Settings for a study of finite rank Butler groups

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Abstract

A finite rank Butler group $G$ is a torsionfree Abelian groups that is the sum of $m$ rank one subgroups; $G$ is a $B(n)$-group if $n$ is the maximum number of independent relations between the $m$ subgroups. After the well-known class $B(0)$, the much studied $B(1)$ and the first approaches to $B(2)$, in this paper we generalize some of the tools used before and introduce new ones to work in every $B(n)$. We study some of the relationships between these tools, and while clarifying some basic settings describe an interesting class of indecomposables.

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Introduction

In this paper, group = torsionfree Abelian group of finite rank.

Once rank 1 groups (i.e. additive subgroups of $\mathbb{Q}$) are known, the next step is the study of their finite sums: they constitute the class of Butler groups, so named because M.C.R. Butler, in his 1967 paper [Bu], gave two equivalent characterizations, which promised rich developments, and generated a vast literature. Structure theorems—though—were scarce: besides the classical completely decomposable groups (direct sums of rank 1 groups, classified by Baer in 1937 [Ba]) the other well-researched subclass is the class of $B(1)$-groups (see [A,AV,FM] for history, and numerous papers by many authors, among which the ones underwriting this paper).

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A Butler group is defined by its representation: $G$ is Butler if it coincides with the sum of a finite number of its rank 1 subgroups, i.e. if it can be represented as the quotient $X/K$ of a completely decomposable group $X$ over a pure subgroup $K$. Completely decomposable groups (called $B(0)$-groups) are those for which $K = 0$; $B(n)$-groups (also denoted $B^{(n)}$ or $B(n)$) are those for which rank $K = n$. The class $B(1)$ turned out to be much more complex than initially expected: this is due to the intricate relations it hosts between linear, order-theoretic and combinatoric structures, prompting the introduction of new tools to deal with them and of new ways of representing its members.

Since $X' \geq K$ and $X$ isomorphic to $X'$ only insures $X/K$ quasi-isomorphic to $X'/K$ [FM, Lemma 1.3], we choose to use, as a basic equivalence, quasi-isomorphism ([FI]: isomorphism up to finite index) instead of isomorphism; in fact, we will write "isomorphic, indecomposable, direct summand, ..." instead of "quasi-isomorphic, strongly indecomposable, quasi-direct summand, ...".

In this paper we aim to set the ground for a study of the structure of general $B(n)$-groups; the latest approaches to $B(2)$-groups (e.g. [DVM 8,DVM 9,DVM 10,VWW]) convinced us that the best way to proceed would be to pursue both paths: the detailed study of $B(2)$, and at the same time the honing of the old tools and the introduction of new ones, that would be better suited to the task if approached from the more general setting. Starting as we did from an acquaintance with $B(1)$-groups, the surprises were continuous: e.g., where the stress had been on the numerator $X$, here the denominator $K$ comes powerfully into play, dictating in its own way a lot of the structure (Section 5); the combinatorial aspects show a geometric side (Section 4); the representation of types as products of primes goes even deeper, but looses the power of determining the group (Section 2).

Our approach preserves the viewpoint it had with $B(1)$-groups, strictly dependent on the actual representation of $G = \langle g_1 \rangle_* + \cdots + \langle g_m \rangle_*$. Here $*$ indicates pure closure; the $m$-tuple $(g_1, \ldots, g_m)$ is a base of $G$, which is a $B(n)$-group if $n$ is the maximum number of independent relations involving the base elements; the type of an element $g$ is the isomorphism class of the pure subgroup $\langle g \rangle_*$, and the types $t_1, \ldots, t_m$ of the base elements constitute a type-base of $G$. A fundamental role is played by the set of indices $I = \{1, \ldots, m\}$ and its families of subsets: in Section 1 we introduce a relation on $\mathcal{P}(\mathcal{P}(I))$ that extends the ordering of partitions of $I$. We sketch here the main tools introduced in this paper; only the first interactions between them are investigated, but they seem to offer a rich and promising starting point.

- The tent (Sections 1, 2.C). We extend to $\mathcal{P}(\mathcal{P}(I))$ the definition of tent, a function that was basically responsible for the structure of $B(1)$-groups. Here is how it goes now:

1. The completely decomposable numerator $X$ of $G$ (suitably chosen following ($\ast$)), or equivalently the $m$-tuple of its base types, determines the tent, a function $t : \mathcal{P}(\mathcal{P}(I)) \to \mathbb{T}$ (the set of all types).
2. The denominator $K$ attaches to each $g \in G$ a set $\text{maxfam}(g)$ of subsets of $I$, thus determining a subset $\text{Maxfam}(G)$ of $\mathcal{P}(\mathcal{P}(I))$.
3. The image $t(\text{maxfam}(g))$ is the type of $g$, and $t(\text{Maxfam}(G)) = \text{typeset}(G)$, the set of types of all pure rank 1 subgroups of $G$.

In other words, the numerator $X$ determines the tent $t$, while the denominator $K$ determines the restriction of the domain that allows the tent to operate on $G$. (Let us just hint that $\text{maxfam}(g)$ yields all maximal solvable subsystems of a linear system, whose homogeneous part depends on $K$, while the constant terms are given by $g$.)
– The Primes (Section 2.D). The typeset of a Butler group is a finite lattice, which can be represented as a sub-$\wedge$-semilattice of $(\mathbb{N}, \text{l.c.m., g.c.d.})$, thus viewing each type as a squarefree product of primes. While primes simply mark the sup-irreducible elements of the lattice, once $X$ is given—hence the base types are chosen—the primes become columns in a table with base types as rows; thus each prime $p$ acquires a support in $I$ (crucial, e.g., in deciding if $p$ divides the type of a given element). This table represents the tent, and provides an invaluable tool for computations and examples. The question of which $m$-tuple of types—hence, which $m$-tuple of products of primes—can be the type-base of a $B(n)$-group, was called ‘regularity’ in $B(1)$, and sometimes shunned as insubstantial. It acquires weight when one recognizes that its answer is not general (as in $B(1)$) but depends on $K$. It turns out that every $K (\neq 0)$ prevents certain primes from occurring (if primes were not in the picture, this would spell: “prevents the lattice from having certain sup-irreducible elements,” and would probably look intractable). If we build the widest table of primes allowed by a given $K$, the $X$ determined by the rows (the types) of the table completes a $B(n)$-group $X/K$ that is in a certain sense minimal among those with denominator $K$ (Section 5). Theorem 2.14 classifies the tents (i.e. the type structure) of $B(n)$-groups with a given denominator (i.e. linear structure) $K$.

– The basic partition (Section 2.B). The linear part of the game is played in $V$, a $\mathbb{Q}$-vector space of dimension $m$, containing $X$ and $K$. A maximal independent set of vectors of $K$ (i.e. of relations of $G$) produces an $n \times m$ matrix of rank $n$. It is useful to consider the partition $\mathcal{A} = \{A_1, \ldots, A_k\}$ of $I$ obtained by declaring two indices equivalent if the corresponding columns of the matrix are equal. Deleting repeated columns we obtain a narrower $n \times k$ matrix $M$ which—together with its transpose $M^T$—is responsible for the linear behavior of $G$. The basic partition can then add considerably to the complication; the most tractable nontrivial cases occur when $\mathcal{A} = \text{min}(\text{singleton blocks})$.

– The Configuration (Section 4). The matrix $M^T$, interpreted as a $k$-ple of points in the projective space $PS(n-1)$, determines a configuration consisting of the subspaces generated by the points (the interesting cases start at $n = 3$). Much of the effect of $K$ on $G$ can be drawn from this configuration, but unfortunately not all: see the beautiful groups in Example 6.b, where still another geometric interpretation comes to help.

A main difficulty we encountered in treating this subject has been the insufficiency of alphabets; wanting to maintain a certain consistency in symbols, without falling into nightmarish complications, we have adopted a balance point which chooses perception over perfection: for instance, we often prove properties for the base element $g_1$ without bothering to write the ensuing general result for $g_i$, which would need more positions and more notation (see e.g. Section 2.E). We also chose to stay at an elementary level, both because the subject “sums of rank 1 groups” is elementary, and because only after the structures have shown more of their intricacies and interactions it will become clear which abstract notions are really useful and appropriate.

The amount of open questions is, understandably, staggering. The main ones are, of course: determining classes of indecomposables, and conditions for decomposability. But the definition of $G$, dependent on its representation, provides another set of problems: that of isomorphism across representations, i.e. the problem of base changes, both inside a class $B(n)$ or across classes (see Example 6.b). At the end of each section we have evidenced those problems that we felt we should have answered before closing it; but we have learned that—particularly in this subject—the difference between trivial and hard is difficult to gauge in advance; Section 7 summarizes other problems. As usual in our work, we have not stinted on examples: many inside the sections, and Section 6 devoted to some crucial ones. We hope that our efforts on (relative) ease...
of presentation and richness of examples will tempt old and new readers towards this field: the time is ripe for Butler groups to hit the water again.

1. Notation and first results

Lower case Greek letters (with the exception of \(\sigma, \tau, \phi, \psi\)) will denote rational numbers. We will keep notation and tools introduced in our previous papers (in particular [DVM 4, DVM 10]) on \(B(1)\)- and \(B(2)\)-groups. \(\mathbb{Q}\) is the field of rationals; for the \(i\)th prime \(p_i\), \(\mathbb{Q}_{p_i} = \{r/p_i^s | r \in \mathbb{Z}, s \in \mathbb{N}\}\) is a rank one group of type \((0, 0, \ldots, 0, \infty, 0, 0, \ldots)\) with \(\infty\) at the \(i\)th place.

As usual, \(\mathbb{T}(\wedge, \vee)\) denotes the lattice of all types, with the added maximum, denoted by the symbol \(\infty\), for the (improper) type of the 0 group. If \(w\) is an element of a group \(W\), the type \(t_W(w)\) of \(w\) in \(W\) denotes the isomorphism type of the pure subgroup \(\langle w \rangle^*\), and is called a type of \(W\); \(\text{typeset}(W) = \{t_W(w) | w \in W\}\) ensures [A, Lemma 3.1.3] that it is a sub-\(\wedge\)-semilattice of \(\mathbb{T}\), hence (having \(\infty\) as a maximum) a lattice.

If \(H\) is a subset of \(W\) we set \(t_W(H) = \bigvee\{t_W(w) | w \in H\}\); we will use this notation in particular when \(H\) is a coset.

Throughout, \(I = \{1, \ldots, m\}\); if \(E \subseteq I\), set

\[E^{-1} = I \setminus E.\]

For the group \(W = \langle w_1 \rangle^* + \cdots + \langle w_m \rangle^*\), and \(E \subseteq I\), let

\[w_E = \sum\{w_i | i \in E\}.\]

Given the element \(w = \gamma_1 w_{C_1} + \cdots + \gamma_h w_{C_h}\) of \(W\), when \(\gamma_i \neq \gamma_j\) if \(i \neq j\), \(C = \{C_1, \ldots, C_h\}\) is called a partition of \(I\) into equal-coefficient blocks for \(w\), or shortly a partition of \(w\), with respect to the elements \(w_1, \ldots, w_m\); when these elements are fixed, we set \(C = \text{part}_W(w)\).

The lattice \(\mathbb{P}(I)\) of partitions of the set \(I\) is ordered by “bigger = coarser”; we will now try to extend the ordering of \(\mathbb{P}(I)\) to \(\mathbb{P}(\mathbb{P}(I))\).

**Definition 1.1.** For \(\mathcal{E}, \mathcal{F} \in \mathbb{P}(\mathbb{P}(I))\), set \(\mathcal{E} \leq \mathcal{F}\) (\(\mathcal{E}\) is finer than \(\mathcal{F}\), \(\mathcal{F}\) is coarser than \(\mathcal{E}\)) if for each \(E \in \mathcal{E}\) there is an \(F \in \mathcal{F}\) such that \(E \subseteq F\). Moreover, define \(\mathcal{E} \wedge \mathcal{F} = \{E \cap F | E \in \mathcal{E}, F \in \mathcal{F}\}\).

For a family \(\mathcal{S}\) of \(I\), that is a set \(\mathcal{S} \in \mathbb{P}(\mathbb{P}(I))\), let

\[\mathcal{S}^\downarrow = \{L | L \subseteq S \text{ for some } S \in \mathcal{S}\}\]

be the lower set of \(\mathcal{S}\), also called the down closure of \(\mathcal{S}\); if \(\mathcal{S} = \mathcal{S}^\downarrow\), \(\mathcal{S}\) is called a lower family of \(I\), or down closed (i.e. closed with respect to \(\subseteq\)). Denote by

\[\max(\mathcal{S})\]

the set of maximal elements of \(\mathcal{S}\) (maximal with respect to \(\subseteq\) in \(I\)); \(\mathcal{S}\) is a maxfam of \(I\) if \(\mathcal{S} = \max(\mathcal{S})\), that is if the elements of \(\mathcal{S}\) are pairwise incomparable. In particular, partitions of \(I\) are maxfams. We have

\[\max(\mathcal{S}) = \max(\mathcal{S}^\downarrow); \quad \mathcal{S}^\downarrow = \max(\mathcal{S})^\downarrow.\]
Finally, set $\text{Downfam}(I) = \{ S \in \mathcal{P}(\mathcal{P}(I)) \mid S = S^\downarrow \}$, the set of families of $I$ that are down closed; $\text{Maxfam}(I) = \{ S \in \mathcal{P}(\mathcal{P}(I)) \mid S = \max(S) \}$, the set of maxfams of $I$.

Note that on $\mathcal{P}(\mathcal{P}(I))$, $\leq$ is reflexive, transitive, but not antisymmetric (e.g., $\{1\}, \{1, 2\} \leq \{2\}, \{1, 2\}$; rather than quotienting $\mathcal{P}(\mathcal{P}(I))$, observe that $\leq$ induces a partial order on both $\text{Downfam}(I)$ and $\text{Maxfam}(I)$; in fact, the maps

$$\max : \text{Downfam}(I) \to \text{Maxfam}(I), \quad S = S^\downarrow \to \max(S),$$

$$\downarrow : \text{Maxfam}(I) \to \text{Downfam}(I), \quad S = \max(S) \to S^\downarrow$$

are order bijections. For $S, S' \in \text{Maxfam}(I)$, $S \land S'$ need not be a maxfam: e.g. $\{1, 2\}, \{1, 3, 4\} \land \{1, 3\}, \{1, 2, 4\} = \{1\}, \{1, 2\}, \{1, 3\}, \{1, 4\} \not\in \text{Maxfam}(I)$. But we can use the above map to define an infimum in $\text{Maxfam}(I)$:

$$S \inf S' = \max(S \land S')$$

which turns it into an inf-semilattice isomorphic to $\text{Downfam}(I)$. The coming Corollary 1.5 will show that the two infima will not cause undue trouble. We have

\textbf{Lemma 1.2.} With the above definition of $\land$, $\text{Downfam}(I)$ is an $\land$-semilattice; in the lattice $\mathcal{P}(I) \leq \text{Maxfam}(I)$ of partitions of $I$ we have $\inf = \land$, and $\mathcal{P}(I)$ is a sub-$\land$-semilattice of $\text{Maxfam}(I)$.

In the study of $B(1)$-groups, an essential role was played by a map $t : \mathcal{P}(I) \to \mathbb{T}$ called tent (see e.g. [DVM 4]). We will extend it via the following

\textbf{Definition 1.3.} If $(t_1, \ldots, t_m)$ is a fixed $m$-tuple of types and $E \subseteq I$, set

$$\tau(E) = \bigwedge \{ t_i \mid i \in E \},$$

in particular, $\tau(\emptyset) = \infty$; if $\mathcal{E}$ is a family of $I$ define

$$t(\mathcal{E}) = \bigvee \{ \tau(E^{-1}) \mid E \in \mathcal{E} \}.$$  

the ensuing map $t : \mathcal{P}(\mathcal{P}(I)) \to \mathbb{T}$ is called tent, and $(t_1, \ldots, t_m)$ is its base.

This definition extends our old definition of the tent of a $B(1)$-group, which is the restriction of $t$ to $\mathcal{P}(I)$.

From $t(\{E\}) = \tau(I \setminus E)$ we have $t_i = \tau(\{i\}) = t(\{i\}^{-1})$, thus the base is determined by the tent. In general $t(\{i\}) = \tau(\{i\}^{-1}) \not\leq t_i$, thus $t_i \not\leq t(\{i\}, \{i\}^{-1})$. In previous papers we called $(t_1, \ldots, t_m)$ “regular” when $t_i = t(\{i\}, \{i\}^{-1})$ for all $i \in I$; here this definition will apply to $B(1)$-groups, as explained at the beginning of Section 3.

\textsuperscript{1} This is not to be confused with the previous definition of $t_W(H)$ if $H$ is a subset of a group $W$.  

Lemma 1.4. In the above setting, if $E, F \in \mathcal{P}(\mathcal{P}(I))$, we have

$$t(E) \land t(F) = t(E \land F).$$

Proof. Let $E \in \mathcal{E}, F \in \mathcal{F}$. We have $\tau(E^{-1}) \land \tau(F^{-1}) = t(E^{-1} \cup F^{-1}) = \tau((E \cap F)^{-1})$; therefore $t(E) \land t(F) = \sqrt{\{\tau((E \cap F)^{-1}) | E \in \mathcal{E}, F \in \mathcal{F}\}} = t(E \land F)$. □

Note that, if $t' : \mathcal{P}(\mathcal{P}(I)) \to \mathbb{T}$ is a map satisfying (♯), and we define $t_i = t'([i])$, for each $E \subseteq I$ we get $t'(E) = \tau(E^{-1}) = t(E)$ (hence $t'$ and $t$ share the same $\tau$); and $t'(E) \geq t(E)$ for each family $E$ of $I$. A tent is thus the lowest among the maps $\mathcal{P}(\mathcal{P}(I)) \to \mathbb{T}$ satisfying (♯).

Corollary 1.5. The restriction of $t$ to $\text{Downfam}(I)$ is a morphism of $\land$-semilattices; the restriction

$$t : (\text{Maxfam}(I), \inf) \to (\mathbb{T}, \land),$$

$$\mathcal{E} \to t(\mathcal{E})$$

is a morphism of $\inf \land$-semilattices. In fact, $t(E) = t(E^\land) = t(\text{max}(\mathcal{E}))$.

2. The basic setting

We settle here all the details of our representation of a $B(n)$-group $G$. Set $G = \langle g_1 \rangle * + \cdots + \langle g_m \rangle *$; then $(g_1, \ldots, g_m)$ is a base of $G$; for $i \in I$, the types $t_i = t_G(g_i)$ are base types of $G$; $(t_1, \ldots, t_m)$ is a type-base of $G$. $G$ has a nontrivial type-base change if it has another type-base (deriving from another base) that is not obtained from the first by a permutation of the base types.

Let $V = \mathbb{Q}x_1 \oplus \cdots \oplus \mathbb{Q}x_m$. We view $G = X/K_X$ as the quotient of a completely decomposable group

$$X = R_1x_1 \oplus \cdots \oplus R_mx_m \leq V$$

(where, for all $i \in I$, $R_i$ is a subgroup of $\mathbb{Q}$ containing $\mathbb{Z}$ of type $t_i = t_G(g_i)$) over a pure rank $n$ subgroup $K_X = \langle a_1, \ldots, a_n \rangle _* $ of $X$; this is then a representation of $G$ as a $B(n)$-group. Here $g_i = x_i + K_X$ and

$$t_X(x_i) = t(R_i) = t_G(g_i) = t_i. \quad (*)$$

This apparently harmless condition on the base types $t_X(x_i)$ of $X$ will have significant consequences; see Sections 2.G and 2.F, Section 3, and the definition of $K$-total group.

Giving the numerator $X$ of $G$ is equivalent to giving the type-base $(t_1, \ldots, t_m)$, which in turn is equivalent to giving the tent $t$ on that base; this $t$ is called the tent of $G$.

Define creel\(^2\) of (the given representation of) $G$ the subspace $K = K_X \otimes \mathbb{Q}$ of $V$ generated by $\{a_1, \ldots, a_n\}$; then dim $K = n$, and $K_X = K \cap X$.

\(^2\) Creel: a basket of freshly fished Primes, see Section 2.F.
For \( r = 1, \ldots, n \), set \( a_r = \beta_{r,1}x_1 + \cdots + \beta_{r,m}x_m \); then \( \{ \beta_{r,1}g_1 + \cdots + \beta_{r,m}g_m = 0 \mid r = 1, \ldots, n \} \) is a maximal independent set of relations in \( G \); we will also call relations of \( G \) the elements of its creel \( K \), and basic relations the chosen \( a_r \). The coefficients of the basic relations form an \( n \times m \) matrix \( B = (\beta_{r,i}) \).

If the matrix \( B \) is equivalent to a block-diagonal matrix the (decomposable) group \( G \) is called degenerate [DVM 10]. In this definition we include the case where \( K \) misses some base element, which is analyzed next.

### 2.A. The diagonal relation

We show once and for all that without loss of generality we may suppose one of the basic relations of \( G \) to be \( g_1 + \cdots + g_m = 0 \) (the diagonal relation).

If for each \( i \in I \) at least one of the \( \beta_{r,i} \) is nonzero, then a suitable linear combination of the basic relations will yield a relation \( a'_1 = \gamma_{1,1}g_1 + \cdots + \gamma_{1,m}g_m = 0 \) with coefficients \( \gamma_{1,i} \) that are all nonzero; this relation can be completed with \( n - 1 \) suitable relations to get a new set of independent relations for \( K_X \). At this point it is enough to replace each \( x_i \) (respectively \( g_i \)) with \( \gamma_{1,i}x_i \) (respectively \( \gamma_{1,i}g_i \)), and each \( R_i \) with \( \gamma_{1,i}^{-1}R_i \), to have the diagonal relation as the first relation.

If instead for a (minimal) \( I' \subset I \) we have \( K_X \leq \bigoplus \{ R_ix_i \mid i \in I' \} \) (that is, \( K_X \) misses the base elements \( x_i \) for \( i \in I \setminus I' \)), then \( G \cong X/K_X \cong ((\bigoplus \{ R_ix_i \mid i \in I' \})/K_X) \oplus (\bigoplus \{ R_ix_i \mid i \in I \setminus I' \}) \) becomes a direct sum of a \( B(n) \)-group of smaller rank with a completely decomposable group. In this case, set

\[
X' = X \oplus \left( \bigoplus \{ R_iz_i \mid i \in I \setminus I' \} \right),
\]

\[
K' = K \oplus \left( \bigoplus \{ Q(x_i + z_i) \mid i \in I \setminus I' \} \right), \quad \text{hence}
\]

\[
K'_{X'} = K_X + \left( \bigoplus \{ R_i(x_i + z_i) \mid i \in I \setminus I' \} \right)
\]

\[
\leq X' = \left( \bigoplus \{ R_ix_i \mid i \in I' \} \right) \oplus (\bigoplus \{ R_ix_i + R_iz_i \mid i \in I \setminus I' \})
\]

so \( K' \) does not miss any base element of \( X' \). Then \( X'/K'_{X'} = (\bigoplus \{ R_ix_i + R_iz_i \}/R_i(x_i + z_i)) \cong X/K_X \cong G \), since the second summand is \( \cong \bigoplus \{ R_ix_i \mid i \in I \setminus I' \} \). Now we can apply to \( X'/K'_{X'} \) the initial procedure, thus representing \( G \) as a \( B(n + |I \setminus I'|) \)-group with the diagonal relation among its relations. E.g., if \( I' = I \setminus \{ m \} \), the initial matrix

\[
B = \begin{bmatrix}
* & \cdots & * & 0 \\
* & \cdots & * & 0 \\
\vdots & \vdots & \vdots & \vdots \\
* & \cdots & * & 0
\end{bmatrix}
\]

will yield

\[
B' = \begin{bmatrix}
* & \cdots & * & 0 & 0 \\
* & \cdots & * & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
* & \cdots & * & 0 & 0 \\
0 & 0 & 0 & 1 & 1
\end{bmatrix},
\]

thus
\[ B'' = \begin{bmatrix} 1 & \cdots & 1 & 1 \\ * & \cdots & * & 0 & 0 \\ \vdots & \cdots & \vdots & \ddots & \vdots \\ * & \cdots & * & \cdots & \cdots & 0 & 0 \\ 0 & \cdots & 0 & 1 & 1 \end{bmatrix}. \]

Observe now that if one of the relations is of the form \( \alpha g_i = 0 \), \( G \) will be transformed into a \( B(n-1) \)-group by: performing a base change of \( K \), to make that relation a basic relation; and then eliminating that relation, and \( g_i \). This can be done also if one of the relations is of the form \( \alpha g_i + \beta g_{i'} = 0 \) for \( i, i' \in I \) and \( \alpha, \beta \neq 0 \) (in which case \( t_i = t_{i'} \)), by making it a basic relation, then eliminating it and \( g_{i'} \), and replacing \( g_i \) by \( h_i = (1 - \alpha/\beta)g_i (= g_i + g_{i'}) \) (see Example 6.d).

Therefore, from now on, unless otherly stated,

1. If \( n > 0 \), the first relation will always be the diagonal relation; and
2. There will be no relation of the form \( \alpha g_i = 0 \), or \( \alpha g_i + \beta g_{i'} = 0 \), for \( i, i' \in I, i \neq i' \), and \( \alpha, \beta \neq 0 \).

2.B. The basic partition \( A \)

If \( G \) is a \( B(0) \)- or a \( B(1) \)-group, and \( g = \gamma_1 g_{C_1} + \cdots + \gamma_h g_{C_h} \in G \), with \( \gamma_i \neq \gamma_j \) if \( i \neq j \), the partition \( C = \{C_1, \ldots, C_h\} \) of \( g \) into equal-coefficient-blocks with respect to the base of \( G \) is unique, and is called \( \text{part}_G(g) \). In general, the basic relations \( a_r \in X \) have each their \( \text{part}_X(a_r) \).

Define basic partition of (our representation of) \( G \) the partition

\[ A = \{A_1, \ldots, A_k\} = \bigwedge \{\text{part}_X(a_r) \mid r = 1, \ldots, n \}; \]

its blocks \( A_j \) are called sections; \( |A| = k \); set \( J = \{1, \ldots, k\} \). Without loss of generality we will assume throughout \( 1 \in A_1 \).

**Observation 1.** \( A \) does not depend on the chosen basic relations: it is in fact the minimum partition of elements of \( K \) (it is not difficult to show that it is the partition of some relation; in fact, we might choose all basic relations \( a_r \) except the first so that \( \text{part}_X(a_r) = A \)). Clearly, \( n \leq k \leq m \).

We will write our basic relations as follows:

\[ (g_1 =) g_1 + \cdots + g_m = 0 \quad \text{(the diagonal relation)}, \]
\[ \alpha_{2,1} g_{A_1} + \cdots + \alpha_{2,k} g_{A_k} = 0, \]
\[ \alpha_{3,1} g_{A_1} + \cdots + \alpha_{3,k} g_{A_k} = 0, \]
\[ \vdots \]
\[ \alpha_{n,1} g_{A_1} + \cdots + \alpha_{n,k} g_{A_k} = 0 \]

and

\[ K_X = \langle a_1, a_2, \ldots, a_n \rangle, \quad \text{with} \]
\[ a_r = \alpha_{r,1} x_{A_1} + \cdots + \alpha_{r,k} x_{A_k} \quad \text{for } r = 1, \ldots, n \text{ and } \alpha_{1,j} = 1 \text{ for all } j \in J. \]
Note that:

- if \( j, j' \in J \) and \( j \neq j' \) there is at least one basic relation \( a_r \) in which \( \alpha_{r,j} \neq \alpha_{r,j'} \);
- if \( 0 = \beta_1 g_{C_1} + \cdots + \beta_h g_{C_h} \) with \( \beta_i \neq \beta_j \) if \( i \neq j \) then \( C = \{C_1, \ldots, C_h\} \geq \mathcal{A} \); in other words, \( K_X \leq X(\mathcal{A}) = \langle x_{A_1} \rangle_* \oplus \cdots \oplus \langle x_{A_k} \rangle_* \leq X \).

We set \( V(\mathcal{A}) = \mathbb{Q} x_{A_1} \oplus \cdots \oplus \mathbb{Q} x_{A_k} \leq V \).

Clearly, the basic partition—trivial for \( B(1) \)-groups—plays an especially important role in \( B(2) \)-groups; but it cannot replace the role of \( K \), as seen in Example 6.b.

Let us settle an extreme case.

**Proposition 2.1.** If \( n = k \), \( G \) degenerates into a direct sum of \( n \) \( B(1) \)-groups.

**Proof.** \( K_X \) is a pure rank \( n \) subgroup of \( X(\mathcal{A}) = \langle x_{A_1} \rangle_* \oplus \cdots \oplus \langle x_{A_k} \rangle_* \), hence, if \( n = k \), \( K_X = X(\mathcal{A}) \); then \( G \cong X_{A_1}/\langle x_{A_1} \rangle_* \oplus \cdots \oplus X_{A_n}/\langle x_{A_n} \rangle_* \). □

In the other extreme case \( k = m \), \( \mathcal{A} \) is the minimum partition \( \text{min} = \{\{1\}, \ldots, \{m\}\} \) of \( I \).

From now on we set \( n < k \leq m \).

The coefficients of the basic relations form an \( n \times k \) matrix of rank \( n \),

\[
M = \begin{bmatrix}
1 & \ldots & 1 & \ldots & 1 \\
\alpha_{2,1} & \ldots & \alpha_{2,n} & \ldots & \alpha_{2,k} \\
\vdots & \ldots & \vdots & \ldots & \vdots \\
\alpha_{n,1} & \ldots & \alpha_{n,n} & \ldots & \alpha_{n,k}
\end{bmatrix},
\]

called a basic matrix of \( G \) (or of \( K \)).

Note that \( M \) (obtained from \( B \) by eliminating repeated columns) is equivalent to a block-diagonal matrix if and only if so is \( B \), hence it implies \( G \) degenerate.

2.C. Types of elements

We have \( G \cong X/K_X \cong (X + K)/K \); for \( g \in G \),

\[
t_G(g) = t_{(X+K)/K}(x + K) = \sqrt{\{t_{(X+K)}(x + y) \mid y \in K\}}
\]

\[
= \sqrt{\{t_{(X+K)}s(x + y) \mid y \in K, s \in \mathbb{Z}\{0\}\}}
\]

\[
= \sqrt{\{t_{(X+K)}s(x + y) \mid sy \in K_X, s \in \mathbb{Z}\{0\}\}}
\]

\[
\geq \sqrt{\{t_{X}s(x + y) \mid sy \in K_X \}} = t_{X/K_X}(x + K_X) = t_G(g);
\]

thus, to make our life easier, in computing \( t_G(g) \) we will use \( K \) instead of \( K_X \).

For \( x = \gamma_1 x_1 + \cdots + \gamma_m x_m \in X \) set

\[
\text{supp}_X(x) = \{i \in I \mid \gamma_i \neq 0\} \quad \text{(the support of } x)\text{),}
\]

\[
\text{Z}_X(x) = \{i \in I \mid \gamma_i = 0\} \quad \text{(the zero-block of } x)\text{).}
\]

Then we have
\[ t_X(x) = \bigwedge \{ t_i \mid i \in \text{supp}(x) \} = \tau(\text{supp}(x)) = \tau((Z(x))^{-1}); \]

whence the rule: the bigger the zero-block, the bigger the type.

Let \( g = x + K \in G \); then
\[
t_G(g) = \bigvee \{ t(x+y) \mid y \in K \} = \bigvee \{ \tau((Z(x+y))^{-1}) \mid y \in K \}, \]
hence
\[ (** \) the types of \( G \) are suprema of infima of base types. \]

To compute the type of \( g = x + K \) in \( G \) we must thus get hold of the zero-blocks of all its representatives; we will call them zero-blocks of \( g \). (In particular, all blocks of \( \text{part}_X(x) \)—that is all equal-coefficient blocks of \( x \)—will become zero-blocks of \( g \), by adding to \( x \) in turn suitable multiples of \( x_i \); if \( G \) is \( B(1) \), these are the only zero-blocks of \( g \).)

**Definition 2.2.** For \( g \in G \), define \( \text{fam}_G(g) = \{ Z(x) \mid x + K = g \} \), the set of zero-blocks of \( g \); \( \text{maxfam}_G(g) \) the set of maximal elements of \( \text{fam}_G(g) \); \( \text{Maxfam}(G) = \{ \text{maxfam}_G(g) \mid g \in G \} \).

Note that \( \text{maxfam}_G(g) = \text{maxfam}_V(g) \) depends only on \( K \); the same holds for \( \text{Maxfam}(G) \).

**Corollary 2.3.** If \( g \in G \),

(i) \( t_G(g) = t(\text{fam}_G(g)) = t(\text{fam}_G(g)^\downarrow) = t(\text{maxfam}_G(g)) = t(\text{maxfam}_G(g)^\downarrow); \)

(ii) \( \text{typeset}(G) = t(\text{Maxfam}(G)). \)

If \( G \) is \( B(1) \), \( \text{fam}_G(g) = \text{maxfam}_G(g) = \text{part}_G(g) \) for all \( g \in G \), hence \( \text{Maxfam}(G) = \mathbb{P}(I) \): in this case the typeset of \( G \) is determined by the tent. Instead, Example 6.c shows that for \( n > 1 \) the typeset of a \( B(n) \)-group is in general not determined by its tent: \( \text{Maxfam}(G) \) depends on the relations of \( G \), that is on its creel \( K \). Example 6.e will show that for \( n \geq 3 \) \( \text{Maxfam}(G) \) is not in general a sub-inf-semilattice of \( \text{Maxfam}(I) \).

2D. Primes

In studying \( B(1) \)-groups a powerful tool has been the representation of \( \text{typeset}(G) \) as a sub-\( \wedge \)-semilattice of the square-free natural numbers with (l.c.m., g.c.d.): hence, of types as products of Primes. [The capital \( P \) is a precaution due to the fact that, while in \( B(1) \)-groups the only relation was the diagonal one, with no coefficients to take into account, here the relations have coefficients (which are divisible by “real” primes, written with lower case \( p \)); in examples, though, Primes = primes.] We summarize here the construction: associate a Prime \( p_0 \) to the minimum type of \( G \); then, to each type \( \sigma \) of length 1 associate the product \( p_0p \) of \( p_0 \) with a new Prime; if a type \( \sigma \) of length 2 has two types \( p_0p, p_0p' \) under it, associate to it their l.c.m. \( p_0pp' \); while if \( \sigma \) covers only one type—hence is \( \lor \)-irreducible—associate to it the product of this type times a new Prime; analogously for types of length 3, etc. Proceeding thus by finite induction we represent each type of \( \text{typeset}(G) \) as a product of Primes, with the infimum \( \wedge \) (in \( \mathbb{T} \) and in \( \text{typeset}(G) \)) as g.c.d. and the supremum \( \lor \) (in \( \mathbb{T} \)) as l.c.m. If we deal with more than one group the procedure to introduce Primes needs to be slightly modified to accommodate both groups (see [DVM 8]). Note that the supremum in \( \text{typeset}(G) \) is in general \( \geq \lor \). Associating to each Prime \( p \) the smallest
type $\sigma(p)$ in which it occurs creates a bijection of the Primes we used onto the $\lor$-irreducible types of typeset($G$); only the Prime $p_0$ is a type of $G$.

We define $G(p)$ to be the set of elements of $G$ whose type is divisible by $p$, hence $G(p) = G(\sigma(p))$ (= the set of elements of $G$ with type $\geq \sigma(p)$).

**Observation 2.** Adopting the above representation for the typeset of a Butler group, we are in fact studying a class of $2^\aleph_0$ Butler groups, obtained by attributing suitable “real” types to its types, via an $\land$-monomorphism of the “abstract” typeset into $\mathbb{T}$; we call the image a realization of the “abstract” group. Such a class contains $2^\aleph_0$ pairwise non-isomorphic groups, obtainable for instance by varying the attribution of “real” primes to the Primes $p, q, \ldots$ and then writing the types with 0s and $\infty$s and a tail of zeros, e.g.

$$\sigma = \cdots p q = (0, 0, \infty, 0, \ldots, \text{zeros}, \ldots)$$

(see the example below).

If the type $\sigma$ is a product of Primes among which there is $p$, we say $p$ divides $\sigma$ ($p \mid \sigma$), or $p$ is a Prime of $\sigma$, or $\sigma$ has the Prime $p$; if $p$ divides $t_G(g)$, we say $p$ is a Prime of $g$, or $p$ divides $g$.

All the above—in particular, Primes—depends on the typeset, but not on the chosen type-base, hence not on the tent. We revert in the following to our fixed base. Since by (**) all types in typeset($G$) are suprema of $\tau$’s, $\lor$-irreducible types are of the form $\tau(C)$ for some $C \subseteq I$; if $\sigma(p)$ is a $\lor$-irreducible type associated to the Prime $p$, from the construction we have that the maximum such $C$ is $C = \text{supp}(p) = \{i \in I \mid p \text{ divides } t_i\}$, the support of $p$. In particular, different Primes have different supports. Primes and base-types form a finite table, with base-types as rows and Primes as columns: as we did for $B(1)$-groups, we will call tent also this table (which is as well the tent of the overlying $B(1)$-group $X/(x_I)_*$), where for $n \geq 2$ we will mark the sections $A_j$ of $\mathcal{A}$; $\text{supp}(p)$ is then also the support of the column of $p$. $\sigma(p) = \tau(\text{supp}(p))$ can then be computed from the table, as in the following example for $\mathcal{A} = \{A_1, A_2, A_3\} = \{\{1\}, \{2, 3\}, \{4, 5\}\}$ (on the right, the realization for $q = 2, p_0 = 3, p = 5, p' = 7, p'' = 11$):

<table>
<thead>
<tr>
<th>$A_1$</th>
<th>$t_1$</th>
<th>$\cdots$</th>
<th>$q$</th>
<th>$p_0$</th>
<th>$= (0, 0, 0, \infty, \infty, \ldots, \text{zeros}, \ldots)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_2$</td>
<td>$t_2$</td>
<td>$\cdots$</td>
<td>$q$</td>
<td>$p_0$</td>
<td>$= (0, 0, 0, 0, \infty, \ldots, \text{zeros}, \ldots)$</td>
</tr>
<tr>
<td>$t_3$</td>
<td>$p$</td>
<td>$\cdots$</td>
<td>$p_0$</td>
<td>$= (\infty, 0, 0, 0, \infty, \ldots, \text{zeros}, \ldots)$</td>
<td></td>
</tr>
<tr>
<td>$A_3$</td>
<td>$t_4$</td>
<td>$p'$</td>
<td>$q$</td>
<td>$p_0$</td>
<td>$= (0, 0, 0, \infty, \infty, \ldots, \text{zeros}, \ldots)$</td>
</tr>
<tr>
<td>$t_5$</td>
<td>$p''$</td>
<td>$q$</td>
<td>$p_0$</td>
<td>$= (0, 0, 0, \infty, \infty, \ldots, \text{zeros}, \ldots)$</td>
<td></td>
</tr>
</tbody>
</table>

Here $\text{supp}(q) = \{1, 4, 5\}$, so the $\lor$-irreducible type $\sigma(q)$ associated to the prime $q$ is $\sigma(q) = \tau(\{1, 4, 5\}) = t_1 \land t_4 \land t_5 = q p_0$, a minimal type; $\text{supp}(p'') = \{5\}$, so the $\lor$-irreducible type $\sigma(p'') = \tau(\{5\}) = t_5 = p_0 q p''$: the prime $p''$ occurs for the first time in a type of length 2. We stress the fact that, although the tent depends on the type-base, $\lor$-irreducible types (which are independent of the type-base) can be read from the tent: *all Butler groups with a given tent have, whichever the creel $K$, the same $\lor$-irreducible types.*

Note, though, that such a table is not always the tent of a $B(n)$-group: see the next lemma, and Section 3.
Observe, by direct inspection on the tent, that \( E = \text{supp}(p) \) for a Prime \( p \) of \( G \) if and only if \( p \mid \tau(E) \) while \( p \) does not divide \( \tau(F) \) for any \( E \subset F \subset I \); for instance, if \( n \geq 1 \) and \( i \in I \), the set \( I \setminus \{i\} \) is not the support of a Prime, since the diagonal relation implies \( \tau(I \setminus \{i\}) = \tau(I) \).

Let \( G_E = \langle g_i \mid i \in E \rangle \). We have

**Proposition 2.4.**

(i) \( E \subset I \) is the support of a Prime \( p \) of \( G \) if and only if \( (E \subset F \text{ and } \tau(E) = \tau(F)) \) implies \( E = F \). A necessary condition is then that \( (E \subset F \text{ and } G_E = G_F) \) implies \( E = F \).

(ii) If \( E = \text{supp}(p) \), \( G(p) = G_E \).

**Proof.** (i) For the necessary condition we only need to note that \( \tau(E) \) is the minimum type of \( G_E \).

(ii) Recall that, if \( E = \text{supp}(p) \), \( G(p) = G(\tau(E)) = \{ g \in G \mid t_G(g) \geq \tau(E) \} \). The proof of (i) shows that \( G_E \subset G(\tau(E)) \). To show the other inclusion, note that \( g \in G(\tau(E)) \) if and only if \( \tau(E) \leq t_G(g) = \sqrt{\{ \tau(Z^{-1}) \mid Z \in \text{fam}_G(g) \}} \). Then, \( \tau(E) = \sqrt{\{ \tau(Z^{-1}) \mid Z \in \text{fam}_G(g) \}} = \sqrt{\{ \tau(Z^{-1} \cup E) \mid Z \in \text{fam}_G(g) \}} \). But \( \tau(E) \) is \( \lor \)-irreducible, hence for some zero-block \( Z \) of \( g \) we have \( \tau(E) = \tau(Z^{-1} \cup E) \). Then, by (i), \( E = Z^{-1} \cup E \), hence \( Z^{-1} \subset E \), thus \( g \) has a representative whose support \( Z^{-1} \) is contained in \( E \), as wanted. \( \square \)

In the following, when the type-base (hence the table) is fixed, we will also view Primes as columns of the tent, and—identifying them with their support—as subsets of \( I \) (see [BDVM] for a treatment). For instance, in the above example the prime \( q \) is also the fourth column and the set \( \{1, 4, 5\} \). Moreover

– Each Prime \( p \) has a zero-block \( Z(p) = \text{supp}(p)^{-1} \) (zeros can be seen in the \((0, \infty)\)-representation; they are called holes in the representation with Primes): e.g. for the primes \( q \) respectively \( p'' \) above, \( Z(q) = \{2, 3\}, Z(p'') = I \setminus \{5\} \).

– A Prime without holes (like the above prime \( p_0 \)) is called a full Prime and represents the minimum type of the group. In examples we usually do not write the empty Prime, and often skip the full Prime as well.

– A Prime \( p \) with \( \text{supp}(p) = \{i\} \)—like primes \( p, p', p'' \) above—is called a locking Prime; it divides only one (base! see (**)) type of \( G \); the type \( t_i \) having this Prime is a locked type. Above, \( t_3, t_4, t_5 \) are locked types. A tent all of whose base types are locked is a locked tent. We have

**Proposition 2.5.**

(i) A locked type of \( G \) occurs in any type-base of \( G \).

(ii) All locked types are \( \lor \)-irreducible.

(iii) A group with a locked tent has no nontrivial type-base changes.

(iv) If the tent of \( G \) is locked, \( G \) is either degenerate or indecomposable.

**Proof.** (iv) is an immediate consequence of [DVM 10, Proposition 2.1]. \( \square \)

Example 6.a shows what one may expect from locked tents.

Given \( I \), the set of all possible Primes is \( \mathcal{P}(I) \). Since we will deal with properties that are independent of the ordering of Primes, to avoid unnecessary complications we will assume a
fixed order, where Primes with less infinities precede Primes with more, and, when the number of infinities is the same, a lexicographic order is followed, as in the following $m = 4$ example:

\[
\begin{align*}
t_1 &= 0 \infty 0 0 0 \infty \infty \infty 0 0 \infty \infty \infty 0 \infty \\
t_2 &= 0 0 \infty 0 0 \infty \infty 0 0 \infty \infty \infty 0 \infty \\
t_3 &= 0 0 0 \infty 0 0 \infty 0 \infty 0 \infty \infty \infty 0 \infty \\
t_4 &= 0 0 0 0 \infty 0 0 \infty 0 \infty 0 \infty \infty \infty 0 \infty \\
\end{align*}
\]

A table like the one above where all $2^m$ Primes occur is called the $m$-Total Table, and any $m$-tuple of types is obtained by deleting certain Primes (columns) out of the $m$-Total Table. The types $t_1, \ldots, t_m$ yielded by the Total Table (just as any $m$ types) are the base types of a $B(0)$-group; this is not true of $B(n)$-groups for $n \geq 1$, where, as we saw above, we must cancel at least the Primes $I \setminus \{i\}$ for all $i \in I$. The problem of which Primes need to be cancelled in the various cases is treated in Section 2.F. Observe that, with our convention, a permutation of the base types will carry with it a reordering of the Primes.

A Primes-sub-tent of a given tent is one obtained from it by cancelling some Primes (= some columns); the tent obtained by cancelling from the tent of the group $G$ the column of a Prime $p$ can be realized as the tent of the (over)group $G \otimes \mathbb{Q}_p$, that has $p$ as a full prime (which will then be absorbed in the token full Prime of the tent, hence cancelled). Analogously when cancelling a number of Primes.

Observe that if $G \cong G'$, groups obtained from $G$ and $G'$ by cancelling the same Primes are still isomorphic; a curious case of isomorphism of subgroups carrying over to overgroups; but this of course depends on the opposite containment of their tents. Let us state explicitly

**Proposition 2.6.** Let $G$ and $G'$ be $B(n)$-groups. If typeset($G \otimes \mathbb{Q}_p$) $\neq$ typeset($G' \otimes \mathbb{Q}_p$) then typeset($G$) $\neq$ typeset($G'$).

**Proof.** All types of $G \otimes \mathbb{Q}_p$ and $G' \otimes \mathbb{Q}_p$ have $p$ among their primes, thus if a type $\sigma = p_1 \ldots p_r, p$ is—say—in typeset($G \otimes \mathbb{Q}_p$) but not in typeset($G' \otimes \mathbb{Q}_p$), either $\sigma$ or $\sigma' = p_1 \ldots p_r$ is a type of $G$; but clearly not of $G'$. \hfill $\square$

Finally, it may be interesting to observe

**Proposition 2.7.** If $C \subseteq I$, we have $\tau(C) = \bigwedge \{ t_G(g) \mid C^{-1} \in \text{fam}_G(g) \}$; if $C \neq \emptyset$, $\tau(C)$ is the type of infinitely many elements of $G$.

**Proof.** If $C = \{i\}$, $\tau(C) \equiv t_i = t_G(r_1 g_i)$ for all $r_i \in R_i$. If $C = \{1, 2\}$, let $\sigma = t_G(\beta_1 g_1 + \beta_2 g_2) = t_G(\gamma_1 g_1 + \gamma_2 g_2)$. If $\beta_1 = r/s$ and $\gamma_1 = r'/s'$, with $r, s, r', s'$ integers, $\sigma = t_G(r's(\beta_1 g_1 + \beta_2 g_2)) = t_G(r's(\gamma_1 g_1 + \gamma_2 g_2))$ and $r's\beta_1 = r's\gamma_1$. We can then apply [A, 3.1.3] to conclude that $\sigma = t_1 \land t_2 = \tau(\{1, 2\})$. In fact, as soon as two linear combinations of $g_i, g_j$ have the same type— and this happens infinitely many times, since typeset($G$) is finite—this type is $\tau(\{i, j\})$. Let now $\tau(\{1, 2\}) = t_G(\beta_1 g_1 + \beta_2 g_2)$; among the elements $\beta_1 g_1 + \beta_2 g_2 + \beta_3 g_3$ there are (by the above proof) infinitely many of type $\tau(\{1, 2\}) \land t_3 = \tau(\{1, 2, 3\})$. Finite induction completes the proof. \hfill $\square$

**Problem.** Given a finite lattice $L$, determine all tents with image $L$ (see also [DVM 2, DVM 4]).
2.E. Computing $\text{maxfam}_G(g_1)$

As an application, let us start to compute the type of a base element (say, $g_1$); this may seem inane, since we have by initial settings that this type is $t_1$; but it will set the ground for the next section.

We will use this statement, equivalent to Corollary 2.3.

**Lemma 2.8.** The Prime $p$ divides $g$ if and only if $Z(p)$ is contained in a zero-block of $g$ (that is, if and only if $Z(p) \in \text{maxfam}_G(g)$). In particular for $g = g_i$, $i \in \text{supp}(p)$ (iff $i \in Z(p)$) if and only if $Z(p) \in \text{maxfam}_G(g_i)$.

To compute $\text{fam}_G(g_1) = \{Z(x_1 + y) \mid y \in K\}$ we must determine the zero-block of each element $x_1 + y = x_1 + \mu_1a_1 + \cdots + \mu_na_n = x_1 + \mu_1x_1 + \mu_2(\alpha_2,1x_{A_1} + \cdots + \alpha_2,kx_{A_k}) + \cdots + \mu_n(\alpha_n,1x_{A_1} + \cdots + \alpha_n,kx_{A_k})$, with arbitrary $\mu_1, \ldots, \mu_n$. We have

\[
x_1 + y = (1 + \mu_1 + \mu_2\alpha_2,1 + \cdots + \mu_n\alpha_n,1)x_1 + (1 + \mu_2\alpha_2,1 + \cdots + \mu_n\alpha_n,1)x_{A_1\setminus\{1\}} + (\mu_1 + \mu_2\alpha_2,2 + \cdots + \mu_n\alpha_n,2)x_{A_2} + \cdots + (\mu_1 + \mu_2\alpha_2,k + \cdots + \mu_n\alpha_n,k)x_{A_k}.
\]

The zero-blocks of these elements are clearly unions of the sets $\{1\}, A_1\setminus\{1\}, A_2, \ldots, A_k$, to be obtained by equating coefficients to zero; that is, by solutions of subsystems of

\[
\begin{align*}
1 + \mu_1 + \mu_2\alpha_2,1 + \cdots + \mu_n\alpha_n,1 &= 0, \\
\mu_1 + \mu_2\alpha_2,1 + \cdots + \mu_n\alpha_n,1 &= 0, \\
\mu_1 + \mu_2\alpha_2,2 + \cdots + \mu_n\alpha_n,2 &= 0, \\
\vdots & \\
\mu_1 + \mu_2\alpha_2,k + \cdots + \mu_n\alpha_n,k &= 0.
\end{align*}
\]

Since the first and second equation cannot be zero at the same time, we are in fact looking at two systems: a homogeneous system $(Ho)$, obtained by dropping from (1) the first equation, hence involving only the blocks $A_1\setminus\{1\}, A_2, \ldots, A_k$; and a system $(1')$, obtained by dropping the second equation, involving $\{1\}, A_2, \ldots, A_k$:

\[
\begin{align*}
\mu_1 + \mu_2\alpha_2,1 + \cdots + \mu_n\alpha_n,1 &= 0, & \mu_1 + \mu_2\alpha_2,1 + \cdots + \mu_n\alpha_n,1 &= 1, \\
\mu_1 + \mu_2\alpha_2,2 + \cdots + \mu_n\alpha_n,2 &= 0, & \mu_1 + \mu_2\alpha_2,2 + \cdots + \mu_n\alpha_n,2 &= 0, \\
\vdots & & \vdots \\
\mu_1 + \mu_2\alpha_2,k + \cdots + \mu_n\alpha_n,k &= 0, & \mu_1 + \mu_2\alpha_2,k + \cdots + \mu_n\alpha_n,k &= 0.
\end{align*}
\]

The complete matrix of $(Ho)$, that is the incomplete matrix of $(1')$, is the transposed of the basic matrix $M$ of $G$:

\[
M^T = \begin{bmatrix} 1 & \alpha_{2,1} & \cdots & \alpha_{n,1} \\ \cdots & \cdots & \cdots & \cdots \\ 1 & \alpha_{2,n} & \cdots & \alpha_{n,n} \\ \cdots & \cdots & \cdots & \cdots \\ 1 & \alpha_{2,k} & \cdots & \alpha_{n,k} \end{bmatrix}.
\]
System (Ho) has the global solution \( \mu_1 = \mu_2 = \cdots = \mu_n = 0 \), yielding the zero-block \( I \setminus \{1\} = A_1 \setminus \{1\} \cup A_2 \cup \cdots \cup A_k \) of \( x_1 \), hence it contributes \( \tau((I \setminus \{1\})^{-1}) = \tau(\{1\}) = t_X(x_1) = t_1 \) to the type \( t_G(g_1) \). Note that any other solution of (Ho) yields a zero-block contained in \( I \setminus \{1\} \); then by (**) it will contribute a type \( \leq t_1 \) to the supremum, therefore it can be disregarded. Thus the other significant zero-blocks to look for must contain \( \{1\} \), hence come from solutions of subsystems of \( (1') \) containing the first equation; and we are only interested in maximal ones.

A maximal solvable subsystem of \( (1') \) containing the first equation is indexed by a subset \( \{1\} \cup S \) of \( J \), where \( S \subseteq J \setminus \{1\} \) indexes a homogeneous subsystem of \( (1') \) (that is, a subsystem of (Ho)) maximal with respect to having a nonzero solution. Equivalently, \( S \) indexes a maximal-dependent set of rows of \( M^T \), yielding a maximal row-submatrix of rank \( n - 1 \).

**Definition 2.9.** A subset \( S \) of \( J \) is **almost independent** if it indexes a maximal-dependent set of rows of \( M^T \); equivalently, a maximal row-submatrix of rank \( n - 1 \).

We can now answer the initial question:

**Proposition 2.10.**

(i) \( \text{maxfam}_G(g_1) \) consists of the set \( I \setminus \{1\} \) and all sets \( \{1\} \cup (\bigcup \{A_j \mid j \in S\}) \) where \( S \) is almost independent and \( S \subseteq J \setminus \{1\} \).

(ii) The same result holds if we replace everywhere \( \{1\} \subseteq A_1 \) with \( E \subseteq A_1 \), computing \( \text{maxfam}_G(g_E) \). In particular, if \( J \setminus \{1\} \) itself indexes a row-submatrix of \( M^T \) of rank \( n - 1 \) we have \( g_{A_1} = 0 \), and \( G \) is degenerate.

**Proof.** (ii) If \( J \setminus \{1\} \) indexes a row-submatrix of rank \( n - 1 \), then \( I = \{A_1\} \cup (\bigcup \{A_j \mid j \in J \setminus \{1\}\}) \) is a zero-block of \( g_{A_1} \); thus \( g_{A_1} = g_I = 0 \), and \( G \) splits into \( X/(x_{A_1})_* \oplus G' \). \( \square \)

Another version of Proposition 2.10(i) can be found in Proposition 4.2.

Note that if \( E \) is contained in a section, we have from above and from Corollary 2.3(i) that the type \( t_G(g_E) \) depends only on the tent and on the basic partition \( A \).

**Exercise.** As an application, show that if \( G \) is \( B(2) \) we have \( \text{maxfam}_G(g_1) = \{I \setminus \{1\}, \{1\} \cup A_2, \ldots, \{1\} \cup A_k\} \).

**2.F. The K-tent**

What we saw brings forth the following, crucial

**Observation 3.** In the above situation, \( B = \{1\} \cup (\bigcup \{A_j \mid j \in S\}) \) is a maximal zero-block of \( g_1 \), hence its complement \( B^{-1} = (A_1 \setminus \{1\}) \cup (\bigcup \{A_j \mid j \in S^{-1} \setminus \{1\}\}) \) indexes a representative \( y' \) of \( g_1 \), \( y' = y_1 x_{A_1} \setminus \{1\} \) \( + \sum \{y_j x_{A_j} \mid j \in S^{-1} \setminus \{1\}\} \). Since by (**) \( t_G(g_1) = t_X(x_1) = t_1 \), the contribution of \( y' \) to the type of \( g_1 \) cannot exceed \( t_1 \); thus, if \( p \) divides \( y' \) —i.e. if \( \text{supp}(p) \) contains \( B^{-1} - p \) must also divide \( t_1 \), i.e. \( \text{supp}(p) \) must contain \( \{1\} \), that is \( \text{supp}(p) \supseteq \bigcup \{A_j \mid j \in S^{-1}\} \). Thus,

in the tent of \( G \) there cannot be any Prime \( p \) with a single hole on \( \bigcup \{A_j \mid j \in S^{-1}\} \) occurring on \( \{1\} \subseteq A_1 \).
Clearly, though, any \( \{i\} \subseteq A_1 \) can play the role of \( \{1\} \) in the above. Moreover, the property characterizing the section \( A_1 \) with respect to \( S \) is that its index \( 1 \in S^{-1} \), that is, it indexes a row of \( M^T \) not belonging to the maximal row-submatrix of rank \( n - 1 \) indexed by \( S \) (then \( \{1\} \cup S \) indexes a maximal solvable subsystem of \( \{1\}' \)). But any index \( j' \in S^{-1} \) will play the same role, i.e. will yield a section \( A_{j'} \) that is forbidden from hosting a single hole on \( \bigcup \{ A_j | j \in S^{-1} \} \). We can thus formulate the following rule:

**Observation 4.** In this statement there is nothing related to \( X \): the condition is posed solely on \( M^T \), that is on the subspace \( K \) of \( V \). The condition then forbids the stated Primes to all groups \( G' = Y/KY \) with creel \( K \).

In view of the above, we give the following

**Definition 2.11.** A union \( \bigcup \{ A_j | j \in S^{-1} \} \) of sections where \( S \) is almost independent is called a *lobe* of \( K \).

A Prime \( p \) is said to *pierce* a union \( U = \bigcup \{ A_j | j \in J' \} \) of sections if it has exactly one hole on \( U \), that is if \( \text{supp}(p) \cap U = U \setminus \{i\} \) for some \( i \in U \).

Direct computation shows that, if we change the base of \( K \) via an \( n \times n \) nonsingular matrix \( N \), to get a new base \( (b_1, \ldots, b_n) \) of \( K \) with \( NM \) as a basic matrix, the dependence relations that held for the rows of \( M^T \) still hold for the corresponding rows of \( (NM)^T \) (and with the same linear combinators). Therefore the definition of lobe is independent of the chosen base of \( K \).

Our rule then takes the following form:

**Theorem 2.12.** No Prime of \( G \) can pierce a lobe of its creel \( K \).

**Observation 5.** If \( n \geq 1 \), a Prime with only one hole—say, on \( 1 \in A_1 \)—will always pierce a lobe of \( K \), obtained by choosing an almost independent \( S \) not containing \( 1 \); hence, a Prime with only one hole is (as we already knew) always forbidden. A Prime with more than one hole, but all of whose holes are contained in the same section, is always allowed, while a Prime with one hole in a section (say, \( A_1 \)) and all other holes in another (say, \( A_2 \)) is always forbidden, since we can choose \( S \) containing the second row but not the first. These, though, are the only general cases.

Let now \( K \) be any \( n \)-dimensional subspace of \( V \); note that a base change of \( K \) does not change the basic partition \( A \) of \( K \). Theorem 2.12 partitions Primes into two sets, those that (due to \((*)\)) are not allowed by \( K \) and those that are.

**Definition 2.13.** A \( K \)-tent is a tent consisting of all Primes allowed for \( K \). A tent is a total tent if it is a \( K \)-tent for some subspace \( K \) of \( V \).

Realizing now the rows of the \( K \)-tent as actual types \( (t_1, \ldots, t_m) \) (e.g. with zeros and infinities), for each \( i \in I \) let \( R_i \) be a subgroup of \( \mathbb{Q} \) of type \( t_i \) (the \( i \)th row), and set \( X = \bigoplus \{ R_i x_i | i \in I \} \leq V \), \( K_X = K \cap X \), \( G(K) = X/K_X \). The tent of \( G(K) \) coincides with the \( K \)-tent, yielding
Theorem 2.14. If $K$ is an $n$-dimensional subspace of $V$, the $K$-tent is the tent of a $B(n)$-group $G(K)$ with creel $K$. Any $B(n)$-group $G$ with creel $K$ has a tent that is a Primes-sub-tent of the $K$-tent.

Moreover, any Primes-sub-tent of the $K$-tent is the tent of a $B(n)$-group with creel $K$ containing $G(K)$.

Proof. A Primes-sub-tent obtained by cancelling from the $K$-tent the column of a Prime $p$ can be realized as the tent of the group $G = G(K) \otimes \mathbb{Q}_p = (X \otimes \mathbb{Q}_p + K)/K$ since $K \otimes \mathbb{Q}_p = K$. □

Definition 2.15. We call $G(K)$ a $K$-total group; $G$ is a total group if it is a $K$-total group for some subspace $K$ of $V$.

3. Regularity

Consider the following question for $0 \leq n < m$: given an arbitrary $m$-tuple of types $(u_1, \ldots, u_m)$, is it the type-base of some $B(n)$ group? In our treatment of $B(1)$-groups, this condition on an $m$-tuple of types was called regularity. (Besides its theoretical interest, regularity is crucial in building examples: an inadequate $m$-tuple of types can make the most promising construction collapse.) For $B(0)$-groups the answer to the question is Yes; for $B(1)$-groups regularity amounts to the general condition forbidding all Primes with only one hole.

For $B(n)$-groups with $n \geq 2$, the condition is not general any more; as noted in Observation 4, it is independent of $X$, but it depends on the “linear part” of the definition of a Butler group $G$, that is on its relations: we speak of $K$-regularity. In particular, for $B(2)$-groups the answer still depends only on the basic partition $\mathcal{A}$ of $K$: from the exercise at the end of Section 2.E we see that the forbidden Primes are those that pierce a union of $\geq k - 1$ sections. If $n > 2$, as we saw, a tent is $K$-regular if and only if it is a Primes-sub-tent of the $K$-tent.

A connected question is the following: given a completely decomposable group $Y = S_1 y_1 \oplus \cdots \oplus S_m y_m$ with type-base $(u_1, \ldots, u_m)$, and a subspace $K$ of $V$ of dimension $n < m$, with $\dim K = \text{rk}(K \cap Y)$, and $K \cap Y = K_Y$, which is the tent of the $B(n)$-group $G = Y/K_Y$?

Proposition 3.1. In the above setting, the type-base of $G$ is obtained from $(u_1, \ldots, u_m)$ by filling all pierced lobes (while eliminating any ensuing repeated Prime).

Proof. From the discussion at the beginning of Section 2.F we have that if a Prime did pierce a lobe of $K$ at $\{i\}$, then although $p$ does not divide $y_i$, it would divide $g_i = y_i + K$, hence it would divide the base type $t_i$ of $G$. □

Example 3.2. Let $n = 3, k = 4, m = 5$; $\mathcal{A} = \{\{1, 2\}, \{3\}, \{4\}, \{5\}\}$, hence $J = \{1, 2, 3, 4\}$; and set

<table>
<thead>
<tr>
<th></th>
<th>$t_1$</th>
<th>$t_2$</th>
<th>$t_3$</th>
<th>$t_4$</th>
<th>$t_5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_1$</td>
<td>$\cdot$</td>
<td>$\cdot$</td>
<td>$\cdot$</td>
<td>$\cdot$</td>
<td>$\cdot$</td>
</tr>
<tr>
<td>$A_2$</td>
<td>$\cdot$</td>
<td>$\cdot$</td>
<td>$\cdot$</td>
<td>$\cdot$</td>
<td>$\cdot$</td>
</tr>
<tr>
<td>$A_3$</td>
<td>$\cdot$</td>
<td>$\cdot$</td>
<td>$\cdot$</td>
<td>$\cdot$</td>
<td>$\cdot$</td>
</tr>
<tr>
<td>$A_4$</td>
<td>$\cdot$</td>
<td>$\cdot$</td>
<td>$\cdot$</td>
<td>$\cdot$</td>
<td>$\cdot$</td>
</tr>
</tbody>
</table>
Let \( G(a) \), \( G(b) \) be \( B(3) \)-groups with creels \( K(a) \) respectively \( K(b) \) generated by the following relations:

\[
\begin{align*}
(a) & \quad g_{1,2} + g_3 + g_4 + g_5 = 0, \\
0 g_{1,2} + g_3 - 2 g_4 - 3 g_5 = 0, \\
0 g_{1,2} + 2 g_3 - g_4 - 2 g_5 = 0;
\end{align*}
\]

\[
\begin{align*}
(b) & \quad g_{1,2} + g_3 + g_4 + g_5 = 0, \\
0 g_{1,2} + g_3 - 2 g_4 - 3 g_5 = 0, \\
0 g_{1,2} + 2 g_3 - g_4 + 2 g_5 = 0.
\end{align*}
\]

Both have \( A \) as a basic partition. The two transposed matrices are

\[
M_T(a) = \begin{bmatrix}
1 & 0 & 0 \\
1 & 1 & 2 \\
1 & -2 & -1 \\
1 & -3 & -2
\end{bmatrix}, \quad M_T(b) = \begin{bmatrix}
1 & 0 & 0 \\
1 & 1 & 2 \\
1 & -2 & -1 \\
1 & -3 & 1
\end{bmatrix}.
\]

Starting with (a), the subset \( S = \{2, 3, 4\} \) of \( J \) is almost independent for \( M_T(a) \); \( S^{-1} = \{1\} \), hence \( A_1 = \{1, 2\} \) is a lobe for \( K(a) \): thus no prime of \( G(a) \) can pierce it; hence the above prime \( p \)

is forbidden, as would be any prime dividing only one of \( t_1 \) and \( t_2 \). Thus the above tent is not

\( K(a) \)-regular (in this very simple example, the reason is that a consequence of (a) is the relation \( g_1 + g_2 = 0 \), hence \( t_G(g_1) = t_G(g_2) \)).

In particular, if we had started with a completely decomposable \( Y \) with the above base types

and relations (a), in the tent of \( G = Y \setminus K \) the column of \( p \) would have to be filled by \( p \) in its

first entry.

Note that no other \( 3 \times 3 \) submatrix of \( M_T(a) \) has rank 2, therefore all other lobes of \( K \) are unions

of two sections. If a union of two singleton sections \( A_j, A_j' \) is a lobe, then \( t_j = t_j' \). In particular,

a prime \( q \) either divides all three types \( t_3, t_4, t_5 \) or none of them. This helps to complete the

analysis, obtaining the \( K(a) \)-tent:

\[
\begin{array}{c}
A_1 \\
A_2 \\
A_3 \\
A_4
\end{array}
\begin{array}{c}
t_1 = p' \cdot r \\
t_2 = p' \cdot r \\
t_3 = q \cdot r \\
t_4 = q \cdot r \\
t_5 = q \cdot r
\end{array}
\]

As for \( K(b) \), no \( 3 \times 3 \) submatrix of \( M_T(b) \) has rank 2, therefore all lobes of \( K \) are unions of two

sections. The \( K(b) \)-tent is then

\[
\begin{array}{c}
A_1 \\
A_2 \\
A_3 \\
A_4
\end{array}
\begin{array}{c}
t'_1 = p'' \cdot r \\
t'_2 = p \cdot r \\
t'_3 = q \cdot r \\
t'_4 = q \cdot r \\
t'_5 = q \cdot r
\end{array}
\]

allowing in particular the prime \( p \) from which we started.

Having \( m = 5 \), that is groups of rank 2, the two examples describe the two possibilities: decomposable group in case (a), indecomposable in case (b). In fact, if the matrix \( M \) is block-diagonal the group is degenerate; and this is true also if \( NM \) is block-diagonal, where \( N \) is an
$n \times n$ nonsingular matrix, for this is simply a base change of $K$. Example (a) is a particular case of degenerate $B(n)$-group, since

$$M_{(a)} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & -2 & -3 \\ 0 & 2 & -1 & -2 \end{bmatrix}$$

is equivalent to a block-diagonal matrix. Instead, system (b) is equivalent to $g_{1,2} = -9/5 g_3$, $g_4 = 7/5 g_3$, $g_5 = -3/5 g_3$; then $G_{(b)}$ can be represented as the $B(1)$-group $H = \langle h_1 \rangle^* + \langle h_2 \rangle^* + \langle h_3 \rangle^*$ (for $h_1 = 5 g_1$, $h_2 = 5 g_2$, $h_3 = 9 g_3$) with its diagonal relation $h_1 + h_2 + h_3 = 0$, and typeset

$$u_1 = p'' \cdot \cdot \ \cdot \ r$$
$$u_2 = \cdot \ p \ \cdot \ \cdot \ \cdot \ r$$
$$u_3 = \cdot \ \cdot \ q \ \cdot \ r$$

hence indecomposable. (Beware that the linear equivalence always be checked against the types structure!).

4. Configurations

Let us continue with the above examples. Interpret the four row-vectors of $M^T_{(a)}$ as points $P_j$ of the projective space $PS(2)$, by assigning the first coordinate to the $\infty$ line. Linearly dependent sets of rows will yield aligned points (see Fig. 1).

The lobes are easily visualized: they consist of the points left out of some line, that is, left out of some hyperplane of $PS(2)$. Note that the points $P_j$ represent the rows of the matrix, hence the sections $A_j$ of $K$.

The analogous interpretation for $K_{(b)}$ is shown in Fig. 2.

Clearly, what counts here in the construction of the $K$-tent is not the linear system, but the configuration (drawn on the right) imposed to the points (sections) by the system: any subspace $K'$ of $V$ with $m = 5$, $k = 4$ (ensuring an analogous basic partition), $n = 3$ and configuration (a) will yield the same total tent as $K$. In our examples, the two configurations exhaust all possible configurations for $k = 4$, $n = 3$ (4 points in $PS(2)$), since we started with a matrix of maximal rank (hence the points must generate $PS(2)$).
Call splitting configuration one where the points distribute between two complementary subspaces (e.g. case (a)).

**Proposition 4.1.** Degenerate $B(n)$-groups have a splitting configuration.

**Proof.** Let $I = I' \cup I''$ (disjoint union) and $G = G' \oplus G''$, with $G' = \langle g_i \mid i \in I' \rangle_*$ and $G'' = \langle g_i \mid i \in I'' \rangle_*$. The diagonal relation yields then $g_{I'} = -g_{I''} \in G' \cap G'' = 0$; analogously, each relation $a_j$ produces two relations $a_{j}'$, $a_{j}''$. Among them there are $n$ independent ones, two of which may be chosen without loss of generality to be $x_{I'}$ and $x_{I''}$; or, equivalently, $x_I$ and $x_{I'}$. With these generators for $K$, the matrix $M^T$ will be of the form

\[
\begin{bmatrix}
1 & 1 & * & \ldots & * & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
1 & 1 & * & \ldots & * & 0 & \ldots & 0 \\
1 & 0 & 0 & \ldots & 0 & * & \ldots & * \\
\vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \ddots & \vdots \\
1 & 0 & 0 & \ldots & 0 & * & \ldots & * \\
\end{bmatrix}
\]

the points will thus belong to two complementary coordinate subspaces of $PS(n - 1)$. $\square$

Note that by choosing instead $x_{I'}$ and $x_{I''}$ the matrix becomes block-diagonal. The last part of Proposition 2.10(ii) shows that a point-and-hyperplane configuration is always degenerate.

Configurations yield the following better usable version of Proposition 2.10(i), which can be practised in Example 6.e.

**Proposition 4.2.** $\text{maxfam}_{G}(g_1)$ consists of the set $I \setminus \{1\}$ and all $><\text{DEFANGED.1 sets} \{1\} \cup (\bigcup \{A_j \mid j \in S\})$, where $S \subseteq J \setminus \{1\}$ and $S$ indexes a maximal non-generating set of points of $PS(n - 1)$.

(The use of $S$ versus $S^{-1}$ in Proposition 4.2 and Definition 2.11 is an instance of the slippery nature of this subject.)
Rather than defining ‘configuration,’ let us state when two subspaces $K, K'$ of $V$, with partitions $A$, respectively, $A'$ and matrices $M^T$ respectively $M'^T$, have the same configuration. We require:

1. $\dim K = \dim K'$, and $|A| = |A'|$, so that both partitions can be indexed by the same set $J = \{1, \ldots, k\}$;
2. a permutation $f$ of $J$ such that:
   
   (i) $|A_j| = |A'_{f(j)}|$ for all $j \in J$, and
   
   (ii) corresponding sets of row-vectors span subspaces of the same dimension.

We can express this in terms of points by attaching $|A_j|$ to $P_j$, calling it the weight of $P_j$; so we can rephrase (i), (ii) into

(i') $f$ preserves weights, and

(ii') both $f$ and $f^{-1}$ preserve lobes.

To justify the condition on partitions in point (1), note that

**Proposition 4.3.** Let $K, K'$ be subspaces of $V$ with $1 < n = \dim K = \dim K' < m$, and partition $A$ respectively $A'$. If $A \neq A'$, the $K$-tent and the $K'$-tent are different.

**Proof.** If $A \neq A'$, without loss of generality $A_1 \neq A'_1$. If one of the two is contained in the other—say, $A_1 \subseteq A'_1$—then a Prime with two holes in $A'_1$, of which one in $A_1$ and one outside, is allowed for $K'$ but not for $K$ (see Observation 5). If $A_1 \cap A'_1$ is nonempty and different from both $A_1$ and $A'_1$, a Prime with a hole in the intersection and another in $A_1 \setminus (A_1 \cap A'_1)$ is forbidden for $K$, allowed for $K'$. □

We have from the above and from the last part of Example 6.b

**Corollary 4.4.** The $K$-tent depends only on the basic partition and on the configuration of $K$. This is not true in general of the $K$-total group.

Note finally that isomorphism does not imply equal configuration: just think of the homogeneous case: any configuration can be attributed to it, by just cancelling all Primes but the full one. At the other extreme, a way to make configurations more relevant consists in locking base types of $G$.

**Problem.** Inversion of Proposition 4.1.

5. A class of indecomposable Butler groups

We start by investigating when a $K$-tent forbids a locking Prime. For a locking Prime $p$—say, locking $t_1$—to be forbidden, it must be piercing a lobe; this is possible only if $\{1\} = \text{supp}(p) \cap U = U \setminus \{i\}$ for some $i \in U$, say $i = 2$; where $U = \{1, 2\}$ is a lobe of $K$. $U$ must then be a union of sections, hence we have either a Kind 1 configuration: $U = A_1$ and $S = \{2, \ldots, k\}$; or a Kind 2 configuration: $U = A_1 \cup A_2$ (with $\{1\} = A_1, \{2\} = A_2$) and $S = \{3, \ldots, k\}$, with $S$ almost independent.
In Kind 1 the configuration of $K$ consists of a point and a hyperplane (like the one of $K(a)$, with suitable dimension); then the $K$-total group $G(K)$ is degenerate (see Proposition 2.10(ii), or after Proposition 4.1).

In Kind 2, $t_1 = t_2$; the set $\{P_3, \ldots, P_n\}$ generates a hyperplane, while $P_1$ and $P_2$ are not on it. Then $g_1$ has the zero-block $\{1\} \cup A_2 \cup \cdots \cup A_k$, hence $g_1 = \lambda g_2$, against our convention (0). As a conclusion, under (0), we have

**Theorem 5.1.** A total group is either degenerate or indecomposable.

Note that this class of indecomposable $B(n)$-groups is completely determined by $K$.

**Exercise.** Build a degenerate $B(3)$-group with locked tent. (Solution: interpret the (degenerate) configuration of Fig. 1 with its equations (a) by blowing up section $A_1$ to three elements, and $>\text{DEFANGED.2}$ all other sections but at most one to two elements (to avoid Kind 2); e.g. $A = \{\{1, 2, 3\}, \{4, 5\}, \{6, 7\}, \{8\}\}$, and replacing the $g_i$ with the $g_{A_j}$. All locking primes are allowed, while the point-and-hyperplane configuration insures degeneracy.)

**Observation 6.** All of the $2^{\aleph_0}$ realizations of $G(K)$ are (strongly!) indecomposable. Moreover, *locked base types are pairwise different*, and the lock implies by Proposition 2.5(iv) that each group has no nontrivial type-base changes: therefore any two realizations of a non-degenerate $G(K)$ with different *sets of base types* (hence, whose tents do not differ by a permutation of base types) are non-isomorphic.

What can we say of total groups with different creels? When are two “abstract” $B(n)$-groups $G$ and $G'$ isomorphable, that is, when do they allow isomorphic realizations? They are clearly isomorphable if they have the same creel (hence the same partition) and the same tent. But there are less obvious cases:

**Lemma 5.2.** Any two Butler groups with $n = m - 1$ are isomorphable.

**Proof.** An “abstract” rank one group can be realized by any type. \(\square\)

Observation 6 yields

**Proposition 5.3.** Indecomposable total groups with different sets of base-types are non-isomorphable.

6. Examples

In the previous sections we have evidenced three main features of a $B(n)$-group: tent, basic partition and configuration; another relevant feature is the typeset. For the knowledge of $B(n)$-groups it would be useful to determine which of these features determine the others (for instance, in the class of total groups partition + configuration determine the tent); or in which cases some of these features yield isomorphability.

A big problem is the relationship between tent and typeset: not only because, differently from the $B(1)$ case, the tent does not determine the typeset (Observation 3) even if the configuration is the same (Example 6.b), but even more so when the typeset is the same and the tent changes: the
problem of type-base changes is probably the most difficult at this stage (examples of a cross-dressing base change between $B(3)$ and $B(1)$ for $G(b)$ at the end of Section 4, between $B(2)$ and $B(0)$ for $G(c)$ in Example 6.b). Thus, in order to build examples it is better (when possible) to stick to situations with no type-base changes, something that can be secured by suitably chosen locked types. That a locked base, though, does not ensure isomorphic realizations is shown in the next example:

**Example 6.a.** *Same tent, same partition, same typeset, different configuration* for non-isomorphic $B(3)$-groups.

Let $m = k = 6$, $n = 3$ (the minimum for different configurations), type-base

\[
\begin{array}{c|c}
A_1 & t_1 = p_1 & \cdots & \cdots & \cdots & \cdots \\
A_2 & t_2 = p_2 & \cdots & \cdots & \cdots & \cdots \\
A_3 & t_3 = p_3 & \cdots & \cdots & \cdots & \cdots \\
A_4 & t_4 = p_4 & \cdots & \cdots & \cdots & \cdots \\
A_5 & t_5 = p_5 & \cdots & \cdots & \cdots & \cdots \\
A_6 & t_6 = p_6 & \cdots & \cdots & \cdots & \cdots \\
\end{array}
\]

and let $G(a')$ have configuration (a'), $G(b')$ have configuration (b') in the plane (see Fig. 3).

Note that in both cases lobes have cardinality at least 3, therefore Kind 1 and 2 do not occur, and we are allowed all of the above locking Primes. The typeset is determined by the tent, since by (**) every element besides 0 and the base elements has minimum type. But a locked type-base allows only bases of the form $g_i' = \gamma_i g_i$, which preserve the configuration; therefore $G(a')$ and $G(b')$ are non-isomorphic. By Proposition 4.1, $G(a')$ and $G(b')$ are nondegenerate (the only degenerate configuration in the plane is the point-and-line one); hence, by Proposition 2.5(iv), they are both indecomposable.

**Example 6.b.** In $B(1)$-groups, the tent $t$ determines the typeset of the group, which is simply $t(\mathbb{P}(1))$. For $n > 1$ this is not true any more: here is a mean case of *same tent, same partition, same configuration with different typesets*.

The next example is in $B(2)$, hence there is only one configuration. Set $m = 6$ and $K = \langle x_1, a \rangle$, with $a = \alpha_1 x_1 + \cdots + \alpha_6 x_6$, for a rank 4 group $G$. Let the partition be $\mathcal{A} = \min$, so the $\alpha_i$ are pairwise distinct. Let the tent be
In order to determine the type $t_G(g)$ of an element $g = y + K$ with $y = \beta_1 x_1 + \cdots + \beta_6 x_6$, we need to compute maxfam$_G(g)$ from the representatives $y + \lambda x_I + \mu a = \sum((\lambda + \mu \alpha_i + \beta_i)x_i \mid i = 1, \ldots, 6)$; associate to $g$ the matrix $M^T(g)$, obtained from $M^T$ by adding the column of the coefficients $\beta_i$:

$$
M^T(g) = \begin{bmatrix}
1 & \alpha_1 & \beta_1 \\
1 & \alpha_2 & \beta_2 \\
1 & \alpha_3 & \beta_3 \\
1 & \alpha_4 & \beta_4 \\
1 & \alpha_5 & \beta_5 \\
1 & \alpha_6 & \beta_6 \\
\end{bmatrix}.
$$

We show that only a particular choice of the relation $a$ ensures that $t_0 = pp'p''p''' \in$ typeset($G$).

– For $p$ to divide $g$ we must have $Z(p) = \{4, 5, 6\} \subseteq Z(g)$; $g$ must then have a representative $y = \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3$ with $\beta_4 = \beta_5 = \beta_6 = 0$.

– Now, $p'$ divides $g$ if and only if $Z(p') = \{2, 3, 6\} \subseteq Z(g)$, i.e. if and only if there is a representative $y + \lambda x_1 + \mu a$ of $g$ such that

$$
\begin{cases}
\lambda + \mu \alpha_2 + \beta_2 = 0 \\
\lambda + \mu \alpha_3 + \beta_3 = 0, \\
\text{that is iff } \det \begin{bmatrix}
1 & \alpha_2 & \beta_2 \\
1 & \alpha_3 & \beta_3 \\
1 & \alpha_6 & 0 \\
\end{bmatrix} = 0,
\end{cases}
$$

i.e. iff $\beta_2 = \frac{\alpha_2 - \alpha_6}{\alpha_3 - \alpha_6} \beta_3$. Note that this yields $\text{rk } G(pp') = 2$; $g_1$ is obtained for $\beta_3 = 0$. By symmetry, this holds not only for $G(pp') = G(t_1)$, but also for all the other $G(t_i)$.

Call $M^T(g, p')$ the above matrix, i.e. the row-submatrix of $M^T(g)$ whose rows are indexed in $Z(p')$: we have $p' \mid g$ if and only if $\det M^T(g, p') = 0$.

– Analogously, $p''$ divides $g$ if and only if $M^T(g, p'') = 0$, that is—since $Z(p'') = \{1, 3, 5\}$—if and only if $\beta_1 = \frac{\alpha_1 - \alpha_5}{\alpha_3 - \alpha_5} \beta_3$.

Now $\text{rk } G(pp'p''') = 1$, hence $p'''$ divides $g$ if and only if $G(pp'p''') = G(pp'p''p''')$. Whether this happens depends on the choice of the relation $a$: we see it via a geometric interpretation (Fig. 4). View $M^T(g)$ as a sextuple of points $P_i(\alpha_i, \beta_i)$ of the projective space $PS(2)$, with our fixed first coordinates $\alpha_i$. The condition for $p$ sets $P_4$, $P_5$, $P_6$ on the “$x$” axis; fixing arbitrarily $\beta_3 \neq 0$ determines $P_3$, and the condition on $p'$ puts $P_5(\alpha_2, \beta_3)$ on the line $P_3 P_6$, thus determining its second coordinate $\beta_2$. The condition on $p''$ puts $P_1(\alpha_1, \beta_1)$ on the line $P_3 P_5$, but, since $Z(p'''') = \{1, 2, 4\}$, the condition on $p'''$ sets $P_1$ on $P_2 P_4$. Let $Q(\alpha_0, \beta_0)$ be the intersection of $P_3 P_5$ with $P_2 P_4$: the condition for $p'''$ is satisfied if and only if $P_1 = Q$, that is if and only if $\alpha_1 = \alpha_0$, a particular choice for the relation. Let us study the two non-isomorphable groups we found.
Case (c). If $\alpha_1 \neq \alpha_0$, we have a creel $K(c)$ and a group $G(c)$ with four rank 1 subgroups whose types are the product of three primes: $\langle h''' \rangle_* = G(c)(pp'p'')$, $\langle h \rangle_* = G(c)(p'p''p''')$. We have $\langle h''' \rangle_* = G(c)(pp') = G(c) (t_1) \supseteq \langle h''' \rangle_* \oplus \langle h'' \rangle_*$; this last contains a multiple of $g_1$. Proceeding analogously, $H = \langle h''' \rangle_* + \langle h'' \rangle_* + \langle h' \rangle_* + \langle h \rangle_*$ contains a multiple of each base element with its whole type, hence $H = G(c)$; the sum is direct, for otherwise the rank would decrease. Hence $G(c)$ is completely decomposable; the tent going with the $B(0)$-base is

\[
\begin{aligned}
 u''' & = p \quad p' \quad p'' \\
 u'' & = p \quad p' \quad p'' \\
 u' & = p \quad p'' \quad p''' \\
 u & = p \quad p'' \quad p'''
\end{aligned}
\]

This exemplifies the type-base change problem: $G(c)$ as a $B(2)$-group has type-base $(t_1, \ldots, t_6)$; as a $B(0)$-group, $(u, u', u'', u''')$.

Case (d). If $\alpha_1 = \alpha_0$, we have a creel $K(d)$ and a group $G(d)$ where products of three primes do not belong to the typeset, while there is only one rank 1 subgroup $\langle h_0 \rangle_*$ of type $t_0 = pp'p''p'''$, contained in all the $G(t_i)$. In fact, $G(t_i) = \langle g_i, h_0 \rangle_*$; and one can verify that any sextuple $\{ h_i = \gamma_i g_i + \delta_i h_0 \mid \gamma_i \neq 0, i = 1, \ldots, 6 \}$ is a base of $G(d)$. If we had $G(d) = G' \oplus G''$, a base element $g_i$ which is not already in $G'$ or $G''$ would be $g_i = g' + g''$ with $t_i = t' \wedge t''$; then one of $t'$, $t''$ must be $t_i$, and the other $t_0$ ($G(t_i)$ cannot be split into two rank 1 groups of type $t_i$). If—say—$t' = t_i$ then $G'$ contains an $h_i = \gamma_i g_i + \delta_i h_0$; in any case, a base of $G(d)$—without loss of generality the initial base $\{ g_1, \ldots, g_6 \}$—is contained in $G' \cup G''$. The diagonal relation yields then $g_{1'} = \sum \{ g_i \mid g_i \in G' \} = -g_{1''} = \sum \{ g_i \mid g_i \in G'' \} \in G' \cap G'' = 0$, that is $x_{1'} \in K(d)$; but $x_I, x_{1'}$ are independent in $K(d)$, hence $K(d) = \langle x_I, x_{1'} \rangle$, and its basic partition is $\{ I', I'' \}$, not $A = \text{min}$: a contradiction. Thus $G(d)$ is indecomposable.

As always in $B(2)$, the total tents of $G(c)$ and $G(d)$ are the same, containing all primes with $\geq 3$ holes. But the total groups are different, since so are the typesets.

Example 6.b.bis. Same tent, same basic partition and same configuration but different typesets in $B(3)$. Add to $G(c)$ (to $G(d)$) a rank 2 summand, forming $(\langle g_1 \rangle_* + \cdots + \langle g_6 \rangle_*) \oplus (\langle g_7 \rangle_*$ +
\[(g_8)_* + (g_9)_*\] with \(g_7, g_8\) and \(g_9\) of minimal type, and adding the relation \(g_7 + g_8 + g_9 = 0\).
In both cases the basic partition becomes \(\{1, 2, \ldots, 6, 7, 8, 9\} = \{A_1, \ldots, A_7\}\), and if we write \(M^T\) we see that the configuration is splitting point-and-line with \(P_7\) as the point. The new groups are clearly non-isomorphic.

**Example 6.c.** Same total tent, same partition, same configuration, different typeset for non-isomorphic total groups. The total tents of the above \(K(c)\) and \(K(d)\) are the same, consisting of all primes with more than two holes. Thus the same numerator \(X'\) yields \(G(K(c)) = X'/K(c)\) and \(G(K(d)) = X'/K(d)\); these total groups have the same total tent (and the same partition and configuration), but they are non-isomorphic: their typesets are different, an inheritance from \(G(c)\) and \(G(d)\) by Proposition 2.6.

**Example 6.d.** A Kind 2 situation. Let \(k = 5, n = 4, \mathcal{A} = \{\{1, 2\}, \{3\}, \{4\}, \{5\}, \{6\}\}\), \(G = G(K)\),

\[
M^T = \begin{bmatrix}
1 & 0 & 0 & 0 & P_1 \\
1 & 1 & 2 & 1 & P_2 \\
1 & -2 & -1 & -2 & P_3 \\
1 & -3 & 1 & 0 & P_4 \\
1 & -3 & 1 & -1 & P_5
\end{bmatrix}
\]

Here \(P_3, P_4, P_5\) generate a plane, hence \(S = \{3, 4, 5\}\) is almost independent: we have a Kind 2 situation. The configuration then yields a relation \(\alpha g_5 + \beta g_6 = 0\) (which might not be evident from the basic relations), with \(t_5 = t_6\); in fact, subtracting the second basic relation (column) from the fourth we get the relation \(3g_5 + 2g_6 = 0\), which can replace the fourth basic relation \(g_3 - 2g_4 - g_6 = 0\). We can then eliminate this relation and \(g_6\), while replacing \(g_5\) by \(h_5 = (1 - 3/2)g_5 = -1/2g_5 = g_5 + g_6\); \(G\) will be represented as a \(B(3)\)-group; in fact, as the previous group \(G(b)\) of Example 3.2 and Fig. 2. Note that both the above \(G(a)\) and \(G(b)\) are Kind 2—say, with respect to the lobe \(A_3 \cup A_4\): Exercise!

**Example 6.e.** \(\text{Maxfam}(G)\) is in general not closed with respect to inf. Let \(m = k = 6\) (so \(A = \text{min}\); \(n = 3\), so we are in the plane \(PS(2)\);

\[
M = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & -1 & 0 & 1 & -1
\end{bmatrix};
\]

and let \(G = G(K)\). The configuration is in Fig. 6.
Exercise. Use Proposition 4.2 to compute $\text{maxfam}(g_1)$. Solution: note that here hyperplanes are lines; the lines that exclude $P_1$ are $P_4P_5P_6$, $P_2P_4$, $P_2P_5$, $P_2P_6$, $P_3P_4$, $P_3P_5$, $P_3P_6$; hence $\text{maxfam}(g_1) = \{[2, 3, 4, 5, 6], [1, 4, 5, 6], [1, 3, 4], [1, 3, 5], [1, 3, 6], [1, 2, 4], [1, 2, 5], [1, 2, 6]\}$, while $\text{maxfam}(g_2)$ is obtained by switching 1 and 2 (a symmetry visible from the configuration). Here $\inf = \wedge$; $\text{maxfam}(g_1) \wedge \text{maxfam}(g_2)$ contains $\{3, 4, 5, 6\}$; starting with an element with that zero-block—say, $\lambda_1g_1 + \lambda_2g_2$—and adding a linear combination of the relations, a simple linear computation shows that imposing to the element to have all other zero-blocks of the infimum forces it to be zero (see Section 2.E); hence the infimum does not belong to $\text{Maxfam}(G)$.

7. Problems

The following schemes (see Fig. 7) are meant for fixed $m, n$; the first is for generic $B(n)$-groups, the second for total groups. They must be read e.g. as follows: Column 8 of the first: are there two $B(n)$-groups of the same rank with the same tent, basic partitions and typeset, but different configurations? Yes, see Example 6.a. We might also have added a fifth row indexed “≈,” which (under equal typesets) would point to the base change problem. In particular it

![Diagram](image)

Fig. 6.

\begin{center}
\begin{tabular}{|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|}
\hline
& 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\
\hline
\text{Conf.} & $\neq$ & $\neq$ & $\neq$ & $\neq$ & $\neq$ & $\neq$ & $\neq$ & $=$ & $=$ & $=$ & $=$ & $=$ & $=$ & $=$ & $=$ \\
\hline
\text{A} & $\neq$ & $\neq$ & $\neq$ & $\neq$ & $\neq$ & $\neq$ & $\neq$ & $\neq$ & $\neq$ & $\neq$ & $\neq$ & $\neq$ & $\neq$ & $\neq$ \smallskip \\
\hline
\text{tent} & $\neq$ & $\neq$ & $\neq$ & $\neq$ & $\neq$ & $\neq$ & $\neq$ & $\neq$ & $\neq$ & $\neq$ & $\neq$ & $\neq$ & $\neq$ & $\neq$ \smallskip \\
\hline
\text{typeset} & $\neq$ & $\neq$ & $\neq$ & $\neq$ & $\neq$ & $\neq$ & $\neq$ & $\neq$ & $\neq$ & $\neq$ & $\neq$ & $\neq$ & $\neq$ & $\neq$ \smallskip \\
\hline
\hline
\text{BCp} & & & & & & & & & & & & & & & \smallskip \\
\hline
\end{tabular}
\end{center}

\begin{center}
\begin{tabular}{|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|}
\hline
& 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\
\hline
\text{Conf.} & $\neq$ & $\neq$ & $\neq$ & $\neq$ & $\neq$ & $\neq$ & $\neq$ & $\neq$ & $\neq$ & $\neq$ & $\neq$ & $\neq$ & $\neq$ & $\neq$ & $\neq$ \\
\hline
\text{A} & $\neq$ & $\neq$ & $\neq$ & $\neq$ & $\neq$ & $\neq$ & $\neq$ & $\neq$ & $\neq$ & $\neq$ & $\neq$ & $\neq$ & $\neq$ & $\neq$ & $\neq$ \smallskip \\
\hline
\text{tent} & $\neq$ & $\neq$ & $\neq$ & $\neq$ & $\neq$ & $\neq$ & $\neq$ & $\neq$ & $\neq$ & $\neq$ & $\neq$ & $\neq$ & $\neq$ & $\neq$ & $\neq$ \smallskip \\
\hline
\text{typeset} & $\neq$ & $\neq$ & $\neq$ & $\neq$ & $\neq$ & $\neq$ & $\neq$ & $\neq$ & $\neq$ & $\neq$ & $\neq$ & $\neq$ & $\neq$ & $\neq$ & $\neq$ \smallskip \\
\hline
\hline
\text{P3.2} & P3.2 & P3.2 & P3.2 & P3.2 & P3.2 & C3.3 & C3.3 & C3.3 & P3.2 & P3.2 & C3.3 & C3.3 & C3.3 & E5.c \smallskip \\
\hline
\end{tabular}
\end{center}

Fig. 7. Conf. = Configuration, BCp = base change problem, C = Corollary, E = Example, P = Proposition, Y = yes, N = no, ? = open problem.
would be interesting to ask (from column 8) whether non-degenerate total groups with different configurations can be isomorphic.

References