Linear Operators Strongly Preserving r-Potent Matrices Over Semirings

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ABSTRACT

A matrix X is said to be *r-potent* if $X^r = X$. We investigate the structure of linear operators on matrices over antinegative semirings that map the set of r-potent matrices into itself and the set of matrices which are not r-potent into itself.

0. INTRODUCTION

Let $\mathcal{M}_n(\mathbb{F})$ be the algebra of $n \times n$ matrices over an algebraic structure $\mathbb{F}$. Problems of the following type have been of interest to several authors: let $\mathcal{V}$ be an algebraic subset of $\mathcal{M}_n(\mathbb{F})$; characterize the semigroup of all linear operators $\mathbf{T}: \mathcal{M}_n(\mathbb{F}) \to \mathcal{M}_n(\mathbb{F})$ that map $\mathcal{V}$ into $\mathcal{V}$. Beasley and Pullman characterized the linear operators that strongly preserve idempotence over an antinegative semiring with no zero divisors [3].

The purpose of this paper is to extend these previous results. A matrix X is said to be *r-potent* if $X^r = X$. In this paper we will characterize the semigroup of linear operators that *strongly preserve* the set of *r-potent matrices*, i.e. that map r-potents to r-potents and non-r-potents to non-r-potents. We find that it is generated by *transposition*, the *similarity operators*, and some *scaling operators* when $\mathbb{F}$ is an antinegative semiring with no zero divisors (in particular a chain semiring).

Preliminary results and definitions are presented in Section 1.

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1. PRELIMINARIES

A *semiring* is a binary system $(\mathbb{S}, +, \times)$ such that $(\mathbb{S}, +)$ is an abelian monoid (identity 0), $(\mathbb{S}, \times)$ is a monoid (identity 1), $\times$ distributes over $+$, $0 \times s = s \times 0 = 0$ for all s in $\mathbb{S}$, and $1 \neq 0$. Usually $\mathbb{S}$ denotes the system and $\times$ is denoted by a juxtaposition. If $(\mathbb{S}, \times)$ is abelian, then $\mathbb{S}$ is *commutative*. If 0 is the only element to have an additive inverse, then $\mathbb{S}$ is *antinegative*. All rings with unity are semirings, but no such ring is antinegative. The two-element *Boolean algebra* $\mathbb{B}$, the nonnegative integers $\mathbb{Z}^+$, the nonnegative rationals $\mathbb{Q}^+$, and the nonnegative reals $\mathbb{R}^+$ all serve as examples of antinegative semirings which occur in combinatorics.

Algebraic terms such as *unit*, *zero divisor*, *linearity*, and *invertibility* are defined for semirings as for rings.

We let $\mathscr{M}_n(\mathbb{S})$ denote the set of $n \times n$ matrices over $\mathbb{S}$. The matrix all of whose entries are 1 is denoted J_n. We will suppress the subscripts on these matrices when the order is evident from the context,.

If A and B are in $\mathscr{M}$, we say B *dominates* A (written $A \leqslant B$) if $b_{ij} = 0$ implies $a_{ij} = 0$ for all i, j.

The number of nonzero entries in a matrix A is denoted $|A|$. The number of elements in a set $\mathscr{S}$ is also denoted $|\mathscr{S}|$.

A matrix S having at least one nonzero off-diagonal entry is a *star matrix* if all its nonzero entries lie on a *line* (a row or a column). An *s-star matrix* is a star matrix having $|S| = s$ and all diagonal entries 0. A zero-one $n \times n$ matrix with only one entry equal to 1, say the (i, j)th entry, is called a *cell*, E_{ij}. A set of cells is *collinear* if their nonzero entries lie in one line.

When X and Y are in $\mathscr{M}_n(\mathbb{B})$, we define $X \setminus Y$ to be the matrix Z such that $z_{ij} = 1$ if and only if $x_{ij} = 1$ and $y_{ij} = 0$. We let $K = J \setminus I$.

We denote the *Hadamard product* of A and B in $\mathscr{M}$ by $A \circ B$. That is, $C = A \circ B$ if and only if $c_{ij} = a_{ij} b_{ij}$ for all i and j.

An operator $\mathbf{L}$ is *nonsingular* if $\mathbf{L}(X) = O$ only when $X = O$; such operators need not be invertible. For example, when $\mathbb{S} = \mathbb{B}$ if $\mathbf{L}(X) = J$ for all $X \neq O$ in $\mathscr{M}_n(\mathbb{S})$ and $\mathbf{L}(O) = O$, then $\mathbf{L}$ is linear and nonsingular, but not invertible unless $m = n = 1$. However, we have Theorem 2.1 below for operators which strongly preserve r-potence.

Henceforth, we will assume that $\mathbf{T}$ is a linear operator on $\mathscr{M}_n(\mathbb{S})$ which strongly preserves the set of r-potent matrices, and $n \geqslant r \geqslant 2$.

2. THE BOOLEAN (0, 1) CASE

Throughout this section $\mathbb{S} = \mathbb{B}$, the Boolean algebra of two elements.

THEOREM 2.1. *$\mathbf{T}$ is nonsingular.*

Proof. The case $r = 2$ was proved in [3, Lemma 1.3]. So we only need to prove the theorem for $n \geqslant r \geqslant 3$. Suppose $\mathbf{T}(X) = O$ and $X \neq O$; then $\mathbf{T}(C) = O$ for some cell C, because $\mathbb{B}$ is antinegative and $\mathbf{T}$ is linear. Now $[\mathbf{T}(J)]^r = \mathbf{T}(J)$, since $J^r = J$ and $\mathbf{T}$ strongly preserves r-potence. But $[\mathbf{T}(J \setminus C)]^r = [\mathbf{T}(J)]^r = \mathbf{T}(J) = \mathbf{T}(J \setminus C)$, a contradiction, since $(J \setminus C)^r = J \neq J \setminus C$. ∎

NOTE. If $r = 1$, then $\mathbf{T}$ may be singular; in fact, all operators preserve 1-potence strongly, since all matrices are 1-potent.

LEMMA 2.1. *If $n \geqslant 3$ and E is a cell, then $\mathbf{T}(E)$ is a cell.*

Proof. Suppose there is a cell C with $|\mathbf{T}(C)| \geqslant 2$. Let $X_1 = C$, and index the cells by $C_1 = C$ and the rest $C_2, C_3, \dots, C_{n^2}$ arbitrarily. For $2 \leqslant j \leqslant n^2$, let $X_j = X_{j-1}$ or $X_{j-1} + C_{j+1}$ according as $\mathbf{T}(C_j) \leqslant \mathbf{T}(X_{j-1})$ or not. Then $|X_j| \leqslant |X_{j-1}| + 1$ for all $j \geqslant 2$. If equality held for every $2 \leqslant j \leqslant n^2$, then $|\mathbf{T}(X_j)| \geqslant j + 1$, in particular for $j = n^2$, which is impossible. Therefore $|X_{n^2}| \leqslant n^2 - 1$ and $\mathbf{T}(H) \leqslant \mathbf{T}(X_{m-1})$ for some cell $H = C_m$ not dominated by X_{m-1}. Therefore $\mathbf{T}(J) = \mathbf{T}(J \setminus H)$. But $[\mathbf{T}(J)]^r = \mathbf{T}(J)$ because $\mathbf{T}$ preserves r-potence, and so $[\mathbf{T}(J \setminus H)]^r = \mathbf{T}(J \setminus H)$. Therefore $(J \setminus H)^r = J \setminus H$, because $\mathbf{T}$ preserves r-potence strongly. But in fact $(J \setminus H)^r = J$, a contradiction. ∎

LEMMA 2.2. *If $n \geqslant 3$, then $\mathbf{T}$ is bijective on the set of cells.*

Proof. Let E and F be different cells. Suppose $\mathbf{T}(E) = \mathbf{T}(F)$; then $\mathbf{T}(J) = \mathbf{T}\{[J \setminus (E + F)] + (E + F)\} = \mathbf{T}[J \setminus (E + F)] + \mathbf{T}(E + F) = \mathbf{T}[J \setminus (E + F)] + \mathbf{T}(E) + \mathbf{T}(F) = \mathbf{T}[J \setminus (E + F)] + \mathbf{T}(E) = \mathbf{T}(J \setminus F)$. But J is r-potent and $J \setminus F$ is not—a contradiction, since $\mathbf{T}$ strongly preserves r-potence. Thus $\mathbf{T}(E) \neq \mathbf{T}(F)$.

LEMMA 2.3. *If $n \geqslant 3$ then $\mathbf{T}(I) = I$ and $\mathbf{T}(K) = K$.*

Proof. If E is a diagonal cell, then $\mathbf{T}(E)$ is a cell by Lemma 2.1. Since $\mathbf{T}$ strongly preserves r-potence and $E^r = E$, we have $\mathbf{T}(E)^r = \mathbf{T}(E)$. Since the only r-potent cells are diagonal cells, $\mathbf{T}(E)$ is a diagonal cell. Therefore $\mathbf{T}(I) = I$, since $\mathbf{T}$ is bijective on the set of cells from Lemma 2.2. Also, since $\mathbf{T}$ is bijective, $\mathbf{T}(K) = K$. ∎

LEMMA 2.4. *Let F, G be distinct off-diagonal cells, and E be a diagonal cell. Then $(E + F + G)^r = E + F + G$ and $F \neq G^t$ if and only if E, F, and G are collinear.*

Proof. The necessity is trivial. Now, without loss of generality, we assume $E = E_{11}$. Let $F = E_{jk}$ and $G = E_{is}$. We have $j \neq k$, $i \neq s$, $(j, k) \neq (i, s)$, and $(j, k) \neq (s, i)$, since F and G are distinct off-diagonal cells and $F \neq G^t$. First we will show that E is collinear with F or G. Suppose not; then $j, k, i, s \neq 1$. Let $X = E + F + G$. Then $X^2 = (E_{11} + E_{jk} + E_{is})^2 = E_{11} + E_{jk}E_{is} + E_{is}E_{jk}$. If $k = i$ and $j \neq s$, then $X^2 = E_{11} + E_{js}$. If $j = s$ and $k \neq i$, then $X^2 = E_{11} + E_{ik}$. Therefore $X^r \neq X$ for all $r \geqslant 2$, a contradiction. Since $F \neq G^t$, the remaining case is $k \neq i$ and $j \neq s$. We now have $X^2 = E_{11}$, so that $X^r = E_{11} \neq X$ for all $r \geqslant 2$, a contradiction. Therefore E is collinear with F or G.

Assume E and F are collinear and $j = 1$, i.e. $F = E_{1k}$. We will show $i = 1$. Suppose $i \neq 1$. Then $X^2 = E_{11} + E_{1k} + E_{1k}E_{is} + E_{is}E_{11} + E_{is}E_{1k}$. If $s \neq 1$, then $X^2 = E_{11} + E_{1k} + E_{1k}E_{is}$. Furthermore, if $k \neq i$, then $X^2 = E_{11} + E_{1k}$, so that $X^3 = E_{11} + E_{1k} = X^2$. It follows that $X^r = X^2 \neq X$ for all $r \geqslant 2$, a contradiction. If $k = i$, then $X^2 = E_{11} + E_{1k} + E_{1s}$, so that $X^3 = E_{11} + E_{1k} + E_{1s} = X^2 \neq X$, since $k \neq 1$. Hence $X^r \neq X$ for all $r \geqslant 2$, a contradiction. We thus must have $s = 1$. Now $X^2 = E_{11} + E_{1k} + E_{i1} + E_{ik}$, since $(1,k) \neq (1, i)$, so that $X^3 = E_{11} + E_{ik} + E_{i1} + E_{ik} = X^2 \neq X$. Therefore $X^r \neq X$ for all $r \geqslant 2$, a contradiction.

This contradicts the assumption $i \neq 1$, since each choice of s and k yields a contradiction. Therefore E, F, and G are collinear. The proof that G is also collinear with E and F when $k = 1$ and $j \neq 1$ is parallel. ∎

COROLLARY 2.1. *If F, G are distinct off-diagonal cells and E is a diagonal cell such that $(E + F + G)^r = E + F + G$ and $F = G^t$, then r is odd.*

Proof. Without loss of generality, assume $E = E_{11}$, $F = E_{ij}$, and $G = E_{ji}$. Let $X = E + F + G$. Suppose $i = 1$ then $X^2 = E_{11} + E_{1j} + E_{j1} + E_{jj}$ and $X^r = X^2 \neq X$ for all $r \geqslant 2$, a contradiction. Therefore, $i \neq 1$, and similarly, $j \neq 1$. Thus $X^{2k+1} = X$ for all k, since $X^{2k} = E_{11} + E_{ii} + E_{jj}$. Obviously $X^r = X$ only if r is odd. ∎

COROLLARY 2.2. *If F, G are distinct off-diagonal cells, E is a diagonal cell, and r is even, then $(E + F + G)^r = E + F + G$ if and only if E, F, and G are collinear.*

Proof. This is immediate from Lemma 2.4 and Corollary 2.1. ∎

A star matrix is *maximal* if it has exactly $n - 1$ nonzero off-diagonal entries.

LEMMA 2.5. *If* $|F| = n - 1$ *and* $X = J \setminus F$ *is r-potent, then all of the diagonal entries of* X *are nonzero,* F *is a maximal star matrix, and* X *is idempotent.*

Proof. If X were irreducible, then since one of its diagonal entries is not zero, it would have to be primitive. Being r-potent, X would then have to be J. So X is reducible. Therefore it has a $k \times (n - k)$ submatrix of zeros, and hence $k(n - k) = n - 1$. That quadratic in k has only two roots: 1 and $n - 1$. The line containing the $n - 1$ zeros must have a nonzero diagonal entry, because X is reducible. Therefore F is a maximal star matrix, and hence the diagonal entries of X are all nonzero and $X^2 = X$. ■

REMARK. Since $\mathbf{T}$ is bijective on the set of cells, $\mathbf{T}(X \setminus Y) = \mathbf{T}(X) \setminus \mathbf{T}(Y)$ for each fixed X and Y.

COROLLARY 2.3. *If* $n \geqslant 3$, *then* $\mathbf{T}$ *preserves s-stars for all* $1 \leqslant s \leqslant n - 1$.

Proof. Let S be a s-star matrix, H be a maximal star matrix such that $S \leqslant H$, and A be $\mathbf{T}(H)$. Then $|A| = n - 1$, since $\mathbf{T}$ is bijective and $|H| = n - 1$. Thus $(J \setminus H)^r = J \setminus H$, and so $\mathbf{T}(J \setminus H) = \mathbf{T}(J) \setminus \mathbf{T}(H) = J \setminus \mathbf{T}(H) = J \setminus A$ is r-potent. By Lemma 2.5, A is a maximal star matrix. Therefore, $\mathbf{T}(S)$ is an s-star matrix, because $\mathbf{T}$ is bijective on the set of cells and $\mathbf{T}(S) \leqslant \mathbf{T}(H)$. ■

LEMMA 2.6. *If* $n = 2$, *then* $\mathbf{T}$ *is bijective on the set of cells,* $\mathbf{T}(I) = I$, *and* $\mathbf{T}(K) = K$.

Proof. Let $E = E_{12}$, $E' = E_{21}$, $D = E_{11}$, $D' = E_{22}$, $S = K + D$, and $S' = K + D'$. We know $\mathbf{T}(E) \neq O$ by Theorem 2.1, and every 2×2 Boolean matrix with two or more cells is either idempotent, S, S', or K; the last is r-potent when r is odd. Suppose $\mathbf{T}(E) \geqslant F + G$, where F and G are distinct cells. Then $\mathbf{T}(E) \geqslant K$, since E and $\mathbf{T}(E)$ is non-r-potent. Also we know $\mathbf{T}(E) \neq K$ and hence $|\mathbf{T}(D + E)| \geqslant 3$ when r is odd. Therefore $\mathbf{T}(D + E) \geqslant K$. But $\mathbf{T}(D + E)$ is idempotent, so that $\mathbf{T}(D + E) = J_2$, since J_2 is the only idempotent 2×2 matrix with more than two cells dominating K. Thus $\mathbf{T}(S) = \mathbf{T}(D + E) + \mathbf{T}(E') = J$. But S is non-r-potent and J is r-potent, a contradiction. So $|\mathbf{T}(E)| = 1$. Furthermore, $\mathbf{T}(E)$ is not a diagonal cell, since E is non-r-potent. Therefore $\mathbf{T}(E) = E$ or E'. Similarly $\mathbf{T}(E') = E$ or E'.

Suppose $\mathbf{T}(E) = \mathbf{T}(E')$. Then $\mathbf{T}(D + E + E') = \mathbf{T}(D + E)$. But $D + E + E'$ is not r-potent, while $D + E$ is r-potent, a contradiction. Therefore (i) $\mathbf{T}(E) = E$ and $\mathbf{T}(E') = E'$ or (ii) $\mathbf{T}(E') = E$ and $\mathbf{T}(E) = E'$. Therefore we have shown $\mathbf{T}(K) = K$. We now suppose $\mathbf{T}(D)$ is not a cell. That is, $\mathbf{T}(D) \geqslant F + G$ for some cells F, G. If F and G are distinct off-diagonal cells, then $\mathbf{T}(D) \geqslant \mathbf{T}(E + E')$, so that $\mathbf{T}(D + E + E') = \mathbf{T}(D)$ is idempotent. It follows that $D + E + E'$ must be r-potent, a contradiction. Thus we may assume that F is a diagonal cell. If $\mathbf{T}(D) \geqslant I$ then $\mathbf{T}(S) \geqslant I$. But $\mathbf{T}(S)$ is not r-potent, and every 2×2 matrix which dominates I is r-potent, a contradiction. Therefore $\mathbf{T}(D)$ does not dominate I. It follows that $\mathbf{T}(D) = F + G$, where F is a diagonal cell and G is E or E'. Then $\mathbf{T}(D + A) = F + E + E'$, where $A = E$ or E'. But $D + A$ is r-potent while $F + E + E'$ is not, a contradiction. Thus $\mathbf{T}(D)$ is a diagonal cell. Similarly $\mathbf{T}(D')$ is a diagonal cell. Suppose $\mathbf{T}(D) = \mathbf{T}(D') = F$, where F is a diagonal cell; then $\mathbf{T}(J) = \mathbf{T}(D + D') + \mathbf{T}(K) = F + K$, which is not r-potent, a contradiction. Therefore $\mathbf{T}(I) = I$, $\mathbf{T}(K) = K$, and $\mathbf{T}$ is bijective on the set of cells. ■

LEMMA 2.7 [2, Lemma 3.7]. *If a nonsingular linear operator* $\mathbf{T}$ *on* $\mathcal{M}_n(\mathbb{B})$ *is bijective on the off-diagonal cells,* $\mathbf{T}(I) \leqslant I$, *and* $\mathbf{T}$ *preserves 2-star matrices, then* $\mathbf{T}$ *is one of, or a composition of two or more of, the following operators:*

(a) *transposition (i.e.,* $X \to X^t$*),*

(b) *similarity operators (i.e.,* $X \to PXP^t$ *for some fixed permutation matrix* P *in* $\mathcal{M}$*),*

(c) *nonsingular diagonal replacement (i.e., for some fixed nonsingular linear operator* s *on the diagonal matrices of* $\mathcal{M}$, $X \to X \circ K + \mathrm{s}(X \circ I)$*).*

THEOREM 2.2. *If* $n \geqslant 2$, *the semigroup* $\mathcal{S}$ *of linear operators strongly preserving r-potent matrices over the two-element Boolean semiring is generated by transposition and the similarity operators.*

Proof. Since transposition and all operators $X \to PXP^t$ are in $\mathcal{S}$ ($P^t = P^{-1}$ when P is a permutation matrix), we need only show that $\mathcal{S}$ is contained in the group they generate. Let $\mathbf{T} \in \mathcal{S}$. If $n \geqslant 3$, let S be an $(n-1)$-star and E be a diagonal cell such that they are in a line. Then $S + E$ is r-potent, so $(S + E)^r = S + E$, and hence $[\mathbf{T}(S + E)]^r = \mathbf{T}(S + E)$. Since $\mathbf{T}$ is linear, $[\mathbf{T}(S) + \mathbf{T}(E)]^r = \mathbf{T}(S) + \mathbf{T}(E)$. By Lemma 2.5, $\mathbf{T}$ preserves 2-stars and $\mathbf{T}(S)$ must be a star matrix. Therefore $\mathbf{T}(E)$, $\mathbf{T}(S)$ are collinear. That is, $\mathbf{T}(S)$ is an $(n-1)$-star and $\mathbf{T}(E)$ is a diagonal cell lying in the same line as $\mathbf{T}(S)$. Thus the operator s in (c) of Lemma 2.7 must be the identity.

In case $n = 2$, by Lemma 2.6, $\mathbf{T}$ either fixes the diagonal cells or switches them. Also, $\mathbf{T}$ either fixes the off-diagonal cells or switches them. The only four possible operators are those given, establishing the theorem. ∎

3. THE ANTINEGATIVE-SEMIRING CASE

In this section, $\mathbb{A}$ is an antinegative semiring with no zero divisors, $n \geqslant r \geqslant 2$, and $\mathcal{S} = \mathcal{S}_n(\mathbb{A})$ denotes the semigroup of all linear operators on $\mathcal{M}_n(\mathbb{A})$ strongly preserving r-potence.

The mapping accomplished by associating each matrix A in $\mathcal{M}_n(\mathbb{S})$ with its pattern $\bar{A}$ in $\mathcal{M}_n(\mathbb{B})$ is a semiring homomorphism when $\mathbb{S}$ is antinegative and zero-divisor-free.

If $\mathbf{T}$ is a linear operator on $\mathcal{M}_n(\mathbb{S})$, let $\bar{\mathbf{T}}$, its *pattern*, be the operator on $\mathcal{M}_n(\mathbb{B})$ defined by $\bar{\mathbf{T}}(\bar{E}_{ij}) = \overline{\mathbf{T}(E_{ij})}$ for all (i, j). Then $\overline{\mathbf{T}(A)} \leqslant \bar{\mathbf{T}}(\bar{A})$ for all A in $\mathcal{M}_n(\mathbb{S})$. Equality holds if $\mathbb{S}$ is an antinegative semiring having no zero divisors.

Let $A \in \mathcal{M}_n(\mathbb{S})$. The *scaling operator* $\mathbf{L}_A$ induced by A is defined by $\mathbf{L}_A : X \to A \circ X$.

LEMMA 3.1. *The semigroup $\mathcal{S}$ is generated by the scaling operators in $\mathcal{S}$, transposition, and the similarity operators.*

Proof. Suppose $\mathbf{T} \in \mathcal{S}$. Then $\bar{\mathbf{T}} \in \mathcal{S}_n(\mathbb{B})$, since $\overline{\mathbf{T}(X)} = \bar{\mathbf{T}}(\bar{X})$ whenever $\mathbb{A}$ is an antinegative semiring having no zero divisors. Therefore $\bar{\mathbf{T}}$ is in the semigroup of operators generated by the similarity operators and transposition, by Theorem 2.1.2. Thus $\mathbf{T}(X) = M \circ \bar{\mathbf{T}}(\bar{A})$ for some $M \in \mathcal{M}$, and the lemma follows. ∎

LEMMA 3.2. *If $n \geqslant 3$ and every element of $\mathbb{A}$ is idempotent, then the identity operator is the only scaling operator that strongly preserves r-potence.*

Proof. Clearly, the identity operator is $\mathbf{L}_J$. Suppose $\mathbf{L} = \mathbf{L}_A$ strongly preserves r-potence for some A. Let i, j, and k be distinct positive integers, $i, j, k \leqslant n$. Put $X_{ijk} = a_{ij}E_{ij} + E_{ik} + E_{jj} + E_{jk}$, $J_{ijk} = E_{ij} + E_{ik} + E_{jj} + E_{jk}$, $X_{jk} = a_{jj}E_{jj} + E_{jk}$, and $J_{jk} = E_{jj} + E_{jk}$. It is easily seen that J_{ijk} and J_{jk} are r-potent. Since $\mathbf{L}(X_{ijk}) = \mathbf{L}(J_{ijk})$ and $\mathbf{L}(X_{jk}) = \mathbf{L}(J_{jk})$, we have that X_{ijk} and X_{jk} are r-potent. Then the (i, k) entry of $(X_{ijk})^r$ is a_{ij}, while the (i, k) entry of X_{ijk} is 1. Thus, $a_{ij} = 1$. Also, the (j, k) entry of $(X_{jk})^r$ is a_{jj}, while the (j, k) entry of X_{jk} is 1. Thus, $a_{jj} = 1$. Since i, j, and k were arbitrary, we have $A = J$. ∎

NOTE. In Lemma 3.2, $\mathbb{A}$ need not be antinegative.

THEOREM 3.1. *If $n \geqslant 3$ and every member of $\mathbb{A}$ is idempotent, then $\mathcal{S}$ is generated by transposition and the similarity operators; $\mathcal{S}$ is therefore a group.*

Proof. This is immediate with Lemmas 3.1 and 3.2. ■

The permutation matrices are the only invertible matrices over those antinegative semirings that have only one unit 1 such as: the nonnegative integers, any chain semiring (such as the fuzzy scalars), or the two-element Boolean algebra. For any antinegative semiring $\mathbb{A}$, Q is invertible in $\mathcal{M}_n(\mathbb{A})$ if and only if $Q = PD$ for some permutation matrix P and some diagonal matrix D whose diagonal entries are all units in $\mathbb{A}$.

COROLLARY 3.1. *If $n \geqslant 3$, the semigroup of linear operators on the $n \times n$ matrices over chain semiring that strongly preserves r-potence is generated by transposition and the operators $X \to PXP^t$, P a permutation matrix.*

LEMMA 3.3. *If $\mathbf{L}_A$ preserves r-potence on $\mathcal{M}_n(\mathbb{A})$, then each diagonal entry in A is r-potent.*

Proof. Since $I^r = I$, we must have that $\mathbf{L}_A(I)$ is r-potent. Thus $(A \circ I)^r = [\mathbf{L}_A(I)]^r = \mathbf{L}_A(I) = A \circ I$, and thus $a_{ii}^r = a_{ii}$ for all i, $1 \leqslant i \leqslant n$. ■

LEMMA 3.4. *Suppose $\mathbb{A}$ is an antinegative, commutative semiring with only one $(r-1)$th root of unity, 1, having the multiplicative cancellation property.*

(i) *If $\mathbf{L}_A$ strongly preserves r-potence, then*

  (a) *each diagonal entry in A is 1, and*

  (b) *when $n \geqslant 3$, there exist units a_i in $\mathbb{A}$ such that for all i, j,*

$$a_{ij} = a_i a_j^{-1}.$$

(ii) *If $a_{ij} = a_i a_j^{-1}$ for all i, j, then $\mathbf{L}_A$ strongly preserves r-potence.*

Proof. Since each diagonal entry in A is r-potent (Lemma 3.3) and none are 0 by Theorem 2.1, it follows by the cancellation property that $a_{ii}^{r-1} = 1$. This implies $a_{ii} = 1$ for all i, since $\mathbb{A}$ has only one $(r-1)$th root of unity, 1. This establishes (i)(a).

Next we fix i and choose $j \neq i$. Let $R = \sum_{k \neq i}(E_{ik} + E_{jk})$; then by direct computation we have $(A \circ R)^2 = (A \circ R)^3$ and hence $(A \circ R)^m = (A \circ R)^2$ for all $m > 1$. In particular, $(A \circ R)^r = (A \circ R)^2$. But R is r-potent, so its image, $A \circ R = \mathbf{L}_A(R)$, is r-potent too. Consequently $A \circ R = (A \circ R)^r = (A \circ R)^2$, i.e., $A \circ R$ is idempotent. Therefore

$$\sum_{k \neq i}\left(a_{ik}E_{ik} + a_{jk}E_{jk}\right) = \sum_{k \neq i}\left(a_{ij}a_{jk}E_{ik} + a_{jk}E_{jk}\right).$$

Therefore

$$a_{ik} = a_{ij}a_{jk} \qquad \text{for all} \quad k \neq i, \tag{3.1}$$

and by interchanging the roles of i and j in (3.1), we obtain

$$a_{jk} = a_{ji}a_{ik} \qquad \text{for all} \quad k \neq j. \tag{3.2}$$

Since $n \geqslant 3$, we can choose $g \neq 1$, j, obtaining $a_{ig} = a_{ij}a_{ji}a_{ig}$ from (3.1) and (3.2), and hence no entry in A is 0 by Theorem 2.1. Thus if $k \neq i$, then $a_{ij} = a_{ig}a_{kg}^{-1}$ for all k. Let $a_i = a_{i1}$. This completes the proof of part (i).

The verification of part (ii) is a straight forward computation. ∎

Let $\mathbb{P}^+$ be the nonnegative members of a nontrivial subring $\mathbb{P}$ of the reals. That is, if $\mathbb{P} = \mathbb{R}$ (reals) then $\mathbb{P}^+ = \mathbb{R}^+$; if $\mathbb{P} = \mathbb{Z}$ (integers) then $\mathbb{P}^+ = \mathbb{Z}^+$.

NOTE. If $\mathbb{A} = \mathbb{P}^+$, then Lemma 3.4(i) implies that all $a_{ij} = 1$.

THEOREM 3.2. *The semigroup $\mathcal{S} = \mathcal{S}_n(\mathbb{P}^+)$ is generated by transposition and the similarity operators, unless $n = 2$ and $\mathcal{M}_2(\mathbb{P}^+)$'s r-potent matrices are triangular and hence are on a single line. In that case, an additional family of generators is required, namely, the set of scaling operators*

$$X \to \begin{bmatrix} 1 & x \\ y & 1 \end{bmatrix} \circ X \qquad \text{with} \quad xy > 0.$$

Proof. Because a scaling operator induced by a matrix A satisfying Lemma 3.4(i) is the similarity operator $X \to DXD^{-1}$, where $D = \operatorname{diag}(a_1, a_2, \ldots, a_n)$, the theorem is immediate from Lemmas 3.1 and 3.4 unless $n = 2$ and $\mathcal{M}_2(\mathbb{P}^+)$'s r-potent matrices are triangular. In that case, suppose $\mathbf{T}$ is in $\mathscr{S}$. Lemma 2.6 implies that we may assume $\mathbf{T}$ is a scaling operator, say $\mathbf{T} = \mathbf{L}_A$. According to Lemma 3.4,

$$A = \begin{bmatrix} 1 & x \\ y & 1 \end{bmatrix} \qquad \text{for some} \quad x, y \text{ in } \mathbb{P}^+.$$

Then $xy > 0$: otherwise

$$A \circ \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad A \circ \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

are not r-potent because

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

are not, a contradiction. Conversely, the scaling operators

$$X \to \begin{bmatrix} 1 & x \\ y & 1 \end{bmatrix} \circ X$$

are in $\mathscr{S}$ whenever $xy > 0$. ■

COROLLARY 3.2. *The semigroup* $\mathscr{S} = \mathscr{S}_n(\mathbb{R}^+)$ *is generated by transposition, permutation similarity, and* $X \to DXD^{-1}$, *where* D *is a diagonal matrix and all* $d_{ii} > 0$.

COROLLARY 3.3. *The semigroup* $\mathscr{S} = \mathscr{S}_n(\mathbb{Z}^+)$ *is generated by transposition and permutation similarity, unless* $n = 2$. *If* $n = 2$, *an additional family of generators is needed, namely, all the scaling operators*

$$X \to \begin{bmatrix} 1 & x \\ y & 1 \end{bmatrix} \circ X \qquad \text{with} \quad xy \geqslant 1.$$

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