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A min–max relation for K_3-covers in graphs noncontractible to $K_5 \backslash e$ [☆]

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Abstract

In Euler and Mahjoub (1991) it is proved that the triangle-free subgraph polytope of a graph noncontractible to $K_5 \backslash e$ is completely described by the trivial inequalities and the so-called triangle and odd wheel inequalities. In this paper we show that the system defined by those inequalities is TDI for a subclass of that class of graphs. As a consequence we obtain the following min–max relation: If G is a graph noncontractible to $K_5 \backslash e$, then the minimum number of edges covering all the triangles of G equals the maximum profit of a partition of the edge set of G into edges, triangles and odd wheels. Here the profit of an edge is 0, the profit of a triangle is 1 and the profit of a $2k + 1$-wheel ($k \in \mathbb{N}$) is equal to $k + 1$.

Keywords: Graphs noncontractible to $K_5 \backslash e$; K_i-covers; Total dual integrality; Polytopes

1. Introduction

We consider graphs which are finite, undirected, loopless and without multiple edges. We denote a graph by $G = (V, E)$, where V is the *node set* and E is the *edge set* of G. A graph G is said to be *contractible* to a graph H, if H may be obtained from G by a sequence of elementary removals and contractions of edges. A contraction consists of identifying a pair of adjacent nodes and of preserving all other adjacencies between nodes (multiple edges arising from the identification are replaced by single edges and loops are deleted).

A *K_3-cover* of a graph $G = (V, E)$ is an edge subset of E which intersects all the triangles of G. Given a graph $G = (V, E)$ and a weight function $w: E \to \mathbb{R}$, the *K_3-cover problem* in G consists of finding a K_3-cover in G whose total weight is minimum. This problem is NP-complete in general [9]. It has been shown to be polynomial in chordal

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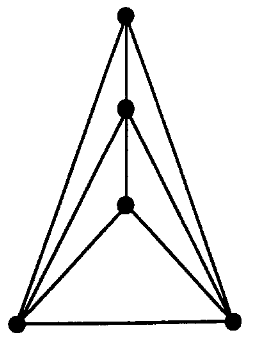

K5\e

Fig. 1.

graphs whose maximum clique size is fixed, by Conforti et al. [3] and in graphs noncontractible to $K_5\backslash e$ (the complete graph on 5 nodes minus one edge, see Fig. 1) by Euler and Mahjoub [5].

A graph is called an *n-wheel* (denoted W_n) if it consists of a cycle of n nodes and a node (so-called *universal*) which is adjacent to every node of the cycle.

Given a graph $G = (V, E)$, we associate with every edge (resp. triangle, $2k + 1$-wheel) a profit equal to 0 (resp. 1, $k + 1$). Define a *Δ-partition* of G to be a partition of the edge set E into edges, triangles and odd wheels. And let the profit of a Δ-partition of G be the sum of the profits of its elements. In this paper we are going to show, using a polyhedral approach, that, for a graph G noncontractible to $K_5\backslash e$, the minimum cardinality of a K_3-cover equals the maximum profit of a Δ-partition of G.

Given a graph $G = (V, E)$ and an edge subset $F \subseteq E$, the 0–1 vector $x^F \in \mathbb{R}^E$, where $x^F(e) = 1$ if $e \in F$ and 0 if not, is called the *incidence vector* of F. The convex hull $P(\Delta(G))$ of the incidence vectors of all the edge sets of triangle-free subgraphs of G is called the *triangle-free subgraph polytope* of G i.e.

$$P(\Delta(G)) = \{x^F \in \mathbb{R}^E \mid F \subseteq E,\ (V,\ F) \text{ is triangle-free}\}.$$

Thus the K_3-cover problem in G is equivalent to the following linear program:

$$\begin{array}{ll} \max & w^T x, \\ \text{subject to} & x \in P(\Delta(G)). \end{array}$$

The polytope $P(\Delta(G))$ is full dimensional. This implies that (up to multiplication by a positive constant) there is a unique nonredundant inequality system $Ax \leqslant b$ such that $P(\Delta(G)) = \{x \in \mathbb{R}^E \mid Ax \leqslant b\}$.

Let $G = (V, E)$ be a graph. Clearly, any incidence vector x^F of a triangle-free edge set F of G satisfies the constraints:

$$x(C) \leqslant 2 \qquad \text{for all triangles } C \text{ in } G, \tag{1}$$

$$x(W_n) \leqslant 3k + 1 \quad \text{for all } n\text{-wheels, } n = 2k + 1 \text{ and } k \in \mathbb{N}, \tag{2}$$

$$0 \leqslant x(e) \leqslant 1 \quad \text{for all edges } e \text{ of } G. \tag{3}$$

Inequalities (1)–(3) are called respectively *triangle*, *odd wheel* and *trivial inequalities*. Here $b(F)$, where $b: E \to \mathbb{R}$ and $F \subseteq E$, denotes $\sum_{e \in F} b(e)$.

Conforti et al. [2] showed that the inequalities (1)–(3), for W_n with $n \geqslant 4$ and odd, define facets for $P(\Delta(G))$. For $P(\Delta(W_3))$, inequality (2) is redundant, this inequality can be obtained by summing the four inequalities associated with the triangles of W_3.

A system $Ax \leqslant b$ is called *totally dual integral* (*TDI*) [4, 6] if the dual of the linear program

$$\begin{array}{ll} \max & wx, \\ \text{subject to} & Ax \leqslant b \end{array}$$

has an integer optimal solution for every integer vector w such that the maximum exists. In the following section we are going to show that the system (1)–(3) is TDI for a subclass of the class of graphs noncontractible to $K_5 \backslash e$. Using this together with a nonminimal description of the polytope $P(\Delta(W_3))$ (given by the system (1)–(3)), we show in Section 3 that for a graph G noncontractible to $K_5 \backslash e$ the minimum cardinality of a K_3-cover of G equals the maximum profit of a Δ-partition of G.

2. Graphs noncontractible to $K_5 \backslash e$ and TDI'ness

A graph $G = (V, E)$ is called *k-sum* ($1 \leqslant k \in \mathbb{N}$) of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ if G is obtained from G_1 and G_2 by identifying a complete graph on k nodes, that is, $V = V_1 \cup V_2$, $|V_1 \cap V_2| = k$ and for every nodes $i, j \in V_1 \cap V_2$, the edge ij belongs to E. Clearly, the graphs shown in Fig. 2 are noncontractible to $K_5 \backslash e$.

Wagner [8] gave the following constructive characterization for graphs noncontractible to $K_5 \backslash e$.

Theorem 1. *Each maximal* (*with respect to its edge set*) *graph $G = (V, E)$ noncontractible to $K_5 \backslash e$ can be obtained by means of* 1- *and* 2-*sums starting from the graphs of Fig.* 2.

Let Γ be the class of graphs noncontractible to $K_5 \backslash e$ that do not contain W_3 as an induced subgraph. In other words, by Wagner's theorem, the class Γ is just the class of graphs that is obtained by means of 1- and 2-sums from $K_1, K_2, K_3, K_{3,3}$, the prism and W_n, $n \geqslant 4$.

Euler and Mahjoub [5] studied, within the framework of a general composition of independence systems, the polytope $P(\Delta(G))$ in graphs which are decomposable by means of 1- and 2-sums. They showed that if a graph G decomposes into two graphs G_1 and G_2, then one can derive a linear system of inequalities which defines the polytope $P(\Delta(G))$ from the linear systems defining $P(\Delta(G_1))$ and $P(\Delta(G_2))$. Using this, they proved that for a graph G which is noncontractible to $K_5 \backslash e$, the polytope $P(\Delta(G))$ is completely described by the inequalities (1)–(3). They also showed that if

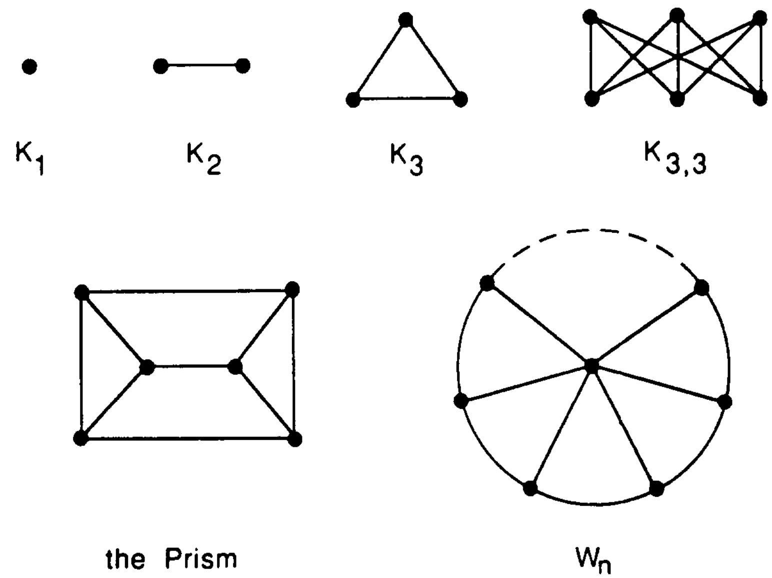

Fig. 2.

the two systems defining $P(\Delta(G_1))$ and $P(\Delta(G_2))$ are TDI then the system defining $P(\Delta(G))$ is as well (see also [1]). In what follows we are going to use this together with Wagner's theorem to show that the system (1)–(3) is TDI for the class Γ. For this we just need to show that the system is TDI for the basic graphs of Γ. This is, in fact, easily seen to hold for the graphs $K_1, K_2, K_3, K_{3,3}$, the prism and the even wheels. For these graphs, the polytope $P(\Delta(G))$ is just defined by the trivial and triangle inequalities. The matrix of the system given by these inequalities, for each of these graphs, is totally unimodular. In what follows we show that the system (1)–(3) is TDI for the odd wheels W_{2k+1}, $k \geqslant 2$.

To this end, let us denote by P_w the linear programming problem

$$P_w = \begin{cases} \max & w^{\mathrm{T}}x, \\ \text{subject to} & (1),(2),(3). \end{cases}$$

By associating a dual variable y_C, δ_n, γ_e, with a constraint of type (1), (2), $x(e) \leqslant 1$, respectively, the dual of P_w, D_w can be written as follows:

$$D_w = \begin{cases} \min & \sum 2y_C + \sum (3k+1)\delta_n + \sum \gamma_e, \\ \text{subject to} & \sum_{C \ni e} y_C + \sum_{W_n \ni e} \delta_n + \gamma_e \geqslant w(e), \quad \text{for all } e \in E, \\ & y \geqslant 0,\ \delta \geqslant 0,\ \gamma \geqslant 0. \end{cases}$$

Theorem 2. *The system* (1)–(3) *is TDI for every* $2k+1$*-wheel*, $2 \leqslant k \in \mathbb{N}$.

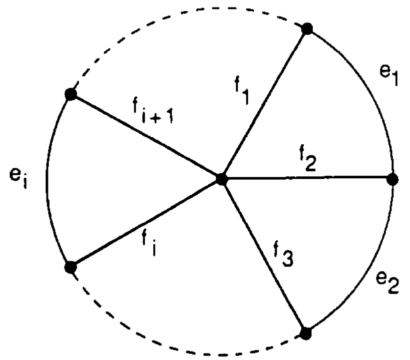

Fig. 3.

Proof. Let F be the edge set of a $2k + 1$-wheel, $k \geqslant 2$. Let $e_1, \dots, e_{2k+1}, f_1, \dots, f_{2k+1}$ be the edges of F such that e_i and e_{i+1} are incident to a common node and $\{e_i, f_i, f_{i+1}\}$ forms a triangle, for $i = 1, 2, \dots, 2k + 1$ (see Fig. 3, where the indices are taken modulo $2k + 1$).

Let $\lambda(w)$ denote the optimal value of the objective function of P_w (D_w). Since the system (1)–(3) is integral and, consequently, the problem P_w has always an integer optimal solution, it follows that $\lambda(w)$ is integer whenever w is integer valued. Now, to show the theorem, we shall use ideas similar to those of Barahona et al. [1] for acyclic spanning subgraphs.

We shall proceed by induction on w. Obviously, for $w \leqslant 0$, D_w has the trivial zero optimal solution. Now assume that D_w has an integer optimal solution for every integer vector w, $w \leqslant z$, $w \neq z$, and let us show that D_w has an integer optimal solution for $w = z$. For this we may assume that $w > 0$ (if $w(e) \leqslant 0$ for some edge $e \in F$, then the system (1), (2) is easily seen to be TDI for the graph obtained from W_{2k+1} by removing the edge e).

Now consider the set of inequalities among (1)–(3), that are satisfied with equality by every optimal solution of P_w. Let us denote this set by T_w.

Case 1: $x(e_0) \leqslant 1$ is in T_w for some edge e_0.

Let w' be the vector given by

$$w'(e) = \begin{cases} w(e) & \text{if } e \in F \setminus \{e\}, \\ w(e) - 1 & \text{if } e = e_0. \end{cases}$$

We claim that $\lambda(w') = \lambda(w) - 1$. Indeed, it is clear that $\lambda(w') \leqslant \lambda(w)$. If $\lambda(w') = \lambda(w)$, then e_0 cannot be in any optimal solution of $P_{w'}$, otherwise $\lambda(w)$ would not be maximum. Since every optimal solution for $P_{w'}$ is at the same time optimal for P_w, this contradicts the fact that $x(e_0) \leqslant 1$ is in T_w and our claim is proved. Now by the induction hypothesis, there is an integer optimal solution to $D_{w'}$. Consider the solution obtained from that solution by increasing by one the value of the dual variable associated with $x(e_0) \leqslant 1$. This solution is integer and optimal for D_w.

Case 2: $x(C) \leqslant 2$ is in T_w for some triangle C.

Let w' be the vector given by

$$w'(e) = \begin{cases} w(e) & \text{if } e \in F \backslash C, \\ w(e) - 1 & \text{if } e \in C. \end{cases}$$

We claim that $\lambda(w') = \lambda(w) - 2$. Clearly, $\lambda(w) - 2 \leqslant \lambda(w') \leqslant \lambda(w)$.

(i) If $\lambda(w') = \lambda(w)$, then C cannot intersect any optimal solution to $P_{w'}$. But since, in this case, every optimal solution for $P_{w'}$ is also optimal for P_w, we have a contradiction.

(ii) If $\lambda(w') = \lambda(w) - 1$, then we claim that every optimal solution for $P_{w'}$ contains exactly one edge of C. In fact, it is clear that such a solution cannot contain more than one edge of C. Now if, for instance, there is an optimal solution for $P_{w'}$, say F_1, which does not intersect C, then there must exist an edge, say f, such that $F_2 = F_1 \cup \{f\}$ is still triangle-free. In fact, it is easy to see that any maximal triangle-free edge subset of F intersects all the triangles of F. But then F_2 defines an optimal solution for P_w whose incidence vector does not satisfy $x(C) \leqslant 2$ with equality, a contradiction. Consequently, we obtain that $\lambda(w') = \lambda(w) - 2$ and our claim is proved.

Now consider the solution obtained from an integer optimal solution of $D_{w'}$ by increasing by one the value of the dual variable associated with $x(C) \leqslant 2$. We have that this solution is integer and optimal for D_w.

Case 3: $x(F) \leqslant 3k + 1$ is in T_w.

W may assume that no constraints of type (1) or type (2) are satisfied with equality by all the optimal solutions of P_w, otherwise we are either in Case 1 or in Case 2.

Claim. *$w(e) = w(f)$ for all $e, f \in F$.*

Proof of the claim. First remark that, since $w > 0$, any optimal solution for P_w is a maximal triangle-free subset of F and, by the remark above, intersects every triangle $\{e_i, f_i, f_{i+1}\}$, $i = 1, \ldots, 2k + 1$. Moreover, if e_i does not belong to an optimal solution of P_w, then it follows that f_i, f_{i+1} both belong to that solution. Thus from the assumption it follows that for every two edges f_i, f_{i+1}, $i = 1, 2, \ldots, 2k + 1$, there must exist an optimal solution for P_w, say F_i, which contains neither f_i nor f_{i+1}. We claim that $F_i = F \backslash \{f_i, f_{i+1}, f_{i+3}, \ldots, f_{i+2k-1}\}$. For this it suffices to show that F_i cannot contain two consecutive edges f_{i+p}, f_{i+p+1} with $p \in \{2, \ldots, 2k - 1\}$. Indeed, assume that this were not the case and, without loss of generality, that p is odd. Thus $e_{i+p} \notin F_i$. Moreover, the edge set $F \backslash F_i$ intersects each of the edge disjoint triangles $\{e_{i+t}, f_{i+t}, f_{i+t+1}\}$ for $t = 2, 4, \ldots, p - 1, p + 1, p + 3, \ldots, 2k$. Moreover, we have that these triangles all do not contain f_{i+1}. Thus we obtain that $|F \setminus F_i| \geqslant k + 2$, a contradiction.

Now let $\beta = w(F) - \lambda(w)$. By considering the edge sets $F \setminus F_i$ for $i = 1, \ldots, 2k + 1$, we find that the vector $(w(f_1), \ldots, w(f_{2k+1}))^{\mathrm{T}}$ satisfies the system

$$Ax = b,$$

where A is the $(2k + 1, 2k + 1)$-matrix whose rows are respectively the incidence vectors of the sets $F \setminus F_1, \ldots, F \setminus F_{2k+1}$, with respect to the edge set $F \setminus \{e_1, \ldots, e_{2k+1}\}$ and $b = (\beta, \ldots, \beta)^{\mathrm{T}}$. It is not hard to see that the matrix A is nonsingular, implying that the system above has the unique solution given by

$$w(f_i) = \beta/2k + 1 \quad \text{for } i = 1, \ldots, 2k + 1. \tag{4}$$

On the other hand, since there is no constraint of type $x(e) \leqslant 1$ satisfied with equality by every optimal solution of P_w, it follows that for every edge e_i where $i = 1, \ldots, 2k + 1$, there is an optimal solution for P_w, say F^i, which does not contain e_i. Thus $f_i, f_{i+1} \in F^i$. As before, we can show that F^i cannot contain two consecutive edges f_{i+p}, f_{i+p+1} with $p \in \{1, \ldots, 2k\}$, implying that $F^i = F \setminus \{e_i, f_{i+2}, f_{i+4}, \ldots, f_{i+2k}\}$. Since $w(F^i) = w(F_i)$, it follows from (4) that

$$w(e_i) = \beta/2k + 1 \quad \text{for } i = 1, \ldots, 2k + 1,$$

which finishes the proof of the claim.

Now define w' as $w'(e) = w(e) - 1$ for all $e \in F$. From the claim it follows that $\lambda(w') = \lambda(w) - (3k + 1)$. Consider the dual solution obtained from an integer optimal solution of $D_{w'}$ by increasing by one the value of the dual variable associated with $x(F) \leqslant 3k + 1$. This solution is integer and optimal for D_w, which completes the proof of our theorem. □

Thus we can state our main result.

Theorem 3. *The system* (1)–(3) *is TDI for* Γ.

In what follows we shall use Theorem 3 to derive a min–max relation for the K_3-covers in graphs noncontractible to $K_5 \setminus e$.

3. A min–max relation

Let $G = (V, E)$ be a graph noncontractible to $K_5 \setminus e$ and w be an integer weight vector associated with the edges of G. The K_3-cover problem in G is also equivalent to the following linear program

$$P'_w = \begin{cases} \min & w^{\mathrm{T}}x, \\ \text{subject to} & (1)', (2)', (3)', \end{cases}$$

where (1)′, (2)′, (3)′ are obtained respectively from (1), (2), (3) by replacing x by $\mathbf{1} - x$ ($\mathbf{1}$ is the vector whose entries all equal 1).

The dual of P'_w, D'_w is

$$D_w = \begin{cases} \max & \sum y_C + \sum (k+1)\delta_n - \sum \gamma_e, \\ \text{subject to} & \sum_{C \ni e} y_C + \sum_{W_n \ni e} \delta_n - \gamma_e \leqslant w(e), \quad \text{for all } e \in E, \\ & y \geqslant 0,\ \delta \geqslant 0,\ \gamma \geqslant 0. \end{cases} \tag{5}$$

Lemma 4. *If $w \geqslant 0$ and D'_w has an integer optimal solution then such a solution can be chosen so that $\gamma_e = 0$ for all $e \in E$.*

Proof. Let $(y^0, \delta^0, \gamma^0)$ be an integer optimal solution for D'_w. Suppose that for some edge $f \in E$, $\gamma_f^0 > 0$. We shall show that there exists a dual optimal solution to D'_w, say $(y^*, \delta^*, \gamma^*)$, such that $\gamma_f^* = 0$ and $\gamma_e^* = 0$ for all $e \in E$ such that $\gamma_e^0 = 0$. Since $\gamma_f^0 > 0$, it follows that the dual constraint (5) associated with the edge f is satisfied with equality by $(y^0, \delta^0, \gamma^0)$, otherwise one can decrease the value of γ_e by a positive amount and then get a solution whose value is greater than that of $(y^0, \delta^0, \gamma^0)$, contradicting the optimality of the latter one.

Moreover, we may assume that $y_C^0 = 0$ for every triangle C containing f. Indeed, if, for instance $y_{C^0}^0 > 0$ for some triangle C^0 such that $f \in C^0$ and, say, $y_{C^0}^0 \leqslant \gamma_f^0$ (which can be assumed without loss of generality), then one can consider the solution $(y^{0\prime}, \delta^{0\prime}, \gamma^{0\prime})$ such that $\delta^0 = \delta^{0\prime}$ and

$$y_C^{0\prime} = \begin{cases} y_C^0 & \text{if } C \neq C^0, \\ 0 & \text{if } C = C^0, \end{cases} \qquad \gamma_e^{0\prime} = \begin{cases} \gamma_e^0 & \text{if } e \neq f, \\ \gamma_e^0 - y_{C^0}^0 & \text{if } e = f, \end{cases}$$

which is integer and optimal for D'_w.

Consequently, we can suppose that

$$\sum_{W_n \ni f} \delta_n = \gamma_f + w(f).$$

Since $w \geqslant 0$, $\sum_{W_n \ni f} \delta_n \geqslant \gamma_f$. Now, let $n^1, \ldots, n^p$, $p \in \mathbb{N}$, be odd integers, such that $n^i = 2k^i + 1$, $k^i \in \mathbb{N}$, and $f \in W_{n^i}$, for $i = 1, \ldots, p$, and

$$\sum_{i=1,\ldots,p-1} \delta_{n^i} < \gamma_f \quad \text{and} \quad \sum_{i=1,\ldots,p-1} \delta_{n^i} \geqslant \gamma_f.$$

Let

$$\varepsilon = \gamma_f - \sum_{i=1,\ldots,p-1} \delta_{n^i}.$$

Observe that every wheel W_{n^i} contains k^i edge disjoint triangles, say $C_1^i\ C_2^i, \ldots, C_{k^i}^i$. Thus we may consider the solution $(y^*, \delta^*, \gamma^*)$ given by

$$y_C^* = \begin{cases} y_C^0 + \delta_{n^i} & \text{if } C = C_t^i,\ i = 1, \ldots, p-1;\ t = 1, \ldots, k^i, \\ y_C^0 + \varepsilon & \text{if } C = C_t^p,\ t = 1, \ldots, k^p, \\ 0 & \text{otherwise,} \end{cases}$$

$$\delta_n^* = \begin{cases} 0 & \text{if } n = n^1, \ldots, n^{p-1}, \\ \delta_n - \varepsilon & \text{if } n = n^p, \\ \delta_n & \text{otherwise,} \end{cases}$$

$$\gamma_e^* = \begin{cases} 0 & \text{if } e = f, \\ \gamma_e & \text{if } e \neq f. \end{cases}$$

It is easily seen that the solution $(y^*, \delta^*, \gamma^*)$ is feasible and optimal to D'_w. □

Since, by Theorem 3, the system (1)–(3) is TDI for Γ, it follows that the system given by the inequalities (1)', (2)', (3)' is also TDI for Γ. Consequently, the dual D'_w has an integer optimal solution for every integer weight system w and every graph in Γ. Now to establish our relation between the minimum cardinality of a K_3-cover of a graph G noncontractible to $K_5\backslash e$ and the maximum profit of a Δ-partition of G it remains to examine the graph W_3. The system (1)–(3) is not, unfortunately, TDI for that graph. In fact consider a W_3 and let $\{e_1, e_2, e_3, f_1, f_2, f_3\}$ be its edge set (we suppose that W_3 has the form which is shown in Fig. 3). Now let us associate with the edges of W_3 the weight system $w = (1,1,0,\ 0,1,0)$. Let C^1, C^2, C^3 be respectively the cycles $\{e_1, e_2, e_3\}$, $\{e_1, f_1, f_2\}$, $\{e_2, f_2, f_3\}$. It is easy to see that in this case, the problems P_w and D_w have unique optimal solutions which are given by $x(e_1) = x(e_2) = x(f_2) = 1$, $x(e_3) = x(f_1) = x(f_3) = 0$ and $y_{C^1} = y_{C^2} = y_{C^3} = \frac{1}{2}$, $y_C = 0$ otherwise, $\delta = 0$, $\gamma = 0$, respectively. However it is not hard to see that for W_3, the dual problem D'_1 has an integer optimal solution. Moreover, if we compose W_3 by means of 1- or 2-sum with a graph for which the system (1)–(3) is TDI, the dual problem D'_1 has for the resulting graph an integer optimal solution (see [5]). Consequently we have the following lemma.

Lemma 5. *The problem D'_1 has an integer optimal solution for every graph noncontractible to $K_5\backslash e$.*

Now by combining Theorem 3 and Lemmas 4, 5 we obtain the following min–max relation.

Theorem 6. *If G is a graph noncontractible to $K_5\backslash e$, then the minimum cardinality of a K_3-cover of G equals the maximum profit of a Δ-partition of G.*

Let $G = (V, E)$ be a graph. A K_3-packing of G is a collection of pairwise edge disjoint triangles of G. If $\mathscr{T}$ is a collection of triangles of G, the *K_3-cover number* $c_3(\mathscr{T})$ (resp. *K_3-packing number* $p_3(\mathscr{T})$ of $\mathscr{T}$ is the smallest number of edges (K_2's) that cover all the triangles of $\mathscr{T}$ (resp. the largest number of triangles of $\mathscr{T}$ which are pairwise edge disjoint). A graph G is called *K_3-perfect* [3] if $c_3(\mathscr{T}) = p_3(\mathscr{T})$ for any collection $\mathscr{T}$ of triangles of G. (Conforti et al. [3] introduced a more general concept: The K_i-perfect graphs, where i is a fixed positive integer. They called a graph K_i-perfect if for any collection $\mathscr{K}$ of K_i's, $c_i(\mathscr{K}) = p_i(\mathscr{K})$ where $c_i(\mathscr{K})$ and $p_i(\mathscr{K})$ are defined analogously. They gave a characterization for K_i-perfect graphs which is similar to the strong perfect graph conjecture and studied the relationship between K_i-perfect graphs and perfect graphs and perfect and balanced matrices.)

Now let us introduce the class Ω of graphs G such that G is obtained by means of 1- and 2-sums from the graphs $K_1, K_2, K_3, K_{3,3}$, the prism and the even wheels (i.e. G is a graph noncontractible to $K_5 \backslash e$ which does not contain an odd wheel). From Theorem 6 we have that for every graph G of Ω, the minimum number of edges that cover all the triangles of G equals the maximum number of pairwise edge disjoint triangles. Moreover we have the following corollary.

Corollary 7. *Every graph in Ω is K_3-perfect.*

A matrix A is called *perfect* [7] if the associated set packing polytope $\{x \mid Ax \leqslant \mathbf{1}, x \geqslant 0\}$ has all its extreme points in 0–1. Given a graph G, let $T(G)$ denote the transpose of the triangles incidence matrix of G. Conforti et al. [3] showed that a graph is K_3-perfect if and only if the matrix $T(G)$ is perfect. From this and Corollary 7 we obtain the following corollary.

Corollary 8. *For a graph G of Ω, the matrix $T(G)$ is perfect.*

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