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Decomposability of Direct Products of Modules

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1. INTRODUCTION AND THEOREMS

The initial problem in our discussion is to characterize those families $(M_k)_{k\in K}$ of modules over an associative ring R with 1 for which the direct product $\prod_{k\in K} M_k$ is projective (R-modules are unitary right modules unless otherwise stated). A fundamental result in this area is due to Chase [3] who determined the ideal-theoretic equivalent to the requirement that *all* direct products of projective R-modules be projective. In [4] and [13] Chase's result was extended to a description of $\prod$-projective modules (a module M is $\prod - x$ resp. $\sum - x$ in case all direct products resp. direct sums of copies of M have property x). Roughly speaking, the problem can be divided into three separate questions: Which restrictions on the factors are necessary and sufficient for the direct product $\prod_{k\in K} M_k$ to be (1) a direct sum of countably generated submodules, (2) flat, (3) trace-accessible (an R-module M is called trace-accessible if $MT = M$ where $T = \sum\{\mathrm{Im}(f): f \in \mathrm{Hom}_R(M, R)\}$)? This paper primarily deals with the first question. Conditions (2), (3) are fulfilled for *all* families of projective R-modules if R satisfies a finiteness condition which lies between being left coherent and left noetherian [15, 4.2]. For additional information see [4], [13].

We will see that our problem is closely related to the dual one for injectivity. As was shown in [12, 3.4], an R-module M is $\sum$-pure-injective $=\sum$-algebraically compact iff it satisfies the minimum condition on subgroups PM, where P is a subfunctor of the forgetful functor $\mathrm{Mod}\, R \to \mathrm{Mod}\, \mathbb{Z}$ which commutes with direct products. Such functors, called p-functors, serve to unify various chain conditions: The subgroups PM of M include all finitely generated $\mathrm{End}(M_R)$-submodules of M, all annihilators $\mathrm{Ann}_M(X)$ of subsets X of R, and the subgroups $M\mathfrak{a}$ where $\mathfrak{a}$ is a finitely generated left ideal of R. Moreover, they are closed under arbitrary intersections and finite sums. (For more representative examples see Sec. 3, Proof of Theorem 2.)

The following two theorems are of interest beyond the above mentioned problem. The assertion of the first is a natural continuation of [3, 3.1] resp. [7]

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and [13, L.8], where the special cases $M_k \simeq R$ resp. $M_k \simeq M_{k'}$ for all $k, k' \in K$ are considered, but the proof requires new arguments.

THEOREM 1. *Let $\aleph$ be an infinite cardinal, $(M_k)_{k\in K}$ an arbitrary family of R-modules.*

If $\prod_{k\in K} M_k$ is (a direct summand of) a direct sum of submodules, each of cardinality at most $\aleph$, then for each descending chain $(P_n)_{n\in\mathbb{N}}$ of p-functors there is a natural number n_0 such that

$$|\{k \in K: P_n M_k \not\subseteq P_{n_0} M_k \text{ for some } n \in \mathbb{N}\}| < \aleph.$$

($|A|$ is the cardinality of a set A).

Note that for $\aleph \geqslant |R|$ any $\aleph$-generated module is of cardinality at most $\aleph$. Thus projectivity of $\prod_{k\in K} M_k$ forces the above uniform chain condition on the M_k's, with an exceptional set of cardinality less than $|R|$ for each chain of p-functors (this is obvious in case $|R|$ is finite).

The following "converse" does not involve direct products at all.

THEOREM 2. *Every $\sum$-algebraically compact, flat and trace-accessible module is projective.*

Theorem 2 applies to our problem as follows: Suppose $\prod_{k\in K} M_k$ is flat and trace-accessible, and for every descending chain $(P_n)_{n\in\mathbb{N}}$ of p-functors there is a natural number n_0 such that $P_n M_k = P_{n_0} M_k$ for $n \geqslant n_0$ and all k. Then $\prod_{k\in K} M_k$ is projective.

The first of the following corollaries was recently proved in [13, S.1].

COROLLARY 1. *Given a $\prod$-flat and $\prod$-trace-accessible module M, the following are equivalent:*

(1) *M is $\prod$-projective.*

(2) *M is $\sum$-algebraically compact.*

(3) *M satisfies the descending chain condition on subgroups $M\mathfrak{a}$, where $\mathfrak{a}$ is a finitely generated left ideal of R.*

COROLLARY 2. *Suppose M is a simple module. Then M is $\prod$-projective iff M is projective and finite dimensional over its division ring of endomorphisms. (In contrast to this, there are simple, $\sum$-algebraically compact modules which are not finite dimensional over their endomorphism ring.)*

Supplementary remarks. 1. Let M be a projective module with trace T. The equality $P_n M = P_m M$ is equivalent to $T(P_n R) = T(P_m R)$. Moreover, when R is left coherent, the minimum condition on subgroups PM of M is equivalent to that on left ideals of the type $T\mathfrak{a}$, where $\mathfrak{a}$ is a finitely generated left ideal of R. (Compare [3, 3.3]).

2. If R is commutative, then every $\sum$-algebraically compact direct product of projective R-modules is trace-accessible.

The proofs will be deferred till Section 3.

Of course a uniform as well as complete description of projective direct products in terms of the factors is not be be expected. Products over an index set smaller than $|R|$ require an individual treatment, as the following trivial example shows: If R is the direct product of rings R_k, then $R = \prod_{k \in K} R_k$ is a projective R-module without further restrictions on the R_k's. Also note that, in contrast to the case $M = R$, the chain condition in Corollary 1 cannot in general be replaced by the minimum condition on finitely generated $\mathrm{End}(M_R)$-submodules of M: Let R be the endomorphism ring of an infinite dimensional vector space and M the ideal of endomorphisms of finite rank. Then M_R is projective, $\prod$-flat and $\prod$-trace-accessible, $\mathrm{End}(M_R) \cong R$, and ${}_R M$ satisfies the descending chain condition on finitely generated submodules. Yet M_R is not $\prod$-projective.

In a forthcoming paper we will exhibit a large class of examples of $\sum$-algebraically compact rings, i.e. rings for which all projective modules are $\sum$-algebraically compact.

2. Further Applications of Theorem 1

It is immediate that Theorem 1 contains the well-known Faith-Walker-Theorem on decompositions of injective modules:

Corollary 3 (e.g. [1, 25.8]). *R is right noetherian iff there exists a cardinal number $\aleph$ such that every injective R-module is a direct sum of submodules of cardinality at most $\aleph$.*

Proof. One implication is standard. For the converse let $\aleph$ be as above, let M be an injective cogenerator for $\mathrm{Mod}\, R$, and apply Theorem 1 to M^K with $|K| \geqslant \sup(\aleph, \aleph_0)$. This shows that M satisfies the descending chain condition on subgroups $\mathrm{Ann}_M(X)$, $X \subset R$, and equivalently, R satisfies the ascending chain condition on right ideals $\mathrm{Ann}_R(Y)$, $Y \subset M$. Since M is a cogenerator, the latter comprehend all right ideals.

By a classical result in abelian group theory, direct sums of countable (abelian) torsion groups can be characterized by sets of invariants [5, I 23.1 and II 78.4]. The method of proof of Theorem 1 shows that for the torsion subgroup of a direct product of abelian groups this advantageous decomposability occurs only in trivial cases. The first part of the following corollary is due to Mishina and Łoś (for $R = \mathbb{Z}$ and $\aleph = \aleph_0$ [10] and [9]). For the rest of this section $\aleph$ will denote an infinite cardinal.

COROLLARY 4. *Let R be a Dedekind domain and M the direct product of R-modules M_k $(k \in K)$, each with the largest divisible submodule M_k'. For $0 \neq \mathfrak{p} \in \operatorname{Spec} R$, $T_\mathfrak{p}M$ is the $\mathfrak{p}$-component of the torsion submodule of M.*

1. *Suppose M is a direct sum of submodules of cardinality $\leqslant \aleph$. Then each M_k has the same property, and there is a non-zero ideal $\mathfrak{a}$ of R and a subset K' of K with $|K'| < \aleph$ such that $(M_k/M_k')\mathfrak{a} = 0$ for $k \in K \backslash K'$. (For $\aleph = \aleph_0$ and R countable the converse is clearly also true.)*

2. *Let R be countable. Then $T_\mathfrak{p}M$ is a direct sum of countable submodules iff each $T_\mathfrak{p}M_k$ has this property and $(T_\mathfrak{p}M_k/T_\mathfrak{p}M_k')\,\mathfrak{p}^n = 0$ for some $n \in \mathbb{N}$ and almost all $k \in K$.*

Proof. 1. Write $M_k = M_k' \oplus M_k''$. By Kaplansky's Theorem, every direct summand of M is a direct sum of $\aleph$-generated submodules which, again by hypothesis, are even of cardinality $\leqslant \aleph$. In particular, this is true for $\prod_{k \in K} M_k''$. Since the p-functors $M \mapsto M\mathfrak{b}$, where $\mathfrak{b}$ runs through all non-zero ideals of R, form a downward directed system, Theorem 1 guarantees the existence of a non-zero ideal $\mathfrak{a}$ of R and a subset K' of K of cardinality $< \aleph$ such that $M_k''\mathfrak{a}\mathfrak{b} = M_k''\mathfrak{a}$ for all non-zero ideals $\mathfrak{b}$ and all $k \in K\backslash K'$. Since M_k'' is reduced, this means $M_k''\mathfrak{a} = 0$ for $k \in K \backslash K'$.

2. "If" is clear, since $\prod_{k \in K} T_\mathfrak{p}M_k'$ is a direct sum of countable modules. "Only if": It is enough to consider the case where M_k is reduced $\mathfrak{p}$-primary for all k. Assume that for each $i \in \mathbb{N}$ there are infinitely many k with $M_k\,\mathfrak{p}^i \neq 0$. Then there is a sequence $(k_n)_{n \in \mathbb{N}}$ of pairwise different elements of K and a strictly increasing sequence $(l_n)_{n \in \mathbb{N}}$ of natural numbers such that M_{k_n} contains a direct summand isomorphic to $R/\mathfrak{p}^{l_n}$. (If almost all M_k's are bounded, this is clear. Otherwise use the fact that a reduced unbounded $\mathfrak{p}$-primary module contains cyclic direct summands of arbitrarily high order, which is an obvious generalization of [5, I p. 119 Ex. 1].) Thus we are free to start with $K = \mathbb{N}$ and $M_n = R/\mathfrak{p}^{l_n}$ where $l_{n+1} > l_n$. Pick $r_n \in \mathfrak{p}^{l_n - 1}\backslash \mathfrak{p}^{l_n}$. Replacing M by TM we can now proceed as in the proof of Theorem 1, Case 1, with $\aleph = \aleph_0$, P_n: $M \mapsto M\mathfrak{p}^{l_n - 1}$, $x_n = r_n + \mathfrak{p}^{l_n} \in P_n M_n \backslash P_{n+1} M_n$, and $C_n = \{n\}$.

The following specialized version of Theorem 1 is often more manageable than the original one.

COROLLARY 5. *Let $(M_k)_{k \in K}$ be a family of (right) modules over an arbitrary ring R. If $\prod_{k \in K} M_k$ is a direct sum of submodules of cardinality at most $\aleph$, then for each set E of pairwise orthogonal idempotents of R there is a subset K' of K of cardinality smaller than $\aleph$ such that $(\prod_{k \in K \backslash K'} M_k)\,e = 0$ for almost all $e \in E$.*

Proof. Assume the conclusion to fail for some set E of orthogonal idempotents. From this we will derive the existence of a family $(E_n)_{n \in \mathbb{N}}$ of pairwise disjoint nonempty subsets of E with $|\{k \in K : M_k E_n \neq 0\}| \geqslant \aleph$ for all n. With the choice $P_n : M \mapsto \mathrm{Ann}_M(\bigcup_{i \leqslant n} E_i)$, Theorem 1 will then show that the hypothesis cannot be fulfilled: Observe that $xe \neq 0$ implies $xe \in P_n M_k \backslash P_{n+1} M_k$ for $x \in M_k$ and $e \in E_{n+1}$, which means $|\{k \in K: P_{n+1} M_k \subsetneqq P_n M_k\}| \geqslant \aleph$ for all n.

For $e \in E$ let $K(e)$ denote the set of all $k \in K$ with $M_k e \neq 0$ and E_0 the set of all $e \in E$ with $|K(e)| \geqslant \aleph$. In case E_0 is infinite, any partition of E_0 into non-empty subsets E_n $(n \in \mathbb{N})$ has the desired property. Thus we may suppose that E_0 is finite.

We are now interested in the set $K_0 = \{k \in K: M_k(E \backslash E_0) \neq 0\}$. In view of $(\prod_{k \in K \backslash K_0} M_k)\, e = 0$ for $e \in E \backslash E_0$ our assumption implies $|K_0| \geqslant \aleph$. Pick $e_k \in E \backslash E_0$ with $M_k e_k \neq 0$ for $k \in K_0$ and consider the cardinality τ of the set $\{e_k : k \in K_0\}$. If $\tau \geqslant \aleph$, any partition of $\{e_k : k \in K_0\}$ into subsets E_n of cardinality $\geqslant \aleph$ will do. Now suppose $\tau < \aleph$. For convenience, we identify τ with a suitable subset of K_0 so that each e_k $(k \in K_0)$ occurs exactly once among the e_t's $(t \in \tau)$. From $K_0 = \bigcup_{t \in \tau} K(e_t)$ we conclude that $\sup_{t \in \tau} |K(e_t)| \geqslant \aleph$. In particular, since $|K(e_t)| < \aleph$ for $t \in \tau$, the well-ordered set $\{|K(e_t)|: t \in \tau\}$ does not possess a largest element and hence contains a countably infinite family of pairwise disjoint cofinal subsets. But this means that we can find pairwise disjoint subsets K_n $(n \in \mathbb{N})$ of τ with $\sup_{t \in K_n} |K(e_t)| = \aleph$, yielding $E_n = \{e_t : t \in K_n\}$ as required.

Corollary 5 is illustrated by the following

EXAMPLES. In both examples let $\aleph \geqslant |R|$.

1. Let R be the endomorphism ring of an infinite dimensional vector space (or any factor ring of a ring of this type). Then no direct product of non-zero R-modules which extends over an index set of cardinality $\geqslant \aleph$ is a direct sum of $\aleph$-generated submodules. In particular, no such product is projective. (In contrast to this, R^K is clearly a projective right R-module if $|K|$ does not exceed the dimension of the vector space.)

Proof. As is well-known R contains a largest proper two-sided ideal, namely the set of all endomorphisms of rank smaller than the dimension of the vector space (or the homomorphic image of this set). Moreover, there is an infinite set $E = \{e_n : n \in \mathbb{N}\}$ of orthogonal idempotents outside this ideal. Because $Me_n \neq 0$ for each non-zero R-module M, Corollary 5 yields our claim.

2. Let $(S_i)_{i \in I}$ be a family of rings, R a subring of $\prod_{i \in I} S_i$ containing $\bigoplus_{i \in I} S_i$, and let M_k be an R-module with trace T_k for $k \in K$. Suppose that $\prod_{k \in K} M_k$ is a direct sum of $\aleph$-generated submodules. Then there is a subset K' of K with $|K'| < \aleph$ such that $\sum_{k \in K \backslash K'} T_k$ is contained in a finite sum $\bigoplus_{i \in I'} S_i$.

Proof. Choose $E = \{e_i : i \in I\}$ where $e_i \in R$ carries 1 in the i-th place, 0 elsewhere, and apply Corollary 5.

3. Proofs of the Theorems

Proof of Theorem 1. Let $M = \prod_{k\in K} M_k$. By Kaplansky's Theorem (generalized version [1, 26.1]), we may assume $M = \bigoplus_{l\in L} Q_l$ with $|Q_l| \leqslant \aleph$. Let (P_n) be a descending chain of p-functors. If we denote the set $\{k \in K: P_{n+1}M_k \subsetneqq P_nM_k\}$ by E_n, our claim is the existence of a natural number n_0 such that $|\bigcup_{n\geqslant n_0} E_n| < \aleph$. Assume the contrary and distinguish two cases:

(I) There is $n_0 \in \mathbb{N}$ with $|E_n| < \aleph$ for $n \geqslant n_0$.

(II) There are infinitely many $n \in \mathbb{N}$ with $|E_n| \geqslant \aleph$.

In deriving a contradiction from (I), we will use the following set-theoretic lemma. For each ordinal number α we identify the α-th infinite cardinal $\aleph_\alpha$ with the first equivalent ordinal number.

Lemma [2, p. 129 Exercise 19b]. *Let α be a limit ordinal and $c(\alpha)$ the smallest among the cardinals of the cofinal subsets of $\aleph_\alpha$. Then $\aleph_\alpha^{c(\alpha)} > \aleph_\alpha$.*

Case II will be proved impossible in two steps, the second of which is an adaptation of Chase's proof of [3, 3.1]. We will outline it for the sake of completeness.

Case I. Reindex the P_n's so that $n_0 = 1$. From $|\bigcup_{n\in\mathbb{N}} E_n| \geqslant \aleph$ and $|E_n| < \aleph$ for all n we deduce $\aleph = \aleph_0$ or $\aleph = \aleph_\alpha$ with a limit ordinal α and $c(\alpha) = \aleph_0$. In view of the lemma we have $\aleph^{\aleph_0} > \aleph$ in both cases. By induction define pairwise disjoint subsets K_n of E_n respectively: $K_1 = E_1$, $K_{m+1} = E_{m+1}\setminus\bigcup_{i\leqslant m} K_i$. Note that $|K_n| < \aleph$ and $|\bigcup_{n\in\mathbb{N}} K_n| = \aleph$.

For $n \in \mathbb{N}$ choose a subset C_n of the power set $\mathfrak{P}(K_n)$ of K_n of cardinality equal to the minimum of $|\mathfrak{P}(K_n)|$ and $\aleph$. (Note that the adoption of the continuum hypothesis would ensure $|\mathfrak{P}(K_n)| < \aleph$). Moreover, pick $x_k \in P_nM_k\setminus P_{n+1}M_k$ for $k \in K_n$ and all n, and let $M(n)$ be the set of all $y = (y_k) \in M$ such that, for some $T \in C_n$, we have $y_k = x_k$ if $k \in T$ and $y_k = 0$ if $k \in K\setminus T$. Observe $|M(n)| = |C_n| \leqslant \aleph$. Since the K_n's are disjoint, the definition of the subsets $\bigoplus_{n\in\mathbb{N}} M(n)$ and $\prod_{n\in\mathbb{N}} M(n)$ of M is self-evident. The cardinality of the first does clearly not exceed $\aleph$, and therefore it is contained in a direct summand V of M which is also of cardinality at most $\aleph$. On the other hand, observe that the cardinality of $\prod_{n\in\mathbb{N}} M(n)$ exceeds $\aleph$: In case $|M(n)| < \aleph$ for all n, we have $|M(n)| = |\mathfrak{P}(K_n)|$ and conclude $|\prod_{n\in\mathbb{N}} M(n)| = |\mathfrak{P}(\bigcup_{n\in\mathbb{N}} K_n)| > \aleph$. Now suppose $|M(m)| = \aleph$ for some m, i.e. $|\mathfrak{P}(K_m)| \geqslant \aleph$. Then K_m is infinite, hence $|K_m| < \aleph$ implies $\aleph > \aleph_0$, and $|\bigcup_{n\in\mathbb{N}} K_n| = \aleph$ means $\sup_{n\in\mathbb{N}} |K_n| = \aleph$. In particular, there are infinitely many natural numbers n with $|K_n| > |K_m|$ and a fortiori with $|M(n)| = \aleph$. Again it follows that $|\prod_{n\in\mathbb{N}} M(n)| = \aleph^{\aleph_0} > \aleph$.

Let $M = V \oplus W$. In view of the difference in cardinalities there are different elements $x, x' \in \prod_{n\in\mathbb{N}} M(n)$ with the same V-projection, i.e. $z = x - x' \in W$.

Let π_n be the projection of M onto the direct product of the M_k's, $k \in \bigcup_{i \leqslant n} K_i$, along the product of the remaining factors and observe $\pi_n(z) = \pi_n(x) - \pi_n(x') \in V$. From $z - \pi_n(z) = v_n + w_n$ with $v_n \in V$ and $w_n \in W$ we therefore derive $\pi_n(z) + v_n = 0$, that is $z = w_n$ for all n. Because each component z_k equals $\pm x_k$ or 0, we know that $z - \pi_n(z) \in P_{n+1}M$ and infer $w_n \in P_{n+1}M$. But $z \in \bigcap_{n \in \mathbb{N}} P_n M$ contradicts the fact that, for some k and n, we have $z_k = \pm x_k \in P_n M_k \backslash P_{n+1} M_k$.

Case II. In a preliminary step we will construct a strictly increasing sequence $(i_n)_{n \in \mathbb{N}}$ of natural numbers and pairwise disjoint subsets K_n of E_{i_n} respectively, with $|K_n| \geqslant \aleph$ this time. We give the details of the induction step, the beginning being clear then if we set $K_0 = \varnothing$ and $N_0 = \{n \in \mathbb{N} : |E_n| \geqslant \aleph\}$. Let $i_1 < i_2 < \cdots < i_m$ and $K_1, \ldots, K_m$ be as required and let $N_0 \supset N_1 \supset \cdots \supset N_m$ be infinite subsets of $\mathbb{N}$ such that $N_l = \{n \in N_{l-1} : n > i_l \text{ and } |E_n \backslash \bigcup_{j \leqslant l} K_j| \geqslant \aleph\}$ for $1 \leqslant l \leqslant m$. We obtain i_{m+1}, K_{m+1}, N_{m+1} as follows: Let $i_{m+1} = \min N_m$. Divide $E_{i_{m+1}} \backslash \bigcup_{j \leqslant m} K_j$ into two disjoint subsets A and B, each of cardinality at least $\aleph$. In case there are infinitely many $n \in N_m$ with $|(E_n \backslash \bigcup_{j \leqslant m} K_j) \cap A| \geqslant \aleph$, define $K_{m+1} = B$, otherwise define $K_{m+1} = A$. In either case our construction guarantees that $|K_{m+1}| \geqslant \aleph$ and that the set $N_{m+1} = \{n \in N_m : n > i_{m+1}$ and $|E_n \backslash \bigcup_{j \leqslant m+1} K_j| \geqslant \aleph\}$ is infinite.—Simplify the notation to $i_n = n$ by passing to a suitable subsequence of (P_n).

This time, define $M(n) = \prod_{k \in K_n} M_k$. Let $q_l : M = \bigoplus_{j \in L} Q_j \to Q_l$ be the projection. Our aim is to (inductively) construct sequences $(m_n)_{n \in \mathbb{N}}$ of elements $m_n \in P_n M(n)$ and $(l_n)_{n \in \mathbb{N}}$ of elements of L with $q_{l_n}(m_n) \in P_n Q_{l_n} \backslash P_{n+1} Q_{l_n}$ and $q_{l_n}(m_i) = 0$ for $i < n$. Defining $m = \sum_{n \in \mathbb{N}} m_n \in M$, we then conclude $q_{l_n}(m) = q_{l_n}(m_n) + q_{l_n}(\sum_{i > n} m_i) \neq 0$ for each n, since the second summand lies in $P_{n+1} Q_{l_n}$, whereas the first does *not*. This contradicts $m \in \bigoplus_{l \in L} Q_l$.

The case $n = 1$ runs parallel to the induction step. So let m_i, l_i be as above for $i \leqslant n - 1$. We have to show that there is an element $l_n \in L$, outside $L' = \{l \in L : q_l(m_i) \neq 0 \text{ for some } i \leqslant n - 1\}$, such that $q_{l_n}(P_n M(n)) \not\subset P_{n+1} Q_{l_n}$. The contrary would mean that the homomorphism $P_n M(n)/P_{n+1} M(n) \to P_n Q_l / P_{n+1} Q_l$ induced by q_l is zero for all $l \in L \backslash L'$. Since the natural map $P_n M(n)/P_{n+1} M(n) \to P_n M / P_{n+1} M \cong \bigoplus_{l \in L} P_n Q_l / P_{n+1} Q_l$ is injective, we infer that the map $P_n M(n)/P_{n+1} M(n) \to \bigoplus_{l \in L'} P_n Q_l / P_{n+1} Q_l$ is injective. But this is impossible, because the cardinality of the range is at most $\aleph$, whereas the cardinality of $P_n M(n)/P_{n+1} M(n) \cong \prod_{k \in K_n} P_n M_k / P_{n+1} M_k$ exceeds $\aleph$.

Proof of Theorem 2. The following "matrix functors" $[A, \alpha]$ are easily seen to be p-functors: If $A = (a_{ij})_{i \in I, j \in J}$ is a column-finite R-matrix and $\alpha \in I$, then $[A, \alpha] M$ is the α-th projection of the solution set of the system $\sum_{i \in I} X_i a_{ij} = 0$ $(j \in J)$ in M^I. Call A resp. $[A, \alpha]$ finite if J is finite. (According to [12, 3.4], the finite matrix functors are representative in testing the descending chain condition on subgroups PM, where P is an arbitrary p-functor.)

Let M be $\sum$-algebraically compact, flat and trace-accessible. Moreover, let T be the trace ideal of M in R and Δ the ideal of $S = \mathrm{End}(M_R)$ consisting of all finite sums of endomorphisms $mf(-)$ where $m \in M$, $f \in \mathrm{Hom}_R(M, R)$.

First we will check that $x \in \Delta x$ for all $x \in M$. Let $(m_i)_{i \in I}$ be a generating system for M containing $x = m_\alpha$. If the columns of the R-matrix $A = (a_{ij})_{i \in I, j \in J}$ run through the family of relations of the m_i's, i.e. $(a_{ij})_{i \in I} \in R^{(I)}$ with $\sum_{i \in I} m_i a_{ij} = 0$, then $Sx = [A, \alpha] M$. Our aim is to replace A by a finite matrix A_0 in this equation. Then the first claim will be established: Expressing flatness in terms of relations ([3, 2.3]) yields $[A_0, \alpha] M = M([A_0, \alpha] R)$, and we conclude $\Delta x = \Delta M([A_0, \alpha] R) = MT([A_0, \alpha] R) = M([A_0, \alpha] R) = Sx$.—For an arbitrary finite subset J' of J define $A(J') = (a_{ij})_{i \in I, j \in J'}$. By hypothesis, the set of subgroups $[A(J'), \alpha] M$ of M, where J' runs through all finite subsets of J, contains a minimal element $[A(J_0), \alpha] M$ which is even a smallest element. In other words, for $m \in [A(J_0), \alpha] M$, each finite system

$$\sum_{i \in I \setminus \{\alpha\}} X_i a_{ij} = -m a_{\alpha j} \quad (j \in J')$$

can be solved in M. Since M is algebraically compact, the complete system

$$\sum_{i \in I \setminus \{\alpha\}} X_i a_{ij} = -m a_{\alpha j} \quad (j \in J)$$

is solvable, that is $m \in [A, \alpha] M$. Thus $[A, \alpha] M = [A(J_0), \alpha] M$.

Gruson and Raynaud have proved in [6, 2.2.1 and 2.3.4] that every element of a module with the property we have just verified for M is contained in a projective, pure submodule. Since a pure submodule of M is again $\sum$-algebraically compact $= \sum$-pure-injective and therefore a direct summand of M ([12, 3.4] and [11, Theorem 2]), transfinite induction can now be employed to slice off projective direct summands of M until M is exhausted. More precisely: For each ordinal α there is a projective submodule M_α of M such that the family $(M_\beta)_{\beta \leqslant \alpha}$ has the following properties:

1. The sum of the M_β's ($\beta \leqslant \alpha$) is direct, and $\bigoplus_{\beta \leqslant \alpha} M_\beta$ is a direct summand of M.

2. $M_\alpha \neq 0$ in case $\bigoplus_{\beta < \alpha} M_\beta \subsetneqq M$.

We conclude $M = \bigoplus_{\beta \leqslant \alpha} M_\beta$ for $\alpha > |M|$. The details are routine and will be left to the reader. (Note that trace-accessibility is not inherited by direct summands in general, but the property exhibited in step 1 is.)

Proof of Corollary 1. (1) $\Leftrightarrow$ (2) is an immediate consequence of Theorems 1 and 2, (2) $\Rightarrow$ (3) is clear. For (3) $\Rightarrow$ (2) compare the first paragraph of the proof of Theorem 2 and use [12, 1.3].

Proof of Corollary 2. "If": Note that M is $\sum$-algebraically compact and T is a finitely generated left ideal of R. The latter implies that all direct products

M^I are T-accessible and in particular trace-accessible. Since M is also $\prod$-flat by [4, 2.1], Theorem 2 applies.

"Only if": It does not affect our problem to assume that M_R is faithful, i.e. R right primitive. In particular, the left socle of R coincides with the right socle and is of the form $So(R) = \bigoplus_{i\in I} e_iR$, where the e_i's are idempotents and both e_iR and Re_i are simple.

Being simple and projective, M is (up to isomorphism) contained in $So(R)$, and hence $So(R) \neq 0$. Because R is prime, we infer that the left annihilator $\mathrm{Ann}_R So(R) = \bigcap_{i\in I} \mathrm{Ann}_R(e_i)$ equals zero. Besides, since M is faithful, the minimum condition on annihilators $\mathrm{Ann}_M(X)$ of subsets X of R forces the minimum condition on annihilator left ideals upon R. Therefore some *finite* intersection $\bigcap_{i\in I'} \mathrm{Ann}_R(e_i)$ equals zero. Thus R is semisimple and equivalently $\dim({}_sM) < \infty$.

Proofs of the remarks. Let S be the endomorphism ring of M_R.

1. In [15, 2.1 resp. 3.1] the author has shown that $PM = M(PR)$ for each p-functor P resp. that $\mathfrak{a} \mapsto M\mathfrak{a}$ is a bijection between the left ideals $\mathfrak{a} = T\mathfrak{a}$ of R and the left S-submodules of M. This yields the first assertion. The second follows from the first, since R is left coherent iff for each finite matrix functor $[A, \alpha]$ the left ideal $[A, \alpha]\, R$ is finitely generated (see [12, 1.3] and the first paragraph of the proof of Theorem 2).

2. Suppose that R is commutative and that $\prod_{k\in K} M_k$ is a $\sum$-algebraically compact direct product of projective R-modules M_k. If T_k is the trace of M_k respectively, the ideal $T' = \sum_{k\in K} T_k$ is contained in the trace of $\prod_{k\in K} M_k$. It is therefore enough to show that T' is finitely generated, since this implies $(\prod_{k\in K} M_k)\, T' = \prod_{k\in K} M_k$.

First observe $T_k = e_kR$ with $e_k^2 = e_k$: By [14, 2.4 and 3.1] the ideal T_k is pure in R, and $T_k\mathfrak{a} = \mathfrak{a}$ for all ideals $\mathfrak{a}$ contained in T_k. Deduce $T_k(PR) = PT_k$ for each p-functor P. In view of this and 1., the hypothesis implies that T_k is $\sum$-pure-injective and hence a direct summand of R.

The sum $\sum_{k\in K} e_kR$ is equal to a finite subsum: Assume on the contrary that there is a subset $\{k_n : n \in \mathbb{N}\}$ of K with $e_{k_{n+1}} \notin \sum_{i\leqslant n} e_{k_i}R$ for all n. Then none of the idempotents $f_1 = e_{k_1}$, $f_{n+1} = e_{k_{n+1}} \prod_{i\leqslant n} (1 - e_{k_i})$, $n \in \mathbb{N}$, is equal to zero. On the other hand, $f_{n+1} \sum_{k\leqslant n} T_k = 0$. We conclude $P_{n+1}T_{k_{n+1}} \subsetneqq P_nT_{k_{n+1}}$, if P_n is the p-functor $M \mapsto \mathrm{Ann}_M\{f_1, \ldots, f_n\}$. But in view of 1. and the preceding paragraph, this violates the presumed chain condition of $\prod_{k\in K} M_k$.

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