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Additional constraints may soften a non-conservative structural system: Buckling and vibration analysis

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ABSTRACT

The effect of additional kinematic constraints on eigenfrequencies of non conservative systems presenting a non symmetric stiffness matrix is investigated with the use of the second order work criterion. It is shown that there are always additional constraints that may soften structural systems, from both buckling and vibration points of view. The steps for building such constraints are given, consequences on stability are discussed and several illustrating examples are presented.

1. Introduction

As early as 1980, Tarnai discovered a softening effect induced by kinematic constraint in elastic systems. This softening effect was characterized by a decrease of the critical load of divergence instability in conservative systems with equilibrium positions depending on the loading parameters (see for example Tarnai, 1980; Tarnai, 2004). Hence, it was shown to be out of the scope of the usual stability analysis of constrained systems based on Rayleigh's well known theory. In the present paper we show that this surprising phenomenon can be observed in non conservative systems too even if the associated equilibrium positions do not depend on the loading parameters. Recently, Challamel et al. (2010) demonstrated the possible softening effect of additional constraints in the buckling problem of non conservative systems. Here we perform a similar investigation of the spectrum (within the meaning of the set of eigenfrequencies) of the system and we generalize the approach of Challamel et al. (2010) to the vibration analysis. Although the usual framework of investigations involving spectral analysis, vibrations, buckling, divergence or flutter is linear elasticity as, e.g., in Bolotin (1963), the presented approach remains valid in a more general setting, including, e.g., the incrementally piecewise linear evolution. We mean for example the elasto-plasticity if this evolution takes place in a tensorial zone of a purely constitutive problem or external dry friction forces as it was elegantly investigated in Bigoni's recent paper Bigoni and Noselli (2011). In order to be valid, the dynamic evolution has only to be described by an equation similar to Eq. (1) of the present paper.

The effect of constraints on systems the dynamics of which is governed by a symmetric stiffness matrix is actually well known since Rayleigh's and Courant's Minimax theorems: the range of the real spectrum is reduced by a kinematic constraint and the lowest eigenfrequency is then always increasing. We show that the effect of a constraint on systems with the dynamics, governed by any (i.e. nonsymmetric) stiffness matrix may not be a priori forecast. According to the chosen constraint, the lowest eigenfrequency can either increase or decrease. The paper obviously focuses on the decreasing effect, which we call a stiffness softening effect. Section 1 concerns generalities of the spectral analysis of constrained systems investigated with the use of Lagrange multipliers. After performing necessary calculations in Section 3.1, Section 3.2 presents the main result: as long as the second order work criterion is valid, there always exists a constraint (and it may be chosen in the kernel of the corresponding operator) that leads to a decrease of the lowest eigenfrequency of the system. Consequences on both divergence and flutter instabilities are investigated (Sections 3.3, 3.4) and several examples (Section 4) using Ziegler's 2 degree of freedom column as a mechanical model illustrate the results.

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2. Statement of the problem

As it is already mentioned in the introduction, in the following we will only assume that after having started with different possible non linear settings, convenient assumptions and approximations lead to a dynamic evolution governed by the following equation of motion of the free system Σ_{free}:

$$M\ddot{X} + K(p)X = 0 \tag{1}$$

where $K(p)$ is generally a non-symmetric matrix. For circulatory systems like in Bolotin (1963), in Hermann and Bungay (1964) or more recently in Bigoni and Noselli (2011), we often may write $K(p) = K_{int} - pK_{ext}$ and the internal elastic stiffness matrix K_{int} is symmetric and positive definite. Clearly, the non-conservativeness comes from the external loading, meaning that K_{ext} is generally a non-symmetric matrix. In the present general approach, the dependency $p \mapsto K(p)$ is without any importance. M is a symmetric positive definite matrix, p denotes the loading parameter, and X is the perturbation of dimension n. The buckling/vibrations equation of this free system Σ_{free} is obtained for the divergence-type systems as:

$$f(p,s) = \det(K(p) - s^2M) = 0 \tag{2}$$

We assume that for $p = 0$, the stiffness matrix $K(0)$ of the free system is symmetric positive definite (it obviously holds for circulatory systems because $K(0)$ then reduces to a pure elastic stiffness matrix). Let $\omega_{free,k}(p)$ be the $2n$ roots of the polynomial Eq. (2). If they are real, the positive ones are the natural frequencies of the free system. Since for $p = 0$ (eventually counted with their multiplicity) $\omega_{free,k}(0) \in \mathbb{R}$ for all $k = 1,\dots,n$ because of the above assumption about $K(0)$, then, by continuity, there is an interval $[0,p_{max}[$, such that $\omega_{free,k}(p) \in \mathbb{R}$ for all $k = 1,\dots,n$. For the rest of the paper, we assume that $p \in [0,p_{max}[$ and the natural frequencies are ordered: $\omega_{free,1}(p) \leqslant \dots \leqslant \omega_{free,n}(p)$.

We will investigate the spectral properties of such a dynamical system in the presence of an additional kinematic constraint, given by the (linear) holonomic constraint (α is a column vector):

$$\alpha^T \cdot X = 0 \tag{3}$$

The Lagrange multiplier λ can be introduced for the constrained system as:

$$M\ddot{X} + K(p)X + \lambda\alpha = 0 \tag{4}$$

Leading a similar mathematical approach as in Lerbet et al. (2012) for example, the buckling/vibrations equation of the constrained system $\Sigma_{cons}(\alpha)$ is obtained for the divergence-type systems as:

$$h(p,s,\alpha) = \det\begin{pmatrix} K(p) - s^2M & \alpha \\ \alpha^T & 0 \end{pmatrix} = 0 \tag{5}$$

In this paper, we investigate the eventual relationship between the eigenfrequencies $\omega_{cons,k}(p,\alpha)$ of $\Sigma_{cons}(\alpha)$ and the frequencies $\omega_{free,k}(p)$ of Σ_{free}. More precisely, we will compare the lowest eigenfrequencies $\omega_{cons,1}(p,\alpha)$ and $\omega_{free,1}(p)$. It should be reminded that for conservative systems, $\omega_{cons,1}(p,\alpha) \geqslant \omega_{free,1}(p)$ for any constraint (defined by) α. We claim in this paper that for the nonconservative systems with the dynamics governed by Eq. (1), there is at least one peculiar constraint α such that $\omega_{cons,1}(p,\alpha) < \omega_{free,1}(p)$ as long as the symmetric part of $K^s(p)$ is positive definite (second order work criterion), and we present this constraint. We stress however that this dynamic softening effect does not mean that such a constraint leads to divergence-instability of the constraint system. On the contrary, it has been already proved in papers like Challamel et al. (2010) and Lerbet et al. (2012) that, as long as the symmetric part of $K^s(p)$ is positive definite (second order work criterion), no kinematic constraints may induce divergence instabilities. These points are discussed in Sections 3.3 and 3.4.

3. The results

3.1. Preliminary calculations

Let α be an n-column vector. Let us choose $s \notin \{\pm\omega_{free,k}(p), k = 1\dots,n\}$ such that $F(p,s) = K(p) - s^2M$ is not singular. Similar calculations as in Challamel et al. (2010) but for the matrix $F(p,s) = K(p) - s^2M$ give then successively:

$$\begin{pmatrix} F(p,s) & \alpha \\ \alpha^T & 0 \end{pmatrix}\begin{pmatrix} F(p,s)^{-1} & -F(p,s)^{-1}\alpha \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} I_n & 0 \\ \alpha^T F(p,s)^{-1} & -\alpha^T F(p,s)^{-1}\alpha \end{pmatrix}$$

Calculating the determinant of each side of the previous relation leads to:

$$\frac{h(p,s,\alpha)}{f(p,s)} = -\alpha^T F(p,s)^{-1}\alpha \tag{6}$$

Put $\beta = F(p,s)^{-1}\alpha$. (6) reads:

$$h(p,s,\alpha) = -f(p,s)\beta^T F(p,s)\beta$$

which leads, for all s,p such that $f(p,s) \neq 0$ (i.e. $s \notin \{\pm\omega_{free,k}(p), k = 1\dots,n\}$) to:

$$h(p,s,\alpha) = -f(p,s)\beta^T F^s(p,s))\beta \tag{7}$$

with $\beta = F(p,s)^{-1}\alpha$ because obviously $\beta^T F(p,s))\beta = \beta^T F^s(p,s))\beta$ for $F^s(p,s) = K^s(p) - s^2M$ where K^s is the symmetric part of K.

Let Σ_* be the associated conservative system with stiffness matrix $K^s(p)$ and mass matrix M and let $\{\omega_{*,k}(p),\ k = 1\dots,n\}$ be its spectrum which is included in $\mathbb{R}$ for all p. These are the roots of the characteristic polynomial of Σ_* defined as:

$$g(p,s) = \det(K^s(p) - s^2M) = \det(F^s(p,s)) = 0 \tag{8}$$

This spectrum can also be calculated from the generalized Rayleigh's quotient $R^s(X)$ related to the symmetric part K^s of K defined by:

$$R^s(X) = \frac{X^T K^s(p)X}{X^T MX}$$

and the smallest positive root $\omega_{*,1}(p)$ of (8) is for example given by:

$$\omega^2_{*,1}(p) = \inf_{X\in\mathbb{R}^n} \frac{X^T K^s(p)X}{X^T MX}$$

For the rest of the paper, p_{sw} denotes the critical load according to the second order work criterion: this is the smallest positive root of the equation $\det(K^s(p)) = 0$. For $p = 0$, $K^s(0) = K(0)$ is supposed to be positive definite (as it is assumed above). Thus $\det(K^s(0)) > 0$ and there is an open interval $I \subset \mathbb{R}^+$ (by continuity) such that $\det(K^s(p)) > 0$ for all $p \in I$. By definition $p_{sw} = \inf(I)$. If I is bounded then $p_{sw} \in \mathbb{R}^+$. If I is not bounded then $p_{sw} = +\infty$ meaning $\det(K^s(p)) > 0$ for all $p > 0$.

If $p_{free,div}$ is the divergence critical load of the free system Σ_{free} (the smallest positive root of $\det(K(p)) = 0$), it has already been proved that $p_{free,div} \geqslant p_{sw}$ (Lerbet et al., 2009 for example) but it obviously follows from the following result proven by Ostrowski and Taussky (1951):

If A is any definite positive matrix (meaning that its symmetric part is definite positive), then $\det(A) \geqslant \det(A^s) > 0$.

3.2. Main results and stiffness softening effect of kinematics constraints

Preserving the same notation as in the previous sections, two results on the relation of the spectrum of the free system and that of the constrained and associated systems are derived. The first one is the following:

Theorem 1.

$$\omega_{*,1}(p) \leqslant \omega_{free,1}(p) \quad \forall p \in [0, p_{sw}[\tag{9}$$

Proof. Suppose $p \in [0, p_{sw}[$ which means that $K(p)$ (or equivalently $K^s(p)$) is positive definite and let be $s \in [0, \omega_{*,1}(p)[$. Thus, because $g(p,0) = \det(F^s(p,0)) = \det(K^s(p))$, then $f(p,0) \geqslant g(p,0) > 0$. Moreover, as long as $g(p,s) > 0$ the eigenvalues of the real symmetric matrix $F^s(p,s)$ cannot vanish and because for $s = 0$ they are all > 0 (K^s positive definite), they are all > 0 for $s \in [0, \omega_{*,1}(p)[$ which means that $F(p,s)$ remains positive definite for $s \in [0, \omega_{*,1}(p)[$. Applying then the above result of Ostrowski and Taussky, we deduce that $f(p,s) \geqslant g(p,s) > 0$ for $s \in [0, \omega_{free,1}(p)[$. Because $\omega_{free,1}(p)$ is the lowest positive root of $s \mapsto f(p,s)$ we obviously conclude that $\omega_{free,1}(p) \notin [0, \omega_{*,1}(p)[$ or that $\omega_{*,1}(p) \leqslant \omega_{free,1}(p)$. □

Because $\det(F^s(p, \omega_{*,1}(p))) = 0$, $\dim\mathrm{Ker}(F^s(p, \omega_{*,1}(p)) \geqslant 1$ and we may choose $\beta = \beta(p)$ which is nonzero in $\mathrm{Ker}(F^s(p, \omega_{*,1}(p))$. Let now $\alpha = \alpha(p)$ be in $F(\omega_{*,1}(p))(\beta(p))$.

From (7), $h(p, \omega_{*,1}(p), \alpha(p)) = 0$ meaning that $\omega_{*,1}(p)$ lies in the spectrum of $\Sigma_{cons}(\alpha(p))$. From Theorem 1 the second result reads as follows:

Theorem 2. *For all $p \in [0, p_{sw}[$ there is a constraint $\alpha = \alpha(p)$ such that*

$$\omega_{cons,1}(p) \leqslant \omega_{free,1}(p) \tag{10}$$

where $\omega_{cons,1}(p) = \omega_{cons,1}(p, \alpha(p))$ is the smallest eigenfrequency of the constrained system $\Sigma_{cons}(\alpha(p))$.

Proof. Let us choose the constraint $\alpha = \alpha(p)$ as previously. Because $\omega_{*,1}(p)$ lies in the spectrum of $\Sigma_{cons}(\alpha(p))$ it follows that $\omega_{cons,1}(p) \leqslant \omega_{*,1}(p)$. Thus, from (9), we get:

$$\omega_{cons,1}(p) \leqslant \omega_{*,1}(p) \leqslant \omega_{free,1}(p)$$

which allows to conclude. □

Because of the last result, we say that the system is softened by the kinematics constraint meaning stiffness softening because of the direct natural relation between the stiffness of the structure and the lowest eigenfrequency for conservative systems through Rayleigh's quotient. The paradoxical effect resulting from Theorem 2 means that there always exists a kinematics constraint that makes the lowest eigenfrequency smaller in contrast to conservative systems.

Definition 1. The constraint $\alpha(p)$ built as above is called the critical or optimal constraint.

3.3. Divergence instability

The buckling problem is obtained as a particular case of this equation, where the divergence buckling load $p_{free,div}$ of the free system is calculated from $\det(K(p)) = 0$ or equivalently by

$$\omega_{free,1}(p) = 0 \tag{11}$$

while for the critical constraint $\alpha(p)$, the divergence buckling load $p_{cons,div}$ is calculated from the equation $h(p, 0, \alpha(p)) = 0$ or equivalently from

$$\omega_{cons,1}(p) = 0 \tag{12}$$

Thus, from (10)–(12) we deduce that

$$p_{cons,div} \leqslant p_{free,div} \tag{13}$$

Thus, we reproduce by another way the result obtained in Challamel et al. (2010) showing that there always exists a kinematic constraint such that the critical divergence load of the corresponding constrained system is lower than that of the free system and the optimal one is given by the second order work criterion (see e.g. Fig. 2).

3.4. Flutter instability

As mentioned in Challamel et al. (2009), there is no direct relationship between the second order work criterion p_{sw} and the critical flutter load $p_{free,fl}$ meaning that we may meet $p_{sw} < p_{free,fl}$ or $p_{sw} > p_{free,fl}$ according to the considered system while $p_{sw} \leqslant p_{free,div}$ always holds. Thus, similar conclusions hold with $p_{cons} = p_{cons,div}$ and both following situations may occur illustrated in Figs. 3 and 5 obtained with two different mass matrices and two different values of γ. On Fig. 3 calculated for a complete follower force ($\gamma = 1$) and a uniform mass distribution, we observe that $\omega_{cons,1}(p) \leqslant \omega_{*,1}(p) \leqslant \omega_{free,1}(p)$ for all $p \in [0, p_{sw}[$ and $p_{cons} = p_{cons,div} \leqslant p_{free,fl}$ while on the Fig. 5 calculated for a partial follower force ($\gamma = \frac{1}{2}$) and another mass matrix, we observe that $\omega_{cons,1}(p) \leqslant \omega_{*,1}(p) \leqslant \omega_{free,1}(p)$ for all $p \in [0, p_{sw}[$ and $p_{cons} = p_{cons,div} \geqslant p_{free,fl}$.

3.5. Constraint dependency on the load parameter

Without contradiction to the previous results, it is important to stress that the kinematic constraints are themselves depending on the load parameter p. This phenomenon, already investigated in Guran and Plaut (1993), is actually the natural consequence of the used method that leads to the optimal constraints for each state of the system, each state precisely depending on the load parameter. The question to know if there is a fixed family of kinematics constraints α not depending on p such that the corresponding constrained system $\Sigma_{cons}(\alpha)$ still satisfies the condition $\omega_{cons,1}(p) \leqslant \omega_{free,1}(p)$ for all $p \in [0, p_{sw}[$ is a more difficult problem although there are, by continuity, local results in the neighborhood of each value of the load. For example, there is a neighborhood of $p = 0$, i.e. an interval $[0, p_0[$ such that $\omega_{cons,1}(p) \leqslant \omega_{free,1}(p)$ for all $p \in [0, p_0[$ for the constrained system defined by the fixed family of constraints $\alpha = \alpha(0)$ we may call here the fixed constrained system. In Fig. 6, which is similar to Fig. 4, the constrained system with fixed $p = 0$ corresponds to the mixed dashed-dotted gold curve, and we observe that it coincides with the optimal one (dotted blue) on the whole interval $[0, p_{sw}[$ while on Fig. 7 plotted for $\gamma = \frac{3}{4}$, the curve of the fixed constrained system (mixed dashed-dotted gold) coincides only on a subinterval $[0, p_0[$ of $[0, p_{sw}[$ with the optimal constrained system (dotted blue), the eigenfrequency of the fixed constrained system being always higher than (or equal to) the one of the optimal constrained system: this is the property of optimality.

4. Examples

4.1. The mechanical model

To illustrate the previous results, consider the following 2 dof Ziegler column (see Fig. 1) (Ziegler, 1952; Bottema, 1956; Kirillov and Verhulst, 2010). The system Σ consists of two bars OA, AB with $OA = AB = \ell$ linked by two elastic springs with the same stiffness k. The circulatory non conservative load $\vec{P}$ is such that $\widehat{\vec{P}, y} = \gamma\theta_2$ with $0 \leqslant \gamma \leqslant 1$. For $\gamma = 1$, it is a complete follower force while for $\gamma = 0$ it is a conservative load. The equilibrium position is $\theta = (\theta_1, \theta_2) = (0, 0)$ and the bars are supposed to be homogeneous according to their mass distribution except for Fig. 5 obtained for another mass matrix. Adopting a dimensionless format, we use

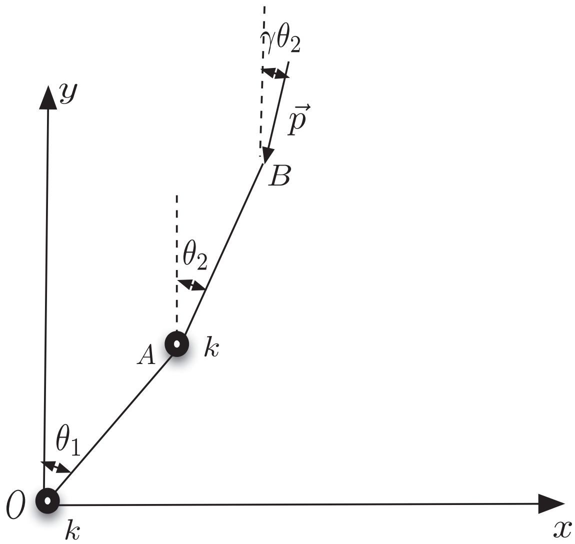

Fig. 1. 2 dof Ziegler's column with partial follower force; $\gamma = 1$ corresponds to complete follower force.

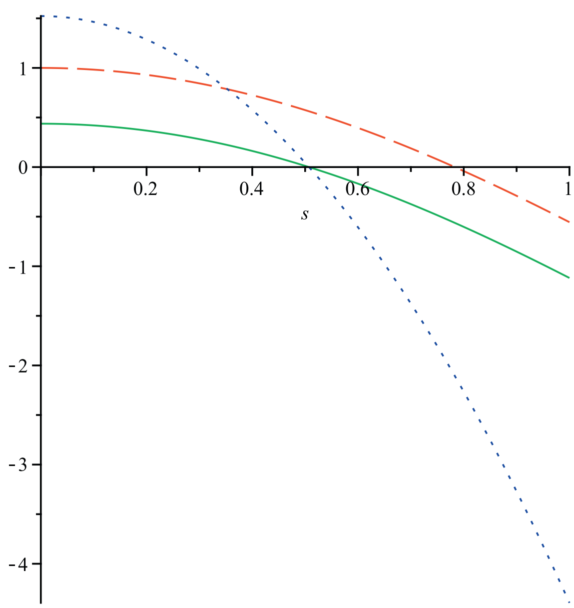

Fig. 2. Comparison of the lowest eigenfrequencies for the free system characteristic polynomial $f(\frac{3}{2}, 1, s)$ (dashed red), the associated system characteristic polynomial $g(\frac{3}{2}, 1, s)$ (plain green) and the (optimal) constrained system characteristic polynomial $h(\frac{3}{2}, 1, s, \alpha)$ (dotted blue): associated and constrained systems characteristic polynomials have the same lowest root that is lower than the one of the free system.(For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

$p = \frac{P\ell}{k}$ as loading parameter. If $\Omega^2 = \frac{k}{m\ell^2}$, Ω is then a pure elastic frequency parameter of Σ and to simplify the presentation, we put $\Omega = 1$.

4.2. Load–frequency boundary for the free system

The equations of motion (1) then read:

$$\begin{pmatrix} \frac{4}{3} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{3} \end{pmatrix} \ddot{X} + \begin{pmatrix} 2 - p & -1 + \gamma p \\ -1 & 1 - (1 - \gamma) p \end{pmatrix} X = 0 \qquad (14)$$

and the buckling/vibrations Eq. (2) of this free system is:

$$\begin{aligned} f(p, \gamma, s) &= \det(K(p, \gamma) - s^2 M) \\ &= 1 - 3p + 3\gamma p + p^2 - \gamma p^2 + \left(-3 + \frac{5}{3} p - \frac{5}{6} \gamma p\right) s^2 \\ &\quad + \frac{7}{36} s^4 \end{aligned}$$

and its smallest positive root is:

$$\omega_{free,1}(p,\gamma) = \frac{1}{7}\sqrt{378 - 210p + 105\gamma p - 21\sqrt{296 - 276p + 96\gamma p + 72p^2 - 72\gamma p^2 + 25\gamma^2 p^2}}$$

4.3. Load–frequency boundary for the associated system

The characteristic polynomial of the associated system Σ_* reads:

$$\begin{aligned} g(p, \gamma, s) &= \det(K^s(p, \gamma) - s^2 M) \\ &= 1 - 3p + 3\gamma p + p^2 - \gamma p^2 - \frac{1}{4} \gamma^2 p^2 \\ &\quad + \left(-3 + \frac{5}{3} p - \frac{5}{6} \gamma p\right) s^2 + \frac{7}{36} s^4 \end{aligned}$$

and its smallest positive root is:

$$\omega_{*,1}(p, \gamma) = \frac{1}{7}\sqrt{378 - 42a - 210p + 105\gamma p}$$

where $a = a(p, \gamma) = \sqrt{74 - 69p + 24\gamma p + 18p^2 - 18\gamma p^2 + 8\gamma^2 p^2}$

4.4. Calculation of the optimal constraint

We follow now the algebraic procedure to find the convenient kinematics constraint given in the previous section.

$$\ker(F^s(p, \gamma, \omega_{*,1})) = \mathrm{Vect}\left\{ x_1 = \begin{pmatrix} -\frac{34 + 4\gamma p - 15p - 3a}{58 - 33p + 20\gamma p - 8a} \\ 1 \end{pmatrix} \right\}$$

and thus the subspace $\mathrm{Vect}\left\{(K_s - \omega_{*,1}^2 .M).x_1\right\}$ generated by the vector $(K_s - \omega_{*,1}^2 .M).x_1$ is the one-dimensional subspace $\mathrm{Vect}\left\{\begin{pmatrix} \frac{1}{2}\gamma p \\ \frac{1}{2}\frac{\gamma p(34 + 4\gamma p - 15p - 3a)}{58 - 33p + 20\gamma p - 8a} \end{pmatrix}\right\}$ so, the constraint is given by the vector:

$$\alpha = \begin{pmatrix} \frac{1}{2}\gamma p \\ \frac{1}{2}\frac{\gamma p(34 + 4\gamma p - 15p - 3a)}{58 - 33p + 20\gamma p - 8a} \end{pmatrix}$$

meaning that the constraint reads:

$$\frac{1}{2}\gamma p \theta_1 + \frac{1}{2}\frac{\gamma p(34 + 4\gamma p - 15p - 3a)}{58 - 33p + 20\gamma p - 8a}\theta_2 = 0$$

or

$$(58 - 33p + 20\gamma p - 8a)\theta_1 + (34 + 4\gamma p - 15p - 3a)\theta_2 = 0$$

4.5. Load–frequency boundary of the optimal constrained system

Because for $n = 2$ it leads to effective and straightforward calculations, we propose here to explicitly calculate the constrained system although obviously it is not the philosophy of the general reasoning.

The constrained system is obviously a one dof system. The general equations of motion (1) of a constrained 2 dof system read:

$$\left\{ \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} \begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} + \begin{pmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} \lambda\alpha_1 \\ \lambda\alpha_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} (\alpha_1 \quad \alpha_2) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \right.$$

By eliminating the variable x_2 in order to keep only one equation in the single variable x_1, we get the remarkable following form:

$$(-\alpha_2 \;\; \alpha_1)\begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}\begin{pmatrix} -\alpha_2 \\ \alpha_1 \end{pmatrix}\ddot{x_1} + (-\alpha_2 \;\; \alpha_1)\begin{pmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{pmatrix}\begin{pmatrix} -\alpha_2 \\ \alpha_1 \end{pmatrix} x_1 = 0 \quad (15)$$

or here ($x_1 = \theta_1$) which gives

$$M_{cons}\ddot{\theta}_1 + K_{cons}\theta_1 = 0$$

with

$$M_{cons} = -1288\,p + 336p^2 + 448\,\gamma\,p - 336\,\gamma\,p^2 + \frac{448}{3}\,\gamma^2 p^2 - \frac{140}{3}\,\gamma\,p\,a + \frac{4144}{3} - \frac{406}{3}\,a + 77\,p\,a$$

and

$$K_{cons} = 19240 - 28744\,p + 14754\,p^2 + 12160\,\gamma\,p - 13704\,\gamma\,p^2 + 4000\,\gamma^2 p^2 - 2628\,p^3 - 2228\,a - 1034\,\gamma\,p\,a + 2278\,p\,a + 653\,\gamma\,p^2\,a - 228\,\gamma^2 p^2\,a - 2608\,\gamma^2 p^3 + 4068\,\gamma\,p^3 - 618\,p^2\,a + 640\,\gamma^3 p^3$$

The buckling/vibrations equation of the constrained system is obtained for the divergence-type systems as:

$$f_{cons}(p, \gamma, s) = K_{cons} - M_{cons}\,s^2$$

whose the smaller positive root obviously is:

$$\omega_{cons,1}(p, \gamma) = \frac{1}{7}\Big(\sqrt{21}((592 + 192\,\gamma p - 552\,p + 64\,\gamma^2 p^2 - 144\,\gamma p^2 + 144p^2 - 58\,a - 20\,\gamma p a + 33\,p a)(19240 - 28744\,p + 12160\,\gamma p + 14754p^2 - 13704\,\gamma p^2 + 653\,\gamma p^2 a + 4000\,\gamma^2 p^2 - 2628\,p^3 - 2228\,a - 1034\,\gamma p a + 2278\,p a - 228\,\gamma^2 p^2 a - 2608\,\gamma^2 p^3 + 4068\,\gamma p^3 - 618\,p^2 a + 640\,\gamma^3 p^3))^{\frac{1}{2}}\Big)/(592 + 192\,\gamma p - 552\,p + 64\,\gamma^2 p^2 - 144\,\gamma p^2 + 144p^2 - 58\,a - 20\,\gamma p a + 33\,p a)$$

4.6. Discussion and analysis of results

4.6.1. Comparison of frequencies at fixed loading parameter p

The validity of the results is restricted to the interval $[0, p_{sw}[$. As already done in Challamel et al. (2010), the critical value p_{sw} of the loading parameter according to the second order work criterion is the lowest root of

$$\det(K^s(p)) = \det\begin{pmatrix} 2 - p & -1 + \frac{1}{2}\,\gamma p \\ -1 + \frac{1}{2}\,\gamma p & 1 - (1 - \gamma)\,p \end{pmatrix} = 0$$

which leads here to

$$p_{sw} = \frac{2\,(-3 + 3\,\gamma + \sqrt{5 - 14\,\gamma + 10\,\gamma^2})}{-4 + 4\,\gamma + \gamma^2}.$$

For $\gamma = 1$, the system is Ziegler's model with complete follower load. In this case, the free system loses its stability by flutter. Let us then choose $p < p_{sw} = 2$, for example $p = \frac{3}{2}$. We then find the corresponding numerical values of the lowest frequencies

$$\omega_{*,1}\left(\frac{3}{2}, 1\right) \approx 0.5073 \leqslant \omega_{free,1}\left(\frac{3}{2}, 1\right) \approx 0.7830$$

More precisely, the positive roots of $\det(F^s(\frac{3}{2}, 1, s) = 0$ are $\sqrt{3} - \frac{1}{2}\sqrt{6}$, $\sqrt{3} + \frac{1}{2}\sqrt{6}$ and thus, with previous notations $\omega_{*,1}(\frac{3}{2}, 1) = \sqrt{3} - \frac{1}{2}\sqrt{6} \approx 0.5073$.

We obviously conclude that

$$\omega_{cons,1}\left(\frac{3}{2}, 1\right) \approx 0.5073 \leqslant \omega_{*,1}\left(\frac{3}{2}, 1\right) \approx 0.5073 \leqslant \omega_{free,1}\left(\frac{3}{2}, 1\right) \approx 0.7830$$

This is illustrated on Fig. 2 where the characteristic polynomials and their lower positive roots are plotted: the optimal constrained system in dotted blue line (unique root $\omega_{cons,1}(\frac{3}{2}, 1) \approx 0.5073$) , the associated system in plain green line (lowest root $\omega_{*,1}(\frac{3}{2}, 1) \approx 0.5073$) and for the free system in dashed red line (lowest root $\omega_{free,1}(\frac{3}{2}, 1) \approx 0.7830$).

For $\gamma = \frac{1}{2}$. Let us choose $p < p_{sw} = \frac{12 - 4\sqrt{2}}{7} \approx 0.9061$, for example $p = \frac{1}{2}$.

Corresponding numerical values are

$$\omega_{cons,1}\left(\frac{1}{2}, \frac{1}{2}\right) \approx 0.3914 \leqslant \omega_{*,1}\left(\frac{1}{2}, \frac{1}{2}\right) \approx 0.3914 \leqslant \omega_{free,1}\left(\frac{1}{2}, \frac{1}{2}\right) \approx 0.3999$$

It is worth mentioning that any kinematic constraint changes a two degree-of-freedom circulatory non-conservative system into a one-degree-of-freedom conservative system. Especially for $\gamma \in [\frac{1}{2}, 1[$ the critical load by divergence of the constrained system $p_{cons,div}$ (for any kinematic constraint) is necessarily strictly lower than the critical load by divergence of the free system $p_{free,div} = +\infty$!!! (see Challamel et al., 2009 for example). Viewed as the decreasing of the critical divergence load, any additional kinematic constraint has then a destabilizing or a sort of stiffness softening effect. This stiffness softening effect however may fail if it is viewed as the decreasing of the lowest eigenfrequency of the system for a given constraint. For example with the constraint $\theta_1 = \theta_2$ and for the loading $p = \frac{1}{2}$, $\gamma = \frac{1}{2}$, we find $\omega_{cons,1}(\frac{1}{2}, \frac{1}{2}) = \frac{\sqrt{3}}{4} \approx 0.4330 > \omega_{free,1}(\frac{1}{2}, \frac{1}{2}) \approx 0.3999$.

4.6.2. Frequencies as functions of the loading parameter, comparison, stability

Concerning the flutter instability, because it depends on the considered system, it has been already discussed in 3.4.

For $\gamma = 1$, we then find

$$\omega_{free,1}(p, 1) = \frac{1}{7}\sqrt{378 - 21\sqrt{296 - 180\,p + 25\,p^2} - 105\,p}$$

$$\omega_{free,2}(p, 1) = \frac{1}{7}\sqrt{378 + 21\sqrt{296 - 180\,p + 25\,p^2} - 105\,p}$$

and

$$\omega_{cons,1}(p, 1) = \frac{1}{7}\sqrt{21}\Big(\Big(19240 - 16584p + 5050p^2 - 528p^3 - 2228\sqrt{74 - 45p + 8p^2}\Big) + 1244p\sqrt{74 - 45p + 8p^2} - 193p^2\sqrt{74 - 45p + 8p^2}\Big)\Big/\Big(592 - 360p + 64p^2 - 58\sqrt{74 - 45p + 8p^2} + 13p\sqrt{74 - 45p + 8p^2}\Big)^{\frac{1}{2}}$$

leading to Fig. 3 where, as mentioned in 3.4:

$$\omega_{cons,1}(p) \leqslant \omega_{free,1}(p) \;\text{ and }\; p_{cons} = p_{cons,div} = 2 \leqslant p_{free,fl} = \frac{18}{5} - \frac{2}{5}\sqrt{7} \approx 2.5416$$

The non-constrained pendulum is stable at positive p when $p < p_{free,fl}$ and unstable by flutter when $p > p_{free,fl}$.

For $\gamma = \frac{1}{2}$, we then find

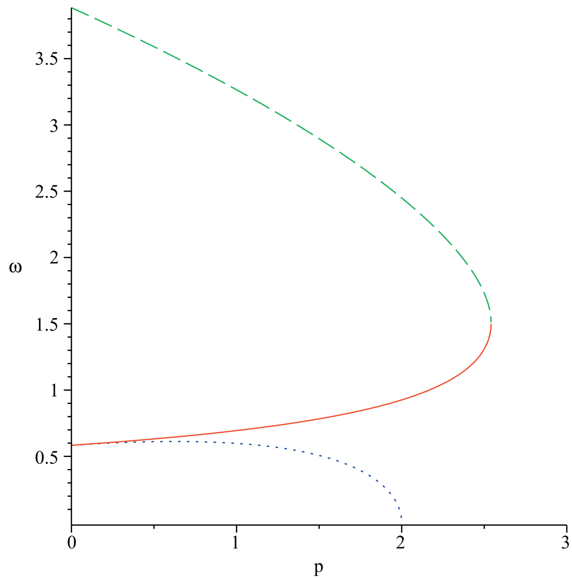

Fig. 3. $\gamma = 1$ Eigenfrequencies ω as functions of the load parameter p: lowest eigenfrequency of the free system (plain red), highest eigenfrequency of the free system (dashed green) and (optimal) constrained system (dotted blue): the free system has no divergence instability but only flutter instability and the constrained system has a divergence load which is smaller than the flutter load of the free system.(For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

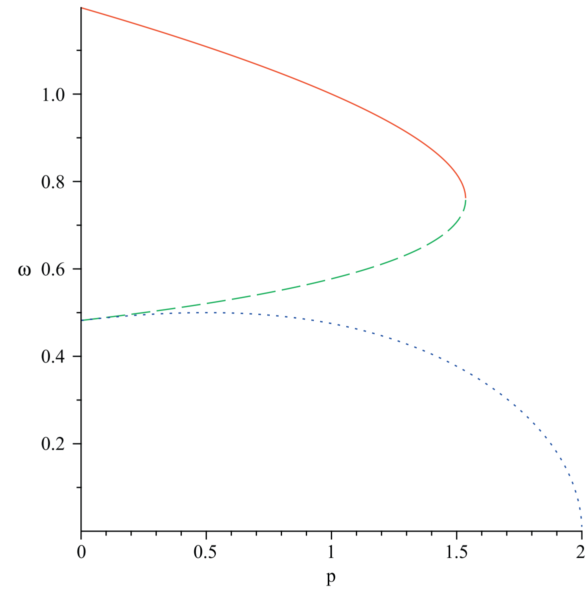

Fig. 5. $\gamma = 1$ and another mass matrix. Eigenfrequencies ω as functions of the load parameter p: lowest eigenfrequency of the free system (dashed green), highest eigenfrequency of the free system (plain red) and (optimal) constrained system (dotted blue): the free system has no divergence instability but only flutter instability and the constrained system has a divergence load which is larger than the flutter load of the free system.(For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

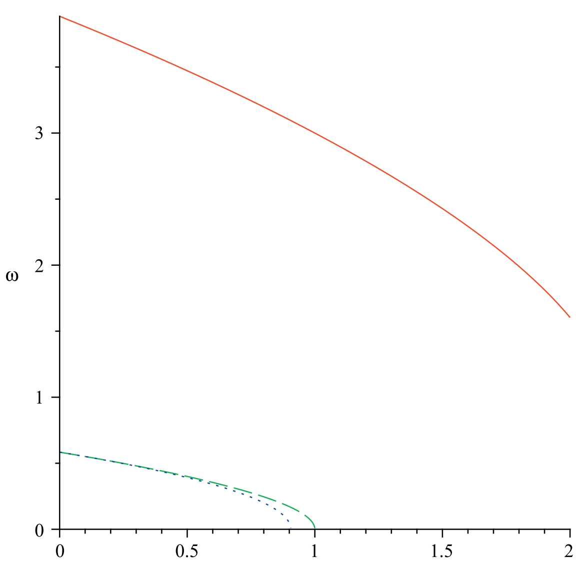

Fig. 4. $\gamma = \frac{1}{2}$ Eigenfrequencies ω as functions of the load parameter p: lowest eigenfrequency of the free system (dashed green), highest eigenfrequency of the free system (plain red) and (optimal) constrained system (dotted blue): the free system has no flutter instability but only flutter instability and the constrained system has a divergence load which is smaller than the divergence load of the free system.(For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

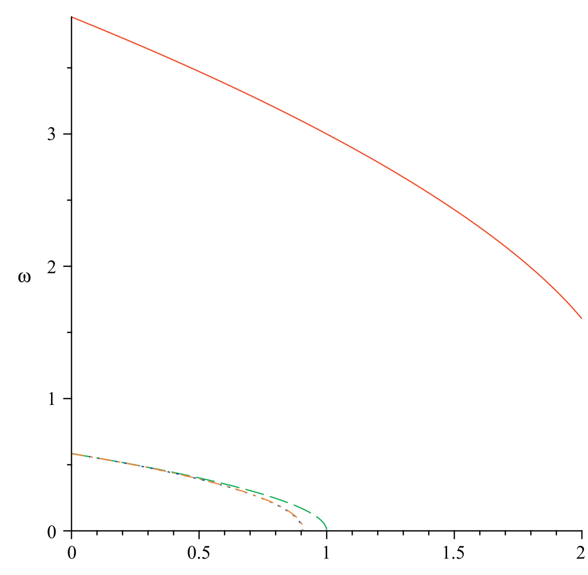

Fig. 6. Eigenfrequencies ω as functions of the load parameter p: lowest eigenfrequency of the free system (dashed green), highest eigenfrequency of the free system (plain red), optimal constrained system (dotted blue) and fixed ($p = 0$) constrained system (mixed dashed-dotted gold): the optimal constraint and the fixed constraint give the same curve.(For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

$$\omega_{free,1}(p,\frac{1}{2}) = \frac{1}{7}\sqrt{378 - 21\sqrt{296 - 228p + \frac{169}{4}p^2} - \frac{315}{2}p}$$

$$\omega_{free,2}(p,\frac{1}{2}) = \frac{1}{7}\sqrt{378 + 21\sqrt{296 - 228p + \frac{169}{4}p^2} - \frac{315}{2}p}$$

and

$$\begin{aligned}\omega_{cons,\frac{1}{2}}(p,\frac{1}{2}) = \frac{1}{7}\sqrt{21}\Big(&\big(19240 - 22664p + 8902p^2 - 1166p^3\\ &-2228\sqrt{74-57p+11p^2} + 1761p\sqrt{74-57p+11p^2}\\ &-\frac{697}{2}p^2\sqrt{74-57p+11p^2}\big)\big(592 - 456p + 88p^2\\ &-58\sqrt{74-57p+11p^2} + 23p\sqrt{74-57p+11p^2}\big)^{-1}\Big)^{\frac{1}{2}}\end{aligned}$$

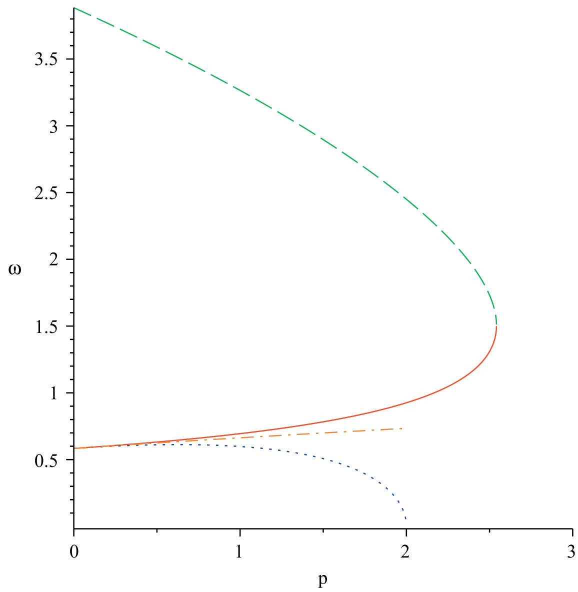

Fig. 7. Eigenfrequencies ω as functions of the load parameter p: highest eigenfrequency of the free system (dashed green), lowest eigenfrequency of the free system (plain red), optimal constrained system (dotted blue) and fixed ($p = 0$) constrained system (mixed dashed-dotted gold): the optimal constraint curve is lower than the fixed constraint curve.(For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

leading to Fig. 4 where, as mentioned in 3.3:

$$\omega_{cons,1}(p) \leqslant \omega_{free,1}(p) \text{ and } p_{cons} = p_{cons,div} = \frac{12}{7} - 4/7\sqrt{2} \approx 0.906 \leqslant p_{free,div} = 1$$

Finally, calculations for the following mass matrix

$$M = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$$

lead to Fig. 5 where, as mentioned in 3.4:

$$\omega_{cons,1}(p) \leqslant \omega_{free,1}(p) \text{ and } p_{cons} = p_{cons,div} = 2 \geqslant p_{free,fl} = 5 - 2\sqrt{(3)} \approx 1.536$$

4.6.3. Dependency on p

As already discussed in (3.5), the variety of the loading dependency of the constraints is illustrated here on Figs. 6 and 7. Comparison of the loading dependency optimal constraint and the fixed constraint is done with the fixed $p = 0$ constraint by adding the curve of the fixed constraint (mixed dashed-dotted gold line) on previous Figs. 3 and 4. In the first case the added curve coincides with the curve of the optimal constrained system: the optimal constraint is exactly the fixed $p = 0$ constraint on the whole interval $[0, p_{sw}[$. In the second one, the added curve does not coincide with the curve of the optimal constrained system

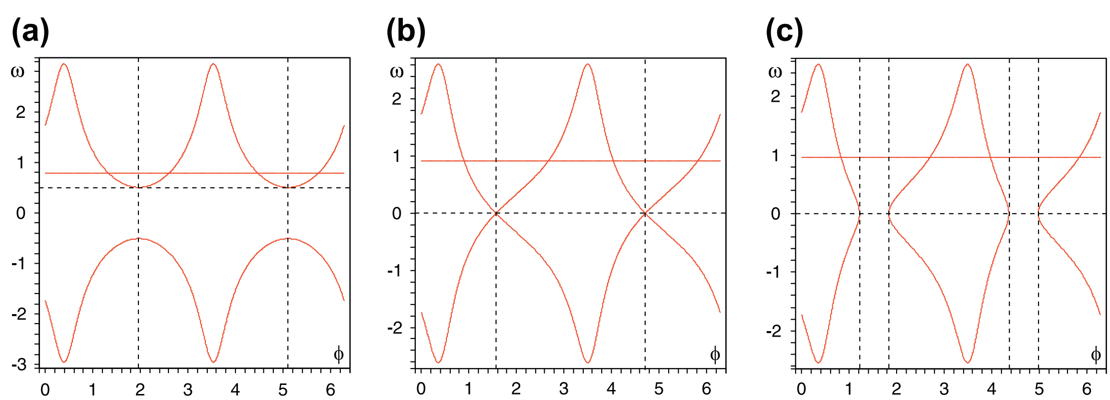

Fig. 8. In the (ϕ, ω) plane the frequencies $\omega_{free,1}$ (horizontal lines) and ω_{cons} for (a) $p = 1.5$, (b) $p = 2$, and (c) $p = 2.1$.

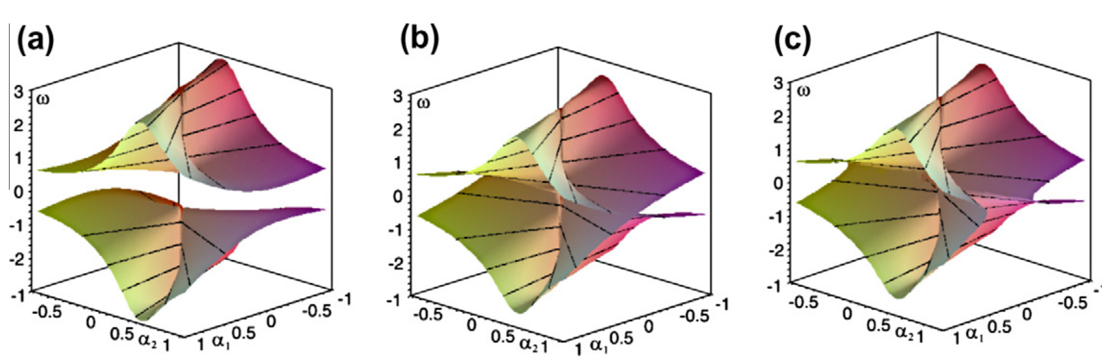

Fig. 9. In the $(\alpha_1, \alpha_2, \omega_{cons})$ space the surfaces of frequencies of the constrained system plotted for (a) $p = 1.5$, (a) $p = 2$, (c) $p = 2.1$.

and is obviously over the optimal one because precisely the optimality of the constraint.

4.7. A singular surface

Finally, we study the frequency of the constrained pendulum as a function of the coefficients of the constraints, α_1 and α_2 in the assumption that $\gamma = 1$. According to Eqs. (14) and (15), this frequency is a root of the polynomial

$$\left(\frac{4}{3}\alpha_2^2 - \alpha_2\alpha_1 + \frac{1}{3}\alpha_1^2\right)\omega^2 = 2\alpha_2^2 - \alpha_2^2 p + 2\alpha_2\alpha_1 - \alpha_2\alpha_1 p + \alpha_1^2.$$

Since the roots depend only on the ratio of the two coefficients, then, introducing $\alpha_1 = \cos(\phi)$ and $\alpha_2 = \sin(\phi)$, we find

$$\omega_{cons}^2 = \frac{-2 - 2\sin(\phi)\cos(\phi) + \sin(\phi)\cos(\phi)p + \cos(\phi)^2 + p - \cos(\phi)^2 p}{\sin(\phi)\cos(\phi) - 4/3 + \cos(\phi)^2}.$$

In the (ϕ, ω) plane let us plot the two frequencies, $\omega_{free,1}$ and ω_{cons}, for different values of p that are inside the stability interval of the free system, see Fig. 8. The horizontal lines in Fig. 8 show $\omega_{free,1}$ for a given p, whereas the curves show the frequencies of the constrained pendulum $\pm\omega_{cons}$.

The frequency $\omega_{cons,1}\left(\frac{3}{2}, 1\right) \approx 0.5073$ is achieved at a unique constraint that corresponds to the minima visible in the panels (a) and (b) of Fig. 8. On the panel (a) the minimum is the frequency $\omega_{cons,1}\left(\frac{3}{2}, 1\right) \approx 0.5073$. For $p = 2$ it is zero. For $p > 2$ there is a *divergence* interval, and the minimal frequencies are zero at its ends, see the panel (c) of Fig. 8.

Therefore, at every p from the stability interval of the free pendulum there is a continuum of the ratios of α_1 to α_2 that yields the effect of softening. The constraint that is found from the general theory developed above is the optimal one, because it gives the minimal possible frequency.

In the $(\alpha_1, \alpha_2, \omega_{cons})$ space we plot the surfaces of frequencies of the constrained system as functions of α_1 and α_2, see Fig. 9. The panel (a) corresponds to $p = 1.5$ and shows two *Plücker conoids* of degree 2 (Berger and Gostiaux, 1988; Kirillov and Stefani, 2012). The panel (b) shows one Plücker conoid of degree 1. The right panel shows the *conical wedge of Wallis* known in the physical literature under the name of the *double coffee-filter* (Berger and Gostiaux, 1988; Kirillov and Stefani, 2012). The three classical ruled surfaces demonstrate how the constraints introduce singularities in the behavior of eigenfrequencies which explains their high sensitivity to the variation of the constraints coefficients.

5. Conclusion

In this paper, we generalized the possible paradoxical softening effect that originates after kinematic constraints are applied to a system. We investigated this stiffness softening effect by means of the spectral analysis of the free and constrained systems. With the use of the second order work criterion, it is established that, for each value of the load parameter, there always exists a (family of) constraint(s) allowing to make the lowest eigenfrequency of the system smaller. Moreover, the second order work criterion provides, in a certain sense, the optimal kinematic constraint. The consequences for the divergence and flutter instabilities are discussed and numerous examples illustrating the results are considered. The behavior of eigenfrequencies as functions of both the load parameter and constraints are also studied. It shows a variety of possible situations for occurrence of flutter and reveals the typical singularities on the eigenfrequency surfaces. The important and difficult problem of finding a global (meaning independent on the load parameter) stiffness softening kinematic constraint is still open.

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