A min–max relation for $K_3$-covers in graphs noncontractible to $K_5\setminus e$

Ali Ridha Mahjoub

Département d’Informatique, Université de Bretagne Occidentale, BP 452, 6 Avenue Victor Le Gorgeu, 29275 Brest Cedex, France

Received 11 May 1992; revised 31 January 1994

Abstract

In Euler and Mahjoub (1991) it is proved that the triangle-free subgraph polytope of a graph noncontractible to $K_5\setminus e$ is completely described by the trivial inequalities and the so-called triangle and odd wheel inequalities. In this paper we show that the system defined by those inequalities is TDI for a subclass of that class of graphs. As a consequence we obtain the following min–max relation: If $G$ is a graph noncontractible to $K_5\setminus e$, then the minimum number of edges covering all the triangles of $G$ equals the maximum profit of a partition of the edge set of $G$ into edges, triangles and odd wheels. Here the profit of an edge is 0, the profit of a triangle is 1 and the profit of a $2k + 1$-wheel ($k \in \mathbb{N}$) is equal to $k + 1$.

Keywords: Graphs noncontractible to $K_5\setminus e$; $K_7$-covers; Total dual integrality; Polytopes

1. Introduction

We consider graphs which are finite, undirected, loopless and without multiple edges. We denote a graph by $G = (V, E)$, where $V$ is the node set and $E$ is the edge set of $G$. A graph $G$ is said to be contractible to a graph $H$, if $H$ may be obtained from $G$ by a sequence of elementary removals and contractions of edges. A contraction consists of identifying a pair of adjacent nodes and of preserving all other adjacencies between nodes (multiple edges arising from the identification are replaced by single edges and loops are deleted).

A $K_3$-cover of a graph $G = (V, E)$ is an edge subset of $E$ which intersects all the triangles of $G$. Given a graph $G = (V, E)$ and a weight function $w: E \to \mathbb{R}$, the $K_3$-cover problem in $G$ consists of finding a $K_3$-cover in $G$ whose total weight is minimum. This problem is NP-complete in general [9]. It has been shown to be polynomial in chordal

* This work was done while the author was visiting Laboratoire ARTEMIS, IMAG Université J. Fourier, Grenoble, France.
A graph is called an n-wheel (denoted \( W_n \)) if it consists of a cycle of \( n \) nodes and a node (so-called universal) which is adjacent to every node of the cycle.

Given a graph \( G = (V, E) \), we associate with every edge (resp. triangle, 2\( k \) + 1-wheel) a profit equal to 0 (resp. 1, \( k + 1 \)). Define a \( \Delta \)-partition of \( G \) to be a partition of the edge set \( E \) into edges, triangles and odd wheels. And let the profit of a \( \Delta \)-partition of \( G \) be the sum of the profits of its elements. In this paper we are going to show, using a polyhedral approach, that, for a graph \( G \) noncontractible to \( K_5 \setminus e \), the minimum cardinality of a \( K_5 \)-cover equals the maximum profit of a \( \Delta \)-partition of \( G \).

Given a graph \( G = (V, E) \) and an edge subset \( F \subseteq E \), the 0–1 vector \( x^F \in \mathbb{R}^E \), where \( x^F(e) = 1 \) if \( e \in F \) and 0 if not, is called the incidence vector of \( F \). The convex hull \( P(\Delta(G)) \) of the incidence vectors of all the edge sets of triangle-free subgraphs of \( G \) is called the triangle-free subgraph polytope of \( G \) i.e.

\[
P(\Delta(G)) = \{ x^F \in \mathbb{R}^E | F \subseteq E, (V, F) \text{ is triangle-free} \}.
\]

Thus the \( K_5 \)-cover problem in \( G \) is equivalent to the following linear program:

\[
\begin{align*}
\text{max} & \quad w^T x, \\
\text{subject to} & \quad x \in P(\Delta(G)).
\end{align*}
\]

The polytope \( P(\Delta(G)) \) is full dimensional. This implies that (up to multiplication by a positive constant) there is a unique nonredundant inequality system \( Ax \leq b \) such that \( P(\Delta(G)) = \{ x \in \mathbb{R}^E | Ax \leq b \} \).

Let \( G = (V, E) \) be a graph. Clearly, any incidence vector \( x^F \) of a triangle-free edge set \( F \) of \( G \) satisfies the constraints:

\[
x(C) \leq 2 \quad \text{for all triangles } C \text{ in } G,
\]

for all triangles \( C \) in \( G \), (1)
Inequalities (1)–(3) are called respectively triangle, odd wheel and trivial inequalities. Here $b(F)$, where $b:E \rightarrow \mathbb{R}$ and $F \subseteq E$, denotes $\sum_{e \in F} b(e)$.

Conforti et al. [2] showed that the inequalities (1)–(3), for $W_n$ with $n \geq 4$ and odd, define facets for $P(A(G))$. For $P(A(W_3))$, inequality (2) is redundant, this inequality can be obtained by summing the four inequalities associated with the triangles of $W_3$.

A system $Ax \leq b$ is called totally dual integral (TDI) [4,6] if the dual of the linear program

$$\begin{align*}
\text{max} & \quad wx, \\
\text{subject to} & \quad Ax \leq b
\end{align*}$$

has an integer optimal solution for every integer vector $w$ such that the maximum exists. In the following section we are going to show that the system (1)–(3) is TDI for a subclass of the class of graphs noncontractible to $K_5\setminus e$. Using this together with a nonminimal description of the polytope $P(A(W_3))$ (given by the system (1)–(3)), we show in Section 3 that for a graph $G$ noncontractible to $K_5\setminus e$ the minimum cardinality of a $K_3$-cover of $G$ equals the maximum profit of a $\Delta$-partition of $G$.

2. Graphs noncontractible to $K_5\setminus e$ and TDI'ness

A graph $G = (V, E)$ is called $k$-sum ($1 \leq k \in \mathbb{N}$) of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ if $G$ is obtained from $G_1$ and $G_2$ by identifying a complete graph on $k$ nodes, that is, $V = V_1 \cup V_2$, $|V_1 \cap V_2| = k$ and for every nodes $i, j \in V_1 \cap V_2$, the edge $ij$ belongs to $E$. Clearly, the graphs shown in Fig. 2 are noncontractible to $K_5\setminus e$.

Wagner [8] gave the following constructive characterization for graphs noncontractible to $K_5\setminus e$.

**Theorem 1.** Each maximal (with respect to its edge set) graph $G = (V, E)$ noncontractible to $K_5\setminus e$ can be obtained by means of 1- and 2-sums starting from the graphs of Fig. 2.

Let $\Gamma$ be the class of graphs noncontractible to $K_5\setminus e$ that do not contain $W_3$ as an induced subgraph. In other words, by Wagner's theorem, the class $\Gamma$ is just the class of graphs that is obtained by means of 1- and 2-sums from $K_1, K_2, K_3, K_{3,3}$, the prism and $W_n$, $n \geq 4$.

Euler and Mahjoub [5] studied, within the framework of a general composition of independence systems, the polytope $P(\Delta(G))$ in graphs which are decomposable by means of 1- and 2-sums. They showed that if a graph $G$ decomposes into two graphs $G_1$ and $G_2$, then one can derive a linear system of inequalities which defines the polytope $P(\Delta(G))$ from the linear systems defining $P(\Delta(G_1))$ and $P(\Delta(G_2))$. Using this, they proved that for a graph $G$ which is noncontractible to $K_5\setminus e$, the polytope $P(\Delta(G))$ is completely described by the inequalities (1)–(3). They also showed that if
the two systems defining $P(\Delta(G_1))$ and $P(\Delta(G_2))$ are TDI then the system defining $P(\Delta(G))$ is as well (see also [1]). In what follows we are going to use this together with Wagner's theorem to show that the system (1)–(3) is TDI for the class $\Gamma$. For this we just need to show that the system is TDI for the basic graphs of $\Gamma$. This is, in fact, easily seen to hold for the graphs $K_1, K_2, K_3, K_{3,3}$, the prism and the even wheels. For these graphs, the polytope $P(\Delta(G))$ is just defined by the trivial and triangle inequalities. The matrix of the system given by these inequalities, for each of these graphs, is totally unimodular. In what follows we show that the system (1)–(3) is TDI for the odd wheels $W_{2k+1}$, $k \geq 2$.

To this end, let us denote by $P_w$ the linear programming problem

$$P_w = \begin{cases} \text{max} & w^T x, \\ \text{subject to} & (1), (2), (3). \end{cases}$$

By associating a dual variable $y_C, \delta, \gamma$, with a constraint of type (1), (2), $x(e) \leq 1$, respectively, the dual of $P_w$, $D_w$ can be written as follows:

$$D_w = \begin{cases} \text{min} & \sum 2y_C + \sum (3k + 1)\delta + \sum \gamma, \\ \text{subject to} & \sum_{e \in E} y_C + \sum_{e \in E} \delta + \gamma \geq w(e), \text{ for all } e \in E, \\ & y \geq 0, \delta \geq 0, \gamma \geq 0. \end{cases}$$

**Theorem 2.** The system (1)–(3) is TDI for every $2k + 1$-wheel, $2 \leq k \in \mathbb{N}$. 
Proof. Let $F$ be the edge set of a $2k + 1$-wheel, $k \geq 2$. Let $e_1, \ldots, e_{2k+1}, f_1, \ldots, f_{2k+1}$ be the edges of $F$ such that $e_i$ and $e_{i+1}$ are incident to a common node and $\{e_i, f_i, f_{i+1}\}$ forms a triangle, for $i = 1, 2, \ldots, 2k + 1$ (see Fig. 3, where the indices are taken modulo $2k + 1$).

Let $\lambda(w)$ denote the optimal value of the objective function of $P_w (D_w)$. Since the system (1)–(3) is integral and, consequently, the problem $P_w$ has always an integer optimal solution, it follows that $\lambda(w)$ is integer whenever $w$ is integer valued. Now, to show the theorem, we shall use ideas similar to those of Barahona et al. [1] for acyclic spanning subgraphs.

We shall proceed by induction on $w$. Obviously, for $w \leq 0$, $D_w$ has the trivial zero optimal solution. Now assume that $D_w$ has an integer optimal solution for every integer vector $w$, $w \leq z$, $w \neq z$, and let us show that $D_w$ has an integer optimal solution for $w = z$. For this we may assume that $w > 0$ (if $w(e) < 0$ for some edge $e \in F$, then the system (1), (2) is easily seen to be TDI for the graph obtained from $W_{2k+1}$ by removing the edge $e$).

Now consider the set of inequalities among (1)–(3), that are satisfied with equality by every optimal solution of $P_w$. Let us denote this set by $T_w$.

Case 1: $x(e_0) < 1$ is in $T_w$ for some edge $e_0$.

Let $w'$ be the vector given by

$$w'(e) = \begin{cases} w(e) & \text{if } e \in F \setminus \{e\}, \\ w(e) - 1 & \text{if } e = e_0. \end{cases}$$

We claim that $\lambda(w') = \lambda(w) - 1$. Indeed, it is clear that $\lambda(w') \leq \lambda(w)$. If $\lambda(w') = \lambda(w)$, then $e_0$ cannot be in any optimal solution of $P_{w'}$, otherwise $\lambda(w)$ would not be maximum. Since every optimal solution for $P_{w'}$ is at the same time optimal for $P_w$, this contradicts the fact that $x(e_0) \leq 1$ is in $T_w$ and our claim is proved. Now by the induction hypothesis, there is an integer optimal solution to $D_{w'}$. Consider the
solution obtained from that solution by increasing by one the value of the dual variable associated with \(x(e_0) \leq 1\). This solution is integer and optimal for \(D_w\).

**Case 2:** \(x(C) \leq 2\) is in \(T_w\) for some triangle \(C\).

Let \(w'\) be the vector given by
\[
    w'(e) = \begin{cases} 
        w(e) & \text{if } e \in F \setminus C, \\
        w(e) - 1 & \text{if } e \in C. 
    \end{cases}
\]

We claim that \(\lambda(w') = \lambda(w) - 2\). Clearly, \(\lambda(w) - 2 \leq \lambda(w') \leq \lambda(w)\).

(i) If \(\lambda(w') = \lambda(w)\), then \(C\) cannot intersect any optimal solution to \(P_{w'}\). But since, in this case, every optimal solution for \(P_{w'}\) is also optimal for \(P_w\), we have a contradiction.

(ii) If \(\lambda(w') = \lambda(w) - 1\), then we claim that every optimal solution for \(P_{w'}\) contains exactly one edge of \(C\). In fact, it is clear that such a solution cannot contain more than one edge of \(C\). Now if, for instance, there is an optimal solution for \(P_{w'}\), say \(F_1\), which does not intersect \(C\), then there must exist an edge, say \(f\), such that \(F_2 = F_1 \cup \{f\}\) is still triangle-free. In fact, it is easy to see that any maximal triangle-free edge subset of \(F\) intersects all the triangles of \(F\). But then \(F_2\) defines an optimal solution for \(P_w\) whose incidence vector does not satisfy \(x(C) \leq 2\) with equality, a contradiction. Consequently, we obtain that \(\lambda(w') = \lambda(w) - 2\) and our claim is proved.

Now consider the solution obtained from an integer optimal solution of \(D_w\) by increasing by one the value of the dual variable associated with \(x(C) \leq 2\). We have that this solution is integer and optimal for \(D_w\).

**Case 3:** \(x(F) \leq 3k + 1\) is in \(T_w\).

We may assume that no constraints of type (1) or type (2) are satisfied with equality by all the optimal solutions of \(P_w\), otherwise we are either in Case 1 or in Case 2.

**Claim.** \(w(e) = w(f)\) for all \(e, f \in F\).

**Proof of the claim.** First remark that, since \(w > 0\), any optimal solution for \(P_w\) is a maximal triangle-free subset of \(F\) and, by the remark above, intersects every triangle \(\{e_i, f_i, f_{i+1}\}, i = 1, \ldots, 2k + 1\). Moreover, if \(e_i\) does not belong to an optimal solution of \(P_w\), then it follows that \(f_i, f_{i+1}\) both belong to that solution. Thus from the assumption it follows that for every two edges \(f_i, f_{i+1}\), \(i = 1, 2, \ldots, 2k + 1\), there must exist an optimal solution for \(P_w\), say \(F_i\), which contains neither \(f_i\) nor \(f_{i+1}\). We claim that \(F_i = F \setminus \{f_i, f_{i+1}, f_{i+3}, \ldots, f_{i+2k-1}\}\). For this it suffices to show that \(F_i\) cannot contain two consecutive edges \(f_{i+p}, f_{i+p+1}\) with \(p \in \{2, \ldots, 2k - 1\}\). Indeed, assume that this were not the case and, without loss of generality, that \(p\) is odd. Thus \(e_{i+p} \notin F_i\). Moreover, the edge set \(F \setminus F_i\) intersects each of the edge disjoint triangles \(\{e_{i+t}, f_{i+t}, f_{i+t+1}\}\) for \(t = 2, 4, \ldots, p - 1, p + 1, p + 3, \ldots, 2k\). Moreover, we have that
these triangles all do not contain \( f_{i+1} \). Thus we obtain that \(| F \setminus F_i | \geq k + 2\), a contradiction.

Now let \( \beta = w(F) - \lambda(w) \). By considering the edge sets \( F \setminus F_i \) for \( i = 1, \ldots, 2k + 1 \), we find that the vector \( (w(f_1), \ldots, w(f_{2k+1}))^T \) satisfies the system

\[
Ax = b,
\]

where \( A \) is the \((2k + 1, 2k + 1)\)-matrix whose rows are respectively the incidence vectors of the sets \( F \setminus F_1, \ldots, F \setminus F_{2k+1} \), with respect to the edge set \( F \setminus \{e_1, \ldots, e_{2k+1}\} \) and \( b = (\beta, \ldots, \beta)^T \). It is not hard to see that the matrix \( A \) is nonsingular, implying that the system above has the unique solution given by

\[
w(f_i) = \beta / 2k + 1 \quad \text{for} \quad i = 1, \ldots, 2k + 1.
\] (4)

On the other hand, since there is no constraint of type \( x(e) \leq 1 \) satisfied with equality by every optimal solution of \( P_w \), it follows that for every edge \( e_i \) where \( i = 1, \ldots, 2k + 1 \), there is an optimal solution for \( P_w \), say \( F_i \), which does not contain \( e_i \). Thus \( f_i, f_{i+1} \in F_i \). As before, we can show that \( F_i \) cannot contain two consecutive edges \( f_{i+p, f_{i+p+1}} \) with \( p \in \{1, \ldots, 2k\} \), implying that \( F_i = F \setminus \{e_{i_1}, e_{i+2}, e_{i+4}, \ldots, e_{i+2k}\} \). Since \( w(F_i) = w(F_i) \), it follows from (4) that

\[
w(e_i) = \beta / 2k + 1 \quad \text{for} \quad i = 1, \ldots, 2k + 1,
\]

which finishes the proof of the claim.

Now define \( w' \) as \( w'(e) = w(e) - 1 \) for all \( e \in F \). From the claim it follows that \( \lambda(w') = \lambda(w) - (3k + 1) \). Consider the dual solution obtained from an integer optimal solution of \( D_w \) by increasing by one the value of the dual variable associated with \( x(F) \leq 3k + 1 \). This solution is integer and optimal for \( D_w \), which completes the proof of our theorem. \( \square \)

Thus we can state our main result.

**Theorem 3.** The system (1)–(3) is TDI for \( \Gamma \).

In what follows we shall use Theorem 3 to derive a min–max relation for the \( K_3 \)-covers in graphs noncontractible to \( K_5 \setminus e \).

### 3. A min–max relation

Let \( G = (V, E) \) be a graph noncontractible to \( K_5 \setminus e \) and \( w \) be an integer weight vector associated with the edges of \( G \). The \( K_3 \)-cover problem in \( G \) is also equivalent to the following linear program

\[
P'_w = \begin{cases} 
\min & w^T x, \\
\text{subject to} & (1)', (2)', (3)', 
\end{cases}
\]
where (1)', (2)', (3)' are obtained respectively from (1), (2), (3) by replacing $x$ by $1 - x$ ($1$ is the vector whose entries all equal 1).

The dual of $P'_w$, $D'_w$ is

$$D'_w = \begin{cases} \max \sum y_C + \sum (k + 1)\delta_n - \sum \gamma_e, \\ \text{subject to } \sum_{c\in e} y_C + \sum w_{n\in c} \delta_n - \gamma_e \leq w(e), \text{ for all } e \in E, \\ y \geq 0, \delta \geq 0, \gamma \geq 0. \end{cases}$$

**Lemma 4.** If $w \geq 0$ and $D'_w$ has an integer optimal solution then such a solution can be chosen so that $\gamma_e = 0$ for all $e \in E$.

**Proof.** Let $(y^0, \delta^0, \gamma^0)$ be an integer optimal solution for $D'_w$. Suppose that for some edge $f \in E$, $\gamma^0_f > 0$. We shall show that there exists a dual optimal solution to $D'_w$, say $(y^*, \delta^*, \gamma^*)$, such that $\gamma^*_f = 0$ and $\gamma^*_e = 0$ for all $e \in E$ such that $\gamma^*_e = 0$. Since $\gamma^0_f > 0$, it follows that the dual constraint (5) associated with the edge $f$ is satisfied with equality by $(y^0, \delta^0, \gamma^0)$, otherwise one can decrease the value of $\gamma_e$ by a positive amount and then get a solution whose value is greater than that of $(y^0, \delta^0, \gamma^0)$, contradicting the optimality of the latter one.

Moreover, we may assume that $\gamma^0_C = 0$ for every triangle $C$ containing $f$. Indeed, if, for instance $\gamma^0_C > 0$ for some triangle $C$ such that $f \in C$ and, say, $\gamma^0_C < \gamma^0_f$ (which can be assumed without loss of generality), then one can consider the solution $(y^{0'}, \delta^{0'}, \gamma^{0'})$ such that $\gamma^0_C$ is integer and optimal for $D'_w$.

Consequently, we can suppose that

$$\sum_{W_n \ni f} \delta_n = \gamma_f + w(f).$$

Since $w \geq 0$, $\sum_{W_n \ni f} \delta_n \geq \gamma_f$. Now, let $n^1, \ldots, n^p$, $p \in \mathbb{N}$, be odd integers, such that $n^i = 2k^i + 1$, $k^i \in \mathbb{N}$, and $f \in W_n i$, for $i = 1, \ldots, p$, and

$$\sum_{i=1}^{p-1} \delta_{n^i} < \gamma_f \quad \text{and} \quad \sum_{i=1}^{p-1} \delta_{n^i} \geq \gamma_f.$$

Let

$$\varepsilon = \gamma_f - \sum_{i=1, \ldots, p-1} \delta_{n^i}.$$
Observe that every wheel $W_n^i$ contains $k^i$ edge disjoint triangles, say $C_1^i, C_2^i, \ldots, C_{k^i}^i$. Thus we may consider the solution $(y^*, \delta^*, \gamma^*)$ given by

$$y_C^* = \begin{cases} y_C^0 + \delta_n^* & \text{if } C = C_i^t, \; i = 1, \ldots, p - 1; \; t = 1, \ldots, k^i, \\ y_C^i + \varepsilon & \text{if } C = C_p^t, \; t = 1, \ldots, k^p, \\ 0 & \text{otherwise}, \end{cases}$$

$$\delta_n^* = \begin{cases} 0 & \text{if } n = n^1, \ldots, n^{p - 1}, \\ \delta_n & \text{otherwise}, \end{cases}$$

$$\gamma_e^* = \begin{cases} 0 & \text{if } e = f, \\ \gamma_e & \text{if } e \neq f. \end{cases}$$

It is easily seen that the solution $(y^*, \delta^*, \gamma^*)$ is feasible and optimal to $D_w$. □

Since, by Theorem 3, the system (1)–(3) is TDI for $\Gamma$, it follows that the system given by the inequalities (1)', (2)', (3)' is also TDI for $\Gamma$. Consequently, the dual $D_w'$ has an integer optimal solution for every integer weight system $w$ and every graph in $\Gamma$. Now to establish our relation between the minimum cardinality of a $K_3$-cover of a graph $G$ noncontractible to $K_5 \setminus e$ and the maximum profit of a $\Delta$-partition of $G$ it remains to examine the graph $W_3$. The system (1)–(3) is not, unfortunately, TDI for that graph. In fact consider a $W_3$ and let \{ $e_1, e_2, e_3, f_1, f_2, f_3$ \} be its edge set (we suppose that $W_3$ has the form which is shown in Fig. 3). Now let us associate with the edges of $W_3$ the weight system $w = (1, 0, 0, 0, 1, 0)$. Let $C^1, C^2, C^3$ be respectively the cycles \{ $e_1, e_2, e_3$ \}, \{ $e_1, f_1, f_2$ \}, \{ $e_2, f_2, f_3$ \}. It is easy to see that in this case, the problems $P_w$ and $D_w$ have unique optimal solutions which are given by $x(e_1) = x(e_2) = x(f_2) = 1$, $x(e_3) = x(f_1) = x(f_3) = 0$ and $y_{C^1} = y_{C^2} = y_{C^3} = \frac{1}{2}$, $y_C = 0$ otherwise, $\delta = 0$, $\gamma = 0$, respectively. However it is not hard to see that for $W_3$, the dual problem $D'_1$ has an integer optimal solution. Moreover, if we compose $W_3$ by means of 1- or 2-sum with a graph for which the system (1)–(3) is TDI, the dual problem $D'_1$ has for the resulting graph an integer optimal solution (see [5]). Consequently we have the following lemma.

**Lemma 5.** The problem $D'_1$ has an integer optimal solution for every graph noncontractible to $K_5 \setminus e$.

Now by combining Theorem 3 and Lemmas 4, 5 we obtain the following min–max relation.

**Theorem 6.** If $G$ is a graph noncontractible to $K_5 \setminus e$, then the minimum cardinality of a $K_3$-cover of $G$ equals the maximum profit of a $\Delta$-partition of $G$. 
Let $G = (V,E)$ be a graph. A $K_3$-packing of $G$ is a collection of pairwise edge disjoint triangles of $G$. If $\mathcal{T}$ is a collection of triangles of $G$, the $K_3$-cover number $c_3(\mathcal{T})$ (resp. $K_3$-packing number $p_3(\mathcal{T})$) of $\mathcal{T}$ is the smallest number of edges ($K_3$'s) that cover all the triangles of $\mathcal{T}$ (resp. the largest number of triangles of $\mathcal{T}$ which are pairwise edge disjoint). A graph $G$ is called $K_3$-perfect [3] if $c_3(\mathcal{T}) = p_3(\mathcal{T})$ for any collection $\mathcal{T}$ of triangles of $G$. (Conforti et al. [3] introduced a more general concept: The $K_i$-perfect graphs, where $i$ is a fixed positive integer. They called a graph $K_i$-perfect if for any collection $\mathcal{X}$ of $K_i$'s, $c_i(\mathcal{X}) = p_i(\mathcal{X})$ where $c_i(\mathcal{X})$ and $p_i(\mathcal{X})$ are defined analogously. They gave a characterization for $K_i$-perfect graphs which is similar to the strong perfect graph conjecture and studied the relationship between $K_i$-perfect graphs and perfect graphs and perfect and balanced matrices.)

Now let us introduce the class $\Omega$ of graphs $G$ such that $G$ is obtained by means of 1- and 2-sums from the graphs $K_1, K_2, K_3, K_3,3$, the prism and the even wheels (i.e. $G$ is a graph noncontractible to $K_5\setminus e$ which does not contain an odd wheel). From Theorem 6 we have that for every graph $G$ of $\Omega$, the minimum number of edges that cover all the triangles of $G$ equals the maximum number of pairwise edge disjoint triangles. Moreover we have the following corollary.

**Corollary 7.** Every graph in $\Omega$ is $K_3$-perfect.

A matrix $A$ is called perfect [7] if the associated set packing polytope $\{x | Ax \leq 1, x \geq 0\}$ has all its extreme points in $0$–$1$. Given a graph $G$, let $T(G)$ denote the transpose of the triangles incidence matrix of $G$. Conforti et al. [3] showed that a graph is $K_3$-perfect if and only if the matrix $T(G)$ is perfect. From this and Corollary 7 we obtain the following corollary.

**Corollary 8.** For a graph $G$ of $\Omega$, the matrix $T(G)$ is perfect.

Acknowledgements

I would like to thank András Sebő and Reinhardt Euler for their constructive comments. Also I would like to thank the referee for useful suggestions concerning this paper.

References