Local gradient estimates and existence of blow-up solutions to a class of quasilinear elliptic equations

Emilio Bello Castillo and René Letelier Albornoz

Departamento de Matemática, Universidad de Concepción, Casilla 160-C, Concepción, Chile

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1. Introduction

In this paper we study the existence of minimal and maximal positive blow-up solutions, that is

$$\lim_{x \in \Omega \atop \text{dist}(x, \partial\Omega) \to 0} u(x) = +\infty,$$

to the quasilinear elliptic equation

$$-\Delta u + H(x, u, \nabla u) = f \quad \text{in} \quad \Omega,$$

where $\Omega$ is a smooth bounded domain, $f \in L^\infty_{\text{loc}}(\Omega)$, and $H$ is an appropriate function that will indicate below.

Semilinear elliptic problems with boundary blow-up conditions of the form

$$-\Delta u + f(u) = 0 \quad \text{in} \quad \Omega,$$

$$\lim_{\text{dist}(x, \partial\Omega) \to 0} u(x) = \infty,$$

have a long history, starting with the results given by Bieberbach [4] in 1916. He considered the function $f(u) = e^u$ and proved that if $\Omega \subset \mathbb{R}^2$ is open regular bounded then there exists a unique $u \in C^2(\Omega)$ which satisfies (2) and the following property: $|u(x) - \sqrt{2\ln(\text{dist}(x, \partial\Omega)^{-2})}|$ is bounded in $\Omega$ as $\text{dist}(x, \partial\Omega) \to 0$. Next, Rademacher [28] in 1943 extended this result to smooth bounded domains in $\mathbb{R}^3$. In both cases, problem (2) has relevant applications: When $\Omega \subset \mathbb{R}^2$, in the theory of automorphic functions and in

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* Corresponding author.

E-mail addresses: ebello@udec.cl (E. Bello Castillo), rletelie@gauss.cfm.udec.cl (R. Letelier Albornoz).

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the theory of Riemannian surfaces of constant negative curvature and when \( \Omega \subset \mathbb{R}^3 \), this equation arises in the study of the electrostatic potential in a glowing hollow metal body.

The existence of positive solutions to (2), in \( \mathbb{N} \) dimensions, was first studied by Keller [14] (see also [15]), and Osserman [26] in 1957. They proved that if \( f \) is locally Lipschitz, nondecreasing on \([0, \infty)\), and \( f(0) \geq 0 \), then the following condition on the growth of \( f \) at infinity is a sufficient condition to guarantee the existence of positive solutions:

\[
\int_0^{\infty} F^{-1/2} < \infty, \quad \text{where } f = F'.
\]

This condition includes the case where \( f(u) = e^u \), which corresponds to results in [4, 28]. Later, Pohozaev [27] motivated by an application to the study of subsonic motion of gasses, in 1960, observed the existence of a positive solution to (2), when \( f(u) = u^2 \) and \( \Omega \) is an open regular bounded of \( \mathbb{R}^N \), \( N \geq 1 \). Uniqueness of solutions to (2) was given in 1974 by Loewner and Nirenberg [23], when \( f(u) = u^{(N+2)/(N-2)} \), \( N \geq 3 \). Next, growth rates and uniqueness of behaviour on the boundary to (2) were obtained simultaneously by Díaz and Letelier [7] and by Kondrat’ev and Nikishkin [16] when \( f(u) = u^m \), \( m > 1 \), and \( \Omega \) is an open bounded regular subset of \( \mathbb{R}^N \), \( N \geq 1 \). After that, Bandle and Giarrusso [1] and Díaz et al. [9] independently obtained existence, uniqueness, and rate explosion on the boundary for the operator

\[
H(x, u, \nabla u) = v|\nabla u|^k + \lambda u^m, \quad m > 1, \ 1 < k < 2, \ v > 0, \ \text{and} \ \lambda > 0.
\]

Related with Eq. (1), under the assumption \( f \in W^{1,\infty}_{\text{loc}}(\Omega) \) Letelier and Ortega [20] obtained \( L^\infty_{\text{loc}} \)-local gradient estimates and existence of solutions to (1) when \( H \) satisfies hypothesis (H1)–(H3) below. Roughly speaking, the restriction on \( f \) considered in [20] seems excessive, because under this assumption we would have \( \Delta u \in W^{1,\infty}_{\text{loc}}(\Omega) \) that gives \( u \in W^{1,\infty}_{\text{loc}}(\Omega) \), while for a partial differential equation (PDE) of second order it is reasonable that the solutions belong to some \( W^{2,s}(\Omega) \), \( s \geq 1 \). Gradient bounds and existence was obtained in Lasry and Lions [18] for the operator

\[
H(x, u, \nabla u) = |\nabla u|^k + \lambda u, \quad k > 1 \text{ and } \lambda > 0.
\]

Local gradient estimates also are obtained, for example, by Gilbarg and Trudinger [12], Boccardo et al. [5], Lasry and Lions [18], Lions [21, 22], Letelier and Ortega [20], and Serre [29]. However, the authors do not know local gradient estimates results for PDE with the generality of Eq. (1).

This way, the main difficulty to obtain blow-up solution to (1) is the weak regularity of the source \( f \in L^\infty_{\text{loc}}(\Omega) \). In this paper, we will contribute in two directions: First, in Section 2, to obtain appropriate estimates for the gradient in (1), which is the main contribution. In this sense, our paper is a generalization of [20]. In concrete, under the assumption \( f \in L^\infty_{\text{loc}}(\Omega) \) and assumptions (H1)–(H3), we obtain \( L^s \)-local gradient estimates for any solution to (1), \( 1 < s < \infty \). We note that assumptions (H1)–(H3) are more general that those considered in [18]. Second, in Section 3, we apply our estimates to prove existence of blow-up solutions to (1). In fact, we construct a minimal blow-up solution to (1) as limit of a family of problems with finite value on the boundary. Finally, we note also that this implies the existence of a maximal explosive solution to (1).
Recent contributions to the development of the elliptic PDE theory whose structures is compatible with blow-up behaviour on the boundary arise, for example, in [2,6,10,11,13,17,19,24,25] and references therein.

2. A local estimate of the gradient

Let us consider the following assumptions on the datum:

1. \( \Omega \) is a bounded open set of \( \mathbb{R}^N \), \( N \geq 1 \).
2. \( f \in L^\infty_{\text{loc}}(\Omega) \).
3. \( H : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^+ \) is a function which satisfies the following structural conditions:
   
   (H1) The function \( x \mapsto H(x, r, q) \) belongs to \( L^\infty_{\text{loc}}(\Omega) \) for all \( (r, q) \) and the function \( (r, q) \mapsto H(x, r, q) \) is continuous for all \( x \in \Omega \);
   
   (H2) \( |H(x, r, q) - H(x, r, q')| \leq \rho_H(|q - q'|) \) a.e. \( x \in \Omega \) for all \( r \) and for \( |q - q'| \) small, where \( \rho_H : \mathbb{R}^+ \to \mathbb{R}^+ \) is an increasing continuous function such that \( \rho_H(0^+) = 0 \);
   
   (H3) \( H(x, r, q) - H(x, s, q) \geq \beta(r - s) \) if \( r \geq s \) a.e. \( x \in \Omega \), where \( \beta : \mathbb{R}^+ \to \mathbb{R}^+ \) is an increasing continuous function such that \( \beta(0^+) = 0 \).

Example 2.1. A particular relevant choice of \( H \) is

\[
-\Delta u + v|\nabla u|^k + \lambda u^p = f \quad \text{in } \Omega,
\]

where \( \Omega \) is a bounded open set of \( \mathbb{R}^N \), \( N \geq 1 \), \( p > 1 \), \( 1 < k < 2 \), \( v > 0 \), \( \lambda > 0 \), and \( f \in L^\infty_{\text{loc}}(\Omega) \).

Definition 2.2. Let \( s > 1 \). We say that \( u \in W^{2,s}_{\text{loc}}(\Omega) \) is a strong solution to (1) if

\[
-\Delta u(x) + H(x, u(x), \nabla u(x)) = f(x) \quad \text{a.e. } x \in \Omega.
\]

For simplicity, by solution we will mean strong solution.

The proof of the following theorem uses the well-known Bernstein’s method (1910) (see, for example, [3,29]).

Theorem 2.1. Let \( u \in W^{2,s}_{\text{loc}}(\Omega) \), \( 1 < s < +\infty \), be a solution to (1) with

\[
f \in L^\infty_{\text{loc}}(\Omega).
\]

Let \( H : \Omega \times \mathbb{R}^+ \times \mathbb{R}^N \to \mathbb{R}^+ \) be a differentiable function satisfying

\[
H_r(x, r, q) \geq 0, \quad q \in \mathbb{R}^N.
\]

Suppose that there exists \( l \in \mathbb{R}^+ \cup \{\infty\} \) such that

\[
\lim_{\|\xi\|_{\mathbb{R}^N} \to +\infty} \frac{\int_{\Omega} H^2(x, \xi, \nabla \xi)}{\int_{\Omega} \left( \frac{2H}{\lambda q} (x, \xi, \nabla \xi) \right) \left( 1 + \|\nabla \xi\|^2 \right)} = l \in \mathbb{R}^+ \cup \{\infty\}.
\]
and suppose that there exists \( m > \max\{1, 2N/l\} \) (\( m > 1 \) if \( l = \infty \)), such that

\[
\lim_{|\nabla \xi|_{L^s(\Omega')}} \frac{\int_{\Omega'} |\nabla \xi|^2m \left( \left\{ \frac{1}{m} \frac{H^2(x, \xi, \nabla \xi)}{m} - \frac{1}{l} |\nabla \xi|^2 \right\} + 8N |\nabla \xi| \frac{\partial H}{\partial q} \right) \right) dx}{\int_{\Omega'} |\nabla \xi|^2m \left( |\nabla \xi|^2 \frac{\partial H}{\partial q} + 1 \right) dx} = +\infty
\]

for any \( \Omega' \Subset \Omega \) and \( \xi \in W^{1,s}(\Omega') \), \( s < +\infty \). Then for every \( \Omega' \Subset \Omega \) we have that

\[
\|\nabla u\|_{L^s(\Omega')} \leq C_0,
\]

where \( C_0 \) is a positive constant which depends only on the upper estimates of \( u, f \) in \( \Omega' \), and \( N \).

**Proof.** Let \( \Omega' \Subset \Omega \) and \( w(x) = |\nabla u(x)|^2 \), \( x \in \Omega \). We consider \( \varphi \in C_0^\infty (\Omega) \) verifying

\[
\varphi \leq 1 \text{ in } \Omega \quad \text{and} \quad \varphi \equiv 1 \text{ in } \Omega',
\]

\[
|\Delta \varphi| \leq C' \varphi^\theta \quad \text{and} \quad |\nabla \varphi|^2 \leq C' \varphi^{1+\theta} \text{ in } \Omega
\]

for some \( \theta \in [0, 1] \), where the constant \( C' = C'(\Omega, \Omega') \) depends only on \( \Omega \) and \( \Omega' \).

Applying the operator \( \partial/\partial x_k \) to both sides of \( (1) \) and multiplying the resulting PDE by \( 2\varphi(\partial u/\partial x_k) \), after some hard calculus we obtain

\[
-\Delta(\varphi w) + 2\varphi |D^2 u|^2 + \frac{2}{\varphi} \nabla \varphi \nabla (\varphi w) + 2\varphi \nabla u \frac{\partial H}{\partial x} + 2\varphi w H_r + \nabla (\varphi w) \frac{\partial H}{\partial q} = -w \Delta \varphi + 2\varphi \nabla u \nabla f + \frac{2}{\varphi} w |\nabla \varphi|^2 + w \nabla \varphi \frac{\partial H}{\partial q}.
\]

As we are searching estimates in \( L^s_{\text{loc}}(\Omega) \), we multiply \( (9) \), in sense of distributions, by \( (\varphi w)^m \) and we integrate on \( \Omega \) to obtain

\[
-\int_\Omega (\varphi w)^m \Delta(\varphi w) + 2\int_\Omega (\varphi w)^m |D^2 u|^2 + 2\int_\Omega (\varphi w)^m \nabla \varphi \nabla (\varphi w) + 2\int_\Omega (\varphi w)^m \nabla u \frac{\partial H}{\partial x} + 2\int_\Omega (\varphi w)^m H_r + \int_\Omega (\varphi w)^m \nabla (\varphi w) \frac{\partial H}{\partial q} = -\int_\Omega (\varphi w)^m w \Delta \varphi + 2\int_\Omega (\varphi w)^m \nabla u \nabla f + \int_\Omega (\varphi w)^m \nabla \varphi \frac{\partial H}{\partial q}.
\]

Since we have

(i) \( -\int_\Omega (\varphi w)^m \Delta(\varphi w) = m \int_\Omega (\varphi w)^{m-1} |\nabla (\varphi w)|^2 \); \n
(ii) \( 2\int_\Omega (\varphi w)^m \nabla \varphi \nabla (\varphi w) \leq \frac{C'}{m+1} \int_\Omega (\varphi w)^{m+\theta} w^{m+1} \);
\( \int_{\Omega} \varphi^m w^{m+1} |\nabla \varphi|^2 \leq C' \int_{\Omega} \varphi^{m+\theta} w^{m+1}; \)

(iv) \( 2 \int_{\Omega} \varphi^{m-1} w^{m+1} |\nabla \varphi|^2 \leq 2C' \int_{\Omega} \varphi^{m+\theta} w^{m+1}; \)

(v) \( \int_{\Omega} (\varphi w)^m \nabla (\varphi w) \frac{\partial H}{\partial q} \geq -\frac{m}{2} \int_{\Omega} (\varphi w)^{m-1} |\nabla (\varphi w)|^2 - \frac{1}{2m} \int_{\Omega} (\varphi w)^{m+1} |\frac{\partial H}{\partial q}|^2; \)

(vi) \( 2 \int_{\Omega} \varphi^{m+1} w^m \nabla u \nabla f \leq \int_{\Omega} \varphi^{m+1} w^m |D^2 u|^2 + \frac{m}{2} \int_{\Omega} (\varphi w)^{m-1} |\nabla (\varphi w)|^2 + C_f^2 (1 + 4m) \int_{\Omega} \varphi^{m+1} w^m + 2C_f \int_{\Omega} \varphi^{m+(1+\theta)/2} w^{m+1/2}, \)

\[ \text{where } C_f = \|f\|_{L^\infty(\Omega')}; \]

(vii) \( |D^2 u|^2 \geq \frac{1}{N} (\Delta u)^2 \geq \frac{1}{4N} [H(x, u, \nabla u) - f]^2 \geq \frac{1}{4N} H^2 - \frac{1}{2N} |f|^2; \)

using hypothesis (5) and (i)–(vi) in (10), we obtain

\[ \int_{\Omega} \varphi^{m+1} w^m |D^2 u|^2 - \frac{1}{2m} \int_{\Omega} (\varphi w)^{m+1} |\frac{\partial H}{\partial q}|^2 \]

\[ + 2 \int_{\Omega} \varphi^{m+1} w^m \left[ \nabla u \frac{\partial H}{\partial x} - \int_{\Omega} \varphi^{m+1} w^m \nabla \varphi \frac{\partial H}{\partial q} \right] \]

\[ \leq 4C' \int_{\Omega} \varphi^{m+\theta} w^{m+1} + 2C_f C^{1/2} \int_{\Omega} \varphi^{m+(1+\theta)/2} w^{m+1/2} + K_f \int_{\Omega} \varphi^{m+1} w^m, \quad (11) \]

where \( K_f = (4C_f^2 + 2C_f). \)

Using (vii) in (11) and after (4), we obtain

\[ \int_{\Omega} \varphi^{m+1} \left\{ \frac{1}{4N} w^m H^2 - \frac{1}{2m} w^{m+1} |\frac{\partial H}{\partial q}|^2 \right\} \]

\[ + 2 \int_{\Omega} \varphi^{m+1} w^m \nabla u \frac{\partial H}{\partial x} - \int_{\Omega} \varphi^{m+(1+\theta)/2} w^{m+1} \left| \frac{\partial H}{\partial q} \right| \]

\[ \leq K_0 \int_{\Omega} \varphi^{m+\theta} w^{m+1} + K_0 \int_{\Omega} \varphi^{m+(1+\theta)/2} w^{m+1/2} + K_0 \int_{\Omega} \varphi^{m+1} w^m, \quad (12) \]

where \( K_0 = \max\{4C', 2C_f C^{1/2}, K_f + C_f^2 / (2N)\}. \)
Finally, for $\xi \in W^{1,s}(\Omega')$, $s < +\infty$, we define
\[
\mathcal{F}(\xi, \nabla \xi) = \int_{\Omega'} |\nabla \xi|^{2m} \left( \left( \frac{1}{2m} H(x, \xi, \nabla \xi)^2 - \frac{1}{m} |\nabla \xi|^2 \left| \frac{\partial H}{\partial q} \right|^2 \right) + 8N \nabla \xi \frac{\partial H}{\partial x} \right)
- \int_{\Omega'} |\nabla \xi|^{2m} \left( |\nabla \xi|^2 \left| \frac{\partial H}{\partial q} \right| + 1 \right). \tag{13}
\]
Since we have (12) and $\{ \xi = u\psi: u$ is a solution of (1) $\} \subset W^{1,s}(\Omega')$, $s < +\infty$, starting from (13) and hypothesis (6) and (7), we obtain the existence of a constant $C_0 > 0$ such that
\[
\| \nabla (u) \|_{L^s(\Omega')} \leq C_0,
\]
where $s < +\infty$ and $C_0$ depends only on the upper estimates of $u, f$ in $\Omega'$, and $N$. \(\square\)

**Remark 2.3.** For Eq. (3), hypothesis (6)–(8) are satisfied with $l = 1/k^2$, $m > 2Nk^2$, and $\theta \in [2/(m+k), 1]$, respectively. In this case inequality (12) is written as
\[
v^2 \left[ \frac{1}{4N} - \frac{k^2}{2m} \right] \| \psi^{m+1} w \|_{L^{m+k}(\Omega)}^m - K \| \psi^{m+1} w \|_{L^{m+1}(\Omega)}^{m+k} + 4C' \| \psi^{m+1} w \|_{L^{m+1}(\Omega)}^{m+k} w \|_{L^{m+k}(\Omega)}^{m+k}
- \left( K + C'_f + 2C' \right) \| \psi^{m+1} w \|_{L^m(\Omega)}^{m+k} \leq 0,
\]
where $K = K(\Omega', \| f \|_\infty, N) > 0$. We note that
\[
m + k > m + \frac{k+1}{2} > m + 1 > m + \frac{1}{2} > m > 1.
\]
**Remark 2.4.** Hypotheses (6) and (7) are close to those used in [21,22,29].

**Remark 2.5.** (1) In terms of the function $f$, our result generalizes that obtained by Letelier and Ortega in [20] where it is considered that $f \in W^{1,\infty}_\text{loc}(\Omega)$.

(2) In terms of the structure to (1), our result generalizes that obtained by Lasry and Lions in [18], who consider $H(x, r, q) = |q|^k + \lambda r$, $k > 1$ and $\lambda > 0$.

### 3. Existence of solutions

**Remark 3.1.** Assuming hypothesis (H1)–(H3), Díaz and Letelier in [8] showed the following results:

(i) Let $u_1, u_2 \in W^{2,\infty}_\text{loc}(\Omega)$ such that
\[
-\Delta u_1 + H(x, u_1, \nabla u_1) \leq -\Delta u_2 + H(x, u_2, \nabla u_2) \quad \text{a.e. in } \Omega
\]
and

\[ \limsup_{\text{dist}(x, \partial \Omega) \to 0} \frac{u_1(x)}{u_2(x)} \leq 1. \]

Then

\[ u_1(x) \leq u_2(x), \quad x \in \Omega. \]

(ii) Moreover, if there exists \( \gamma : \mathbb{R}^+ \to \mathbb{R}^+ \) an increasing continuous function with \( \gamma(0^+) = 0 \) satisfying the following conditions:

\[ \int_{r}^{\infty} \frac{ds}{(\Gamma_{\gamma}(s))^{1/2}} < +\infty, \quad (14) \]

where \( \Gamma_{\gamma}(r) = \int_{0}^{r} \gamma(s) ds \) satisfies the inequalities

\[ \rho_H(r) \leq C \gamma(r)^{-1} \left\{ (2\nu)^{-1} \left( \frac{\beta_{\nu}}{2} \right)^2 r \right\}, \quad r \geq 0, \quad (15) \]

\[ \lambda \beta(r) \geq (\nu + C) \gamma(r - s), \quad r \geq s \geq 0, \quad (16) \]

for some positive constants \( C \) and \( \nu \), then for every solution under bounded \( u \) to (1), we have

\[ u(x) \leq C_{f, \Omega'}, \quad \forall \Omega' \subseteq \Omega, \quad (17) \]

where \( C_{f, \Omega'} \) is a constant depending only on \( \Omega' \) and on the \( L^\infty(\Omega') \)-norm of \( f \).

Consequently, hypothesis (4)–(7), (14)–(16) show the existence of a constant \( C_1 > 0 \) which depends only on the structural datum of (1), \( \Omega' \subseteq \Omega \), and on the \( L^\infty(\Omega') \)-norm of \( f \), such that

\[ \|u\|_{W^{1, s}_{\text{loc}}(\Omega)} \leq C_1, \quad \forall s < +\infty. \]

Step 1. Construction of the function \( u \). Assuming that the gradient growth to (1) is at most quadratic, the sub and supersolution methods and the local estimates (18) allow to demonstrate the following theorem with regular data over the boundary (see, for example, [5]).

**Theorem 3.1.** Let \( \Omega \) be a regular bounded domain in \( \mathbb{R}^N \), suppose also that \( H \) verifies (H1)–(H3),

\[ H(x, 0, 0) = 0, \quad \forall x \in \Omega, \quad (19) \]

and

\[ H(x, r, q) \leq \eta(r) \left( 1 + |q|^2 \right) \]

for some nondecreasing function \( \eta : \mathbb{R}^+ \to \mathbb{R}^+ \). If \( f \in L^\infty(\Omega) \), \( f \geq 0 \), then there is a unique solution \( u_R \in C^2(\Omega) \) to problem
\[ \begin{aligned} -\Delta u_R + H(x, u_R, \nabla u_R) &= f_R \quad \text{in } \Omega, \\
 u_R &= R \quad \text{on } \partial \Omega, \end{aligned} \] where \( f_R(x) = \min\{f(x), R\} \).

**Definition 3.2.** Let \( x \in \Omega \). We define \( u(x) \) by
\[
 u(x) = \lim_{R \to \infty} u_R(x) = \sup_{R > 0} u_R(x) < +\infty. \tag{22} \]

**Remark 3.3.** The monotonicity of the sequence \( \{u_R\}_R \) and (17) allows us to conclude that \( u : \Omega \to \mathbb{R}^+ \) is a well defined function in the open set \( \Omega \).

**Proof.** We will prove that the function \( u \) defined by (22) is a blow-up solution to (1).

Take any \( \Omega' \subset \subset \Omega \). From (18), (22), and the dominated convergence theorem, we obtain
\[
 u \in W^{1,s}(\Omega') \quad \text{and} \quad u_R \rightharpoonup u \quad \text{in } W^{1,s}(\Omega'), \quad R \to \infty, \quad 1 < s < +\infty. \tag{23} \]

From (18) and (20) we have that \( \Delta u_R = H(x, u_R, \nabla u_R) - f_R \) is bounded in \( L^s(\Omega') \) independently of \( R \), and therefore
\[
 \|u_R\|_{W^{2,s}(\Omega'')} \leq C \left[ \|u_R\|_{L^s(\Omega')} + \|\Delta u_R\|_{L^s(\Omega')} \right] \leq M, \quad 1 < s < +\infty, \tag{24} \]
where \( \Omega'' \subset \subset \Omega' \) and \( M \) is a constant independent of \( R \) (see [12, Theorem 9.11]). Therefore, from (23) and (24) we have
\[
 u \in W^{2,s}(\Omega''), \quad 1 < s < +\infty, \\
u_R \rightharpoonup u \quad \text{in } W^{2,s}(\Omega''), \quad R \to +\infty, \quad 1 < s < +\infty. \tag{25} \]

On the other hand, since \( f_R \rightharpoonup f \) in \( L^s(\Omega') \), as \( R \to +\infty, 1 < s < +\infty \), from (23), (25), and the continuity of \( H \), we obtain
\[
 -\Delta u + H(x, u, \nabla u) = f \quad \text{in } L^s(\Omega'), \quad 1 < s < +\infty, 
\]
and therefore \( u \) is solution of (1). It remains to prove only that
\[
 \lim_{\text{dist}(x, \partial \Omega) \to 0} \frac{u(x)}{x \in \Omega} = +\infty. \]

This follows from the continuity of \( u \) in \( \Omega \), the monotonicity of the \( \{u_R\}_R \), and the fact that \( u_R = R \) on \( \partial \Omega \). \( \square \)

**Theorem 3.3.** The function constructed in (22) is the minimal blow-up solution to (1).

That is, if \( v \) is a solution of (1) such that
\[
 \lim_{\text{dist}(x, \partial \Omega) \to 0} v(x) = +\infty, \]

\[
 \begin{aligned} -\Delta u_R + H(x, u_R, \nabla u_R) &= f_R \quad \text{in } \Omega, \\
 u_R &= R \quad \text{on } \partial \Omega, \end{aligned} \tag{21} \]
then
\[ u(x) \leq v(x) \quad \text{in } \Omega. \]

**Proof.** If \( v \) is a solution of (1), then by comparison,
\[ u_R(x) \leq v(x) \quad \text{a.e. } x \text{ in } \Omega, \]
and therefore
\[ \sup \{ u_R(x): R > 0 \} = u(x) \leq v(x) \quad \text{a.e. } x \text{ in } \Omega. \]

\[ \blacksquare \]

**Corollary 3.4.** There exists a solution \( U \) such that \( u(x) < U(x) \), \( \forall x \in \Omega \), for all solution \( u \) of (1). That is, \( U \) is a maximal solution to (1).

**Proof.** Let \( \Omega_n \subset \Omega \) be a sequence of smooth domains that converges to \( \Omega \) and \( \Omega_n \subset \Omega_{n+1} \). If \( u_n \) is a minimal blow-up solution to (1) in \( \Omega_n \), then \( \{ u_n \} \) is monotone decreasing and converges uniformly on compact subsets of \( \Omega \) to a function \( U \). It is easy to prove that \( U \) is a maximal solution of (1).

\[ \blacksquare \]

**References**


