham}(G)$, $\Delta(G)$ be its hamiltonian index and maximal degree, respectively. It is proved that $\mathrm{ham}(G) \leqslant |V(G)| - \Delta(G)$.

1. Introduction

The *line graph* $L(G)$ of a graph G has $E(G)$ as its vertex set and two vertices are adjacent in $L(G)$ if and only if they are adjacent as edges in G. Iterated line graphs are defined recursively: $L^0(G)=G$, $L^n(G)=L(L^{n-1}(G))$ for $n \geqslant 1$. If we disregard isolated vertices, then it is a simple observation that $L(G)$ is connected if and only if G is connected. Since we are studying hamiltonicity of line graphs it makes sense to limit ourselves to connected graphs.

It is known [4] that in an infinite sequence $G, L(G), L^2(G), \ldots$ of iterated line graphs there always exists one that is hamiltonian, except when G is a path. The *hamiltonian index of* G is defined as follows:

$$\mathrm{ham}(G) = \min\{n \colon L^n(G) \text{ is hamiltonian}\}.$$

Since the hamiltonian index does not exist for paths and is obvious for cycles, we will exclude them from the rest of this article. Thus, G will always stand for a connected graph other than a path or a cycle.

A sequence of vertices $u_0 u_1 \ldots u_k$ in G with $u_0 = u_k$ is called a *circuit* if $e_i = u_{i-1} u_i \in E(G)$ $(i=1, \ldots, k)$ are pairwise distinct edges. If $k=0$, then the circuit with the single vertex u_0 has no edges and is said to be *trivial*. A circuit D is *dominating* (resp. *spanning*) if every edge of G is incident with a vertex of D (resp. every vertex of G belongs to D). For instance, a trivial circuit in G is dominating if and only if G is isomorphic to $K_{1,m}$ for some m. A criterion for hamiltonicity of line graphs in terms of dominating circuits appeared in [5].

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Theorem 1. *$L(G)$ is hamiltonian if and only if G contains a dominating circuit.*

If $u \in V(G)$, then the *degree of* u is the number of edges incident with u and is denoted by $\deg(u)$. Sometimes we will write $\deg_G(u)$ to emphasize the relative graph. The *maximal degree of* G is defined as $\Delta(G) = \max\{\deg(u): u \in V(G)\}$. An *eulerian circuit* is a circuit which contains every edge of G. Thus, the eulerian circuit is also dominating. The famous theorem of Euler tells that G possesses an eulerian circuit if and only if all its vertices have even degree.

The aim of this article is to prove the inequality $\mathrm{ham}(G) \leqslant |V(G)| - \Delta(G)$ for every connected graph G except when G is a path.

2. Branch graphs

Let us define

$$E_T(G) = \{e \in E(G): e \text{ belongs to some triangle}\},$$

$$V_D(G) = \{u \in V(G): \deg(u) \neq 2\}.$$

Denote by T_G the subgraph spanned by $E_T(G)$ and define D_G as the subgraph such that $V(D_G) = V_D(G)$, $E(D_G) = \emptyset$. The union of both is the subgraph $T_G \cup D_G$, which may in general be disconnected. Take an arbitrary component H of $T_G \cup D_G$. It is easy to see that H is either a maximal subgraph in G with the property that every edge of H belongs to a triangle in H, or H is trivial, say $V(H) = \{u\}$, such that $u \in V_D(G)$ and u does not lie on any triangle in G. A component of $T_G \cup D_G$ will be called a 3-*component of* G. Let Q be a path in G and suppose that

- no edge of Q belongs to any triangle;
- every internal vertex of Q has degree 2;
- both endvertices of Q belong to $V_D(G)$.

Every such path Q will be called a *branch* (cf. [3]). The *length of* Q is the number of edges contained in Q, i.e. $|E(Q)|$. Note that the edge $e \in E(G)$ is also considered to be a branch (whose length equals 1) in the case when both endvertices of e belong to $V_D(G)$ and e does not lie on any triangle. The set $E(G) - E(T_G \cup D_G)$ is exactly the set of all edges which belong to branches. Let $\{H_1, \ldots, H_s\}$ be the set of all 3-components of G and $\{Q_1, \ldots, Q_t\}$ the set of all its branches. It is clear that

- any two different 3-components are (vertex-) disjoint;
- two different branches can only have their endvertices in common;
- a branch Q and a 3-component can only have endvertices of Q in common;
- if u is an endvertex of some branch, there is exactly one 3-component containing u.

This leads us to construction of the graph $B(G)$ such that

$$V(B(G)) = \{H_1, \ldots, H_s\}, \qquad E(B(G)) = \{Q_1, \ldots, Q_t\},$$

and if Q_i has u, v as its endvertices so that $u \in V(H_j)$, $v \in V(H_k)$, then Q_i is incident with H_j and H_k in $B(G)$. Let

$$\eta(Q_i) = |E(Q_i)|, \quad i = 1, \ldots, t.$$

For $B(G)$ we will usually say that $\eta(Q_i)$ is the *length of the edge* Q_i. The graph $B(G)$ together with the function η defined on $E(B(G))$ is called the *branch graph of* G. $B(G)$ may of course have loops and multiple edges, hence it is not always a simple graph. It should be noted here that in such a graph the loop contributes 2 to the degree of a vertex incident with it.

We would like to find deeper connection between dominating circuits in graphs and circuits in their branch graphs. Obviously, $B(G)$ contains a dominating circuit (D', for instance) if G does. Moreover, D' traverses all vertices which represent nontrivial 3-components. But this argument cannot be reversed. If $B(G)$ possesses a dominating circuit D', then G need not contain one even if D' goes through all 'nontrivial' vertices, i.e. nontrivial 3-components of G.

However, there is a class of graphs for which this can be done, namely, the line graphs. By the known theorem of Beineke [1], if G is the line graph of some other graph, then G cannot have $K_{1,3}$ as its induced subgraph. This means that if G is a line graph and H is a trivial 3-component of G consisting of the single vertex u, then $\deg(u) = 1$ and u is therefore an endvertex of some branch. A branch whose one endvertex has degree 1 is called an *endbranch*. Now let G be a line graph having the dominating circuit D. Obviously D traverses all branches with lengths at least 3. If $H \in V(B(G))$ and $\deg_{B(G)}(H) \geq 2$, then $|V(H)| > 1$ and H contains at least one edge, hence at least one vertex of H lies on D. If $\deg_{B(G)}(H) = 1$, then H is a single vertex of degree 1 in G and $\eta(Q) = 1$ for the branch Q incident with H. Thus, $B(G)$ possesses a special sort of circuit defined below.

Definition 2. A circuit D' in $B(G)$ is said to be *main* if the following hold:

(a) D' traverses every $Q \in E(B(G))$ such that $\eta(Q) \geq 3$;

(b) D' traverses every $H \in V(B(G))$ such that $\deg_{B(G)}(H) \geq 2$;

(c) if $H \in V(B(G))$, $\deg_{B(G)}(H) = 1$, and $Q \in E(B(G))$ is incident with H, then H is a trivial 3-component of G and $\eta(Q) = 1$.

To prove the connection between a dominating circuit in G and a main circuit in its branch graph $B(G)$ we need an additional tool, the so-called graph contractions. For an arbitrary graph F and $e \in E(F)$ define the *simple contraction of* F *by* e (denoted by F/e) as the graph obtained from F by removing the edge e and identifying its endvertices. For any subgraph H of F the *contraction of* F *by* H (shortly F/H) is obtained from F by a sequence of simple contractions using every $e \in E(H)$. Remember that contractions may also result in loops and multiple adjacencies!

Catlin [2] devised a method for searching spanning circuits in graphs by means of circuits in contracted graphs. The following theorem, which is deduced from [3, Theorems 2 and 3], will be useful.

Theorem 3. *Let H be a subgraph of G such that every edge of H lies on a triangle in H. Then G contains a spanning circuit if and only if G/H contains a spanning circuit.*

We are able to prove the following theorem.

Theorem 4. *Let G be the line graph of some graph. Then* (a)$\Leftrightarrow$(b), *where*
(a) *G contains a dominating circuit,*
(b) *$B(G)$ contains a main circuit.*

Proof. Actually we have already seen that if (a) is true, then (b) holds. Conversely, suppose that (b) is true. Denote again by T_G the union of all nontrivial 3-components of G. For every branch Q les us contract all but one edge of Q to obtain the new graph G'. We can say that T_G is a subgraph in G as well as in G'. Let Z' be the set of all vertices of G' whose degree equals 1. The graph $G''=(G'-Z')/T_G$ is clearly isomorphic to $B(G)-Z$ where Z is the set of all vertices of $B(G)$ whose degree is equal to 1. Since $B(G)$ contains a main circuit, it follows from Definition 2 that $B(G)-Z$ (or, equivalently, G'') contains a spanning circuit which traverses every edge with length 3 or more. By Theorem 3, $G'-Z'$ also contains a spanning circuit and G' contains a circuit which traverses all vertices except those of degree 1. Furthermore, there is a circuit D in G which goes through all vertices except those of degree 1 and possibly internal vertices of branches with lengths equal to 2. But no branch with length greater than 1 is an endbranch, by Definition 2, because $B(G)$ contains a main circuit. Hence, D is a dominating circuit in G and (a) holds. $\square$

Remark. If G is not a line graph, then Theorem 4 does not hold in general. However, it is still true that (b)$\Rightarrow$(a).

Let us now study the branch graph of the line graph $B(L(G))$ of G together with $B(G)$. Take an arbitrary 3-component H of G. If H is nontrivial, then $L(H)$ is a subgraph in $L(G)$ contained in some nontrivial 3-component H' of $L(G)$. Suppose H is trivial and $V(H)=\{u\}$. If $\deg_G(u)\geqslant 3$, then all the edges incident with u induce a complete subgraph in $L(G)$ which is contained in a 3-component H'. If $\deg_G(u)=1$ and e is the only edge incident with u, then e is a vertex in $L(G)$ which is either a trivial 3-component or contained in a nontrivial 3-component H'. In all three cases we say that H' is *generated by H*. Now take any branch Q in G, i.e. $Q\in E(B(G))$, and consider the following two cases:

(A) $\eta(Q)=1$. Let H_1, H_2 be the 3-components which are incident with Q in $B(G)$. Denote by H'_1, H'_2 the 3-components generated by H_1 and H_2, respectively. Clearly $L(Q)$ is a trivial subgraph of $L(G)$ which is common to both H'_1 and H'_2. By definition of 3-components, $H'_1=H'_2$;

(B) $\eta(Q)>1$. Then $Q'=L(Q)$ is a branch in $L(G)$, *generated by Q* with length $\eta(Q')=\eta(Q)-1$.

Let $E^*\subseteq E(B(G))$ be the set of edges Q such that $\eta(Q)=1$, and F^* the subgraph spanned by E^*. Define an operator Λ on the class of branch graphs by

$\Lambda(B(G))=B(G)/F^*$. In particular, if E^* is empty then $\Lambda(B(G))=B(G)$. It follows from (A) and (B) that $B(L(G))=\Lambda(B(G))$ and $\eta(Q')=\eta(Q)-1$ (where Q' is generated by Q) for all $Q\in E(B(G))-E^*$. If we write $\Lambda^1(B(G))=\Lambda(B(G))$, then it can be proved by induction that $B(L^n(G))=\Lambda^n(B(G))$ for every positive integer n.

Fix $H_1, H_2\in V(B(G))$. If $\mathscr{P}$ is an arbitrary path from H_1 to H_2, define

$$d(\mathscr{P})=\max_{Q\in E(\mathscr{P})}\eta(Q),$$

$$d(H_1,H_2)=\min\{d(\mathscr{P})\text{: over all paths }\mathscr{P}\text{ from }H_1\text{ to }H_2\}.$$

The function d is a sort of distance on the set $V(B(G))$ and the symbol d will be used for all branch graphs $\Lambda^n(B(G))$ without confusion. Denote by $\Lambda^n(H_1)$ and $\Lambda^n(H_2)$ the 3-components in $\Lambda^n(B(G))$ generated by H_1 and H_2, respectively. It is easy to see that $d(\Lambda(H_1),\Lambda(H_2))=d(H_1,H_2)-1$ if $d(H_1,H_2)>1$, and $\Lambda(H_1)=\Lambda(H_2)$ if $d(H_1,H_2)=1$. This can be generalized for all graphs $\Lambda^n(B(G))$ in the following manner.

Lemma 5. *Fix $n\geqslant 1$ and let $H_1'=\Lambda^n(H_1)$, $H_2'=\Lambda^n(H_2)$. If $d(H_1,H_2)>n$, then $d(H_1',H_2')=d(H_1,H_2)-n$, otherwise $H_1'=H_2'$.*

Proof. The proof goes by induction on n. □

Now take $H'\in V(\Lambda(B(G)))$. In general, $\Lambda^{-1}(H')$ is a set of vertices $H_1,\ldots,H_r$ in $B(G)$ such that $d(H_i,H_j)=1$ for $i\neq j$. Let $E^*=\{Q\in E(B(G))\text{: }Q\text{ is incident with }H_i\text{ for some }i\text{ and }\eta(Q)=1\}$. Then $\sum_{i=1}^{r}\deg(H_i)=\deg(H')+2|E^*|$, and we derive

Lemma 6. *The degree of $H'\in V(\Lambda(B(G))$ is even if the number of odd vertices $H\in\Lambda^{-1}(H')$ is even, and it is odd otherwise.*

3. The main result

For an arbitrary graph F denote the set of odd vertices by $\bar{V}(F)$; note that this set is of even cardinality according to the well-known theorem. Let

$$\omega(G)=\max\{d(H,H')\text{: }H,H'\in\bar{V}(B(G))\}$$

if $\bar{V}(B(G))$ is nonempty, and $\omega(G)=0$ otherwise.

Theorem 7. *The inequality*

$$\mathrm{ham}(G)\leqslant\omega(G)+1. \tag{1}$$

is true for every G.

Proof. Clearly $d(H,H') \leqslant \omega(G)$ for every $H, H' \in \bar{V}(B(G))$ in the case when $\bar{V}(B(G))$ is nonempty. By Lemmas 5 and 6, the branch graph $\Lambda^{\omega(G)}(B(G))$ does not have odd vertices and so it possesses an eulerian circuit. This is also true if $\omega(G)=0$. Thus, $L^{\omega(G)}(G)$ contains a dominating circuit by Theorem 4 which means that $L^{\omega(G)+1}(G)$ is hamiltonian, by Theorem 1. □

Consider the graph G for which the following hold:

(i) G contains an endbranch Q with endvertices v_0, v_1, and $\deg(v_1)=1$;

(ii) if $u \in V(G)$ does not belong to Q, then $uv_0 \in E(G)$.

Clearly $\deg(v_0) = |V(G)| - \eta(Q)$. If (ii) holds for u, then $\deg(u) \leqslant |V(G)| - \eta(Q) - 1$ because u is adjacent to v_0 and at most $|V(G)| - \eta(Q) - 2$ other vertices. This means that v_0 is the vertex of maximal degree.

Lemma 8. *If G satisfies* (i) *and* (ii), *then*

$$\mathrm{ham}(G) = |V(G)| - \Delta(G).$$

Proof. Since $\Delta(G) = \deg(v_0) = |V(G)| - \eta(Q)$, we have to prove that $\mathrm{ham}(G) = \eta(Q)$. Obviously G is not hamiltonian because it satisfies (i), hence $\mathrm{ham}(G) > 0$. Let F be the subgraph induced by v_0 and all its neighbours. If F is isomorphic to $K_{1,m}$ then it contains a trivial dominating circuit. Otherwise, there is an edge $wz \in E(F)$ such that $w, z \neq v_0$. The dominating circuit is now constructed as follows. Start at v_0 and traverse the edges v_0w, wz, zv_0. If there is another edge $w'z'$ such that $w', z' \neq v_0$ have not been traversed yet, then traverse $v_0w', w'z', z'v_0$. Repeat this step many times until every edge of F has at least one endvertex lying on the circuit. This means that F contains the dominating circuit, hence $L(F)$ is hamiltonian by Theorem 1. But $L(F)$ is a nontrivial 3-component of $L(G)$. If $\eta(Q)=1$ then $F=G$ and $\mathrm{ham}(G) = \mathrm{ham}(F) = 1 = \eta(Q)$, otherwise $\eta(Q) > 1$ and the branch graphs $\Lambda^n(B(G))$ $(n = 1, \ldots, \eta(Q)-1)$ are all isomorphic to the complete graph K_2. None of them possesses the main circuit, except when $n = \eta(Q) - 1$. By Theorems 4 and 1, $L^{\eta(Q)}(G)$ is hamiltonian. □

Theorem 9. *Let G be a connected graph other than a path. Then*

$$\mathrm{ham}(G) \leqslant |V(G)| - \Delta(G).$$

Proof. Let $p = |V(G)|$. Clearly $p \geqslant \Delta(G) + 1$ and $p - \Delta(G) \geqslant 1$. If $p - \Delta(G) = 1$, then $G = F$ where F is the graph defined in the proof of Lemma 8. This graph contains a dominating circuit, hence $\mathrm{ham}(G) \leqslant 1 = p - \Delta(G)$. Suppose that $p - \Delta(G) = 2$ and $w \in V(G)$ is the vertex with the maximal degree. Then there exists $u \in V(G)$ different from w such that w is adjacent to all vertices except u (and itself). If $\omega(G) < 2$, then $\mathrm{ham}(G) \leqslant 2 = p - \Delta(G)$ by (1), otherwise $\omega(G) \geqslant 2$ and there exists a branch Q with length equal to $\omega(G)$. But this is only possible when u, w are its endvertices, and in that case $\eta(Q) = \omega(G) = 2$. Moreover, no branch incident with u has length greater than 2.

It follows that the graph $L(G)$ contains exacty two 3-components and an odd number of branches between them. All of these branches have lengths equal to 1. In particular, if there is only one such branch, then one of the 3-components is trivial because it is generated by the endvertex u of the endbranch Q. Anyway, the branch graph $\Delta(B(G))$ contains the main circuit and we conclude that $\mathrm{ham}(G) \leqslant 2 = p - \Delta(G)$, by Theorems 4 and 1.

It remains to see what happens when $p - \Delta(G) \geqslant 3$. Let Q be a branch such that $\eta(Q) = \omega(G)$, and denote by H_0, H_1 the 3-components which are endvertices of Q in $B(G)$. By definition of $\omega(G)$, we can choose Q such that $H_0 \neq H_1$ and no other branch incident with both H_0 and H_1 has length less than $\omega(G)$. Denote the endvertices of Q in G by v_0, v_1. Obviously $v_0 \neq v_1$ and the number of vertices not belonging to Q equals $p - \omega(G) - 1$. On the other hand, at least $\Delta(G) - 1$ vertices do not belong to Q. Thus, $\Delta(G) - 1 \leqslant p - \omega(G) - 1$ and $\omega(G) \leqslant p - \Delta(G)$. If $\omega(G) < p - \Delta(G)$, then $\mathrm{ham}(G) \leqslant p - \Delta(G)$ by (1), otherwise $\omega(G) = p - \Delta(G)$ and there are exactly $\Delta(G) - 1$ vertices which do not lie on Q. Suppose $u \notin V(Q)$ and $\deg(u) = \Delta(G)$. It follows that u should be adjacent to both endvertices v_0, v_1 of Q which contradicts the fact that $\omega(G) = p - \Delta(G) \geqslant 3$. Thus, the only candidates for the maximal degree are the vertices v_0 and v_1, for instance $\deg(v_0) = \Delta(G)$. This means that v_0 is adjacent to all vertices which do not belong to Q, while v_1 is adjacent to none of them. Hence $\deg(v_1) = 1$ and we have thereby proved that the graph G satisfies the conditions (i) and (ii) of Lemma 8. It follows that $\mathrm{ham}(G) = p - \Delta(G)$. $\square$

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