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Tractability of Integration in Non-periodic and Periodic Weighted Tensor Product Hilbert Spaces

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We study strong tractability and tractability of multivariate integration in the worst case setting. This problem is considered in weighted tensor product reproducing kernel Hilbert spaces. We analyze three variants of the classical Sobolev space of non-periodic and periodic functions whose first mixed derivatives are square integrable. We obtain necessary and sufficient conditions on strong tractability and tractability in terms of the weights of the spaces. For the three Sobolev spaces periodicity has no significant effect on strong tractability and tractability. In contrast, for general reproducing kernel Hilbert spaces anything can happen: we may have strong tractability or tractability for the non-periodic case and intractability for the periodic one, or vice versa.

1. INTRODUCTION

In this paper we investigate tractability of integration in the worst case setting. Integration is defined over the d dimensional unit cube for weighted tensor product reproducing kernel Hilbert spaces of non-periodic and periodic functions. Tractability means that the minimal number of function values needed to reduce the initial error by a factor ε by deterministic algorithms in the worst case setting is polynomial in d and ε^{-1}. Here,

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the initial error is given by the norm of the integration operator, and we want to reduce the error for functions from the unit ball of the corresponding Hilbert space. Strong tractability is the special case of tractability in which the minimal number of function values is bounded independently of d and depends polynomially on ε^{-1}.

We consider three variants of the classical Sobolev space of non-periodic and periodic functions whose first mixed derivatives are square integrable. These variants differ by the choice of a norm. More precisely, we consider weighted tensor products of the univariate Sobolev spaces which consist of absolutely continuous functions whose first derivatives belong to $L_2([0, 1])$. The univariate norms are of the form

$$\|f\| = (A^2(f) + \gamma^{-1} \|f'\|^2_{L_2([0,1])})^{1/2}$$

with three different choices of $A(f)$:

$$A(f) = \|f\|_{L_2([0,1])}, \qquad A(f) = f(0), \qquad A(f) = \int_0^1 f(t)\,dt.$$

Here, γ is a positive weight, and we take tensor products of such univariate spaces with possibly different γ_j. The weights γ_j moderate the behavior of functions: a small weight γ_j for a function f of norm 1 means that such a function depends weakly on the jth variable. We assume that $\gamma_1 \geqslant \gamma_2 \geqslant \cdots$, which means that the successive components are of decreasing importance.

Each of these Sobolev spaces is a reproducing kernel Hilbert space with an explicitly known reproducing kernel. Our analysis depends on properties of the reproducing kernels. For the first choice of $A(f)$, the reproducing kernel is especially intriguing. Its explicit form was found in [6].

The Sobolev space with the first choice of $A(f)$ is probably the most common one. Nevertheless, tractability of integration for this space has not been previously studied, whereas the other cases have been studied at least partially. In particular, the second choice of $A(f)$ has been studied in [3, 4], and the third choice of $A(f)$, which leads to the non-periodic and periodic Korobov spaces, has been studied in the periodic case in [2, 5]. For all six Sobolev spaces, we obtain a necessary and sufficient condition for strong tractability and tractability in terms of the weights γ_j.

In particular, strong tractability holds in all six cases iff

$$\sum_{j=1}^{\infty} \gamma_j < \infty,$$

and tractability holds in all six cases iff

$$\limsup_{d\to\infty} \frac{\sum_{j=1}^{d} \gamma_j}{\ln d} < \infty.$$

Such conditions on the weights for strong tractability and tractability are typical, see [2–5].

A key concern of our paper is to investigate the role of periodicity, which is more subtle than might at first appear. This is because we compare the worst case error with the initial error. Periodicity always restricts the class of integrands as compared with the non-periodic case, which of itself makes the problem easier. At the same time periodicity may, and sometimes does, reduce the initial error, and that makes the problem harder. A priori it is not clear which of these two effects has the more significance for tractability. For the three classical Sobolev spaces periodicity has no significant effect for tractability. For general weighted tensor product spaces anything can happen. We provide an example for which the periodic case is tractable whereas the non-periodic case is intractable. The opposite may also happen. That is, we provide an example for which the non-periodic case is tractable for some weights, whereas the periodic case is intractable for all weights.

2. THREE SOBOLEV SPACES

We first define the three Sobolev spaces of not necessarily periodic real functions defined over $[0, 1]^d$. This will be done by taking the weighted tensor product of the Sobolev spaces of univariate real functions defined over $[0, 1]$. All these spaces are Hilbert spaces with reproducing kernels; see [1, 8] for the theory of such spaces. In the second part of this section we present periodic variants of the three Sobolev spaces by imposing the condition for the univariate case that functions have the same values at the boundary points.

2.1. *Non-periodic Case*

The First Sobolev Space. Let $W_{\gamma,1}$ be the Sobolev space of absolutely continuous real functions defined over $[0, 1]$ whose first derivatives belong to $L_2([0, 1])$. The inner product in the space $W_{\gamma,1}$ is defined as

$$\langle f, g\rangle_{W_{\gamma,1}} = \int_0^1 f(t)\, g(t)\, dt + \gamma^{-1} \int_0^1 f'(t)\, g'(t)\, dt \qquad \forall f, g \in W_{\gamma,1},$$

where γ is a positive parameter. The reproducing kernel $K_{\gamma,1}$ of this space can be found in [6], and is of the form

$$K_{\gamma,1}(x,t)=\frac{\sqrt{\gamma}}{\sinh\sqrt{\gamma}}\cosh(\sqrt{\gamma}\,(1-\max(x,t)))\cosh(\sqrt{\gamma}\min(x,t)),$$
$$\forall x,t\in[0,1].$$

It is easy to check the reproducing kernel property,

$$\langle K_{\gamma,1}(x,\cdot),f\rangle_{W_{\gamma,1}}=f(x)\qquad\forall f\in W_{\gamma,1},\ x\in[0,1].$$

It is also easy to see that $K_{\gamma,1}(x,x)\geqslant K_{\gamma,1}(\frac{1}{2},\frac{1}{2})>1$ for all $x\in[0,1]$, and that for small γ we have

$$\begin{aligned}K_{\gamma,1}(x,t)&=1+\frac{\gamma}{2}\left((1-\max(x,t))^2+(\min(x,t))^2-\frac{1}{3}\right)+O(\gamma^2)\\&=1+\frac{\gamma}{2}\left(B_2(|x-t|)+2\left(x-\frac{1}{2}\right)\left(t-\frac{1}{2}\right)\right)+O(\gamma^2)\end{aligned}$$

and

$$K_{\gamma,1}(x,x)=1+\gamma(B_2(x)+\tfrac{1}{6})+O(\gamma^2),$$

where $B_2(x)=x^2-x+1/6$ is the Bernoulli polynomial of degree 2.

Since $f(x)=\langle K_{\gamma,1}(x,\cdot),f\rangle_{W_{\gamma,1}}$ for all $f\in W_{\gamma,1}$, by taking $f=1$ we obtain

$$\int_0^1 K_{\gamma,1}(x,t)\,dt=1\qquad\forall x\in[0,1].$$

This proves that univariate integration

$$I_1(f):=\int_0^1 f(t)\,dt=\langle f,1\rangle_{W_{\gamma,1}}$$

is a continuous linear functional with the representer $h_1=1$. We have, independently of γ,

$$\|I_1\|_{W_{\gamma,1}}=\|h_1\|_{W_{\gamma,1}}=1.$$

We now turn to the d-variate case. We take

$$W_{d,\gamma,1} = W_{\gamma_1,1} \otimes W_{\gamma_2,1} \otimes \cdots \otimes W_{\gamma_d,1}$$

as the tensor product of $W_{\gamma_j,1}$ for possibly different positive γ_j. Throughout this paper we assume that

$$\gamma_1 \geqslant \gamma_2 \geqslant \cdots \geqslant \gamma_d \geqslant \cdots > 0.$$

We then obtain the Sobolev space $W_{d,\gamma,1}$ of d-variate real functions defined over $[0, 1]^d$, which is a Hilbert space with the inner product

$$\langle f, g \rangle_{W_{d,\gamma,1}} = \sum_{u \subseteq \{1,2,\dots,d\}} \prod_{j \in u} \gamma_j^{-1} \int_{[0,1]^d} \frac{\partial^{|u|} f}{\partial x_u}(x) \frac{\partial^{|u|} g}{\partial x_u}(x)\, dx.$$

Here, $x = [x_1, x_2, \dots, x_d]$ and x_u denotes the vector of the $|u|$ components such that $(x_u)_i = x_i$ for all $i \in u$. For $u = \varnothing$, the product $\prod_{j \in u} \gamma_j^{-1}$ is replaced by 1, and the integrand is $f(x)\, g(x)$.

The reproducing kernel of $W_{d,1,\gamma}$ is the product of $K_{\gamma_j,1}$ taken for the successive components of the vectors x and t,

$$K_{d,\gamma,1}(x, t) = \prod_{j=1}^{d} K_{\gamma_j,1}(x_j, t_j) \qquad \forall x, t \in [0, 1]^d.$$

Clearly, multivariate integration

$$I_d(f) := \int_{[0,1]^d} f(t)\, dt = \langle f, 1 \rangle_{W_{d,\gamma,1}}$$

is a continuous linear functional with the representer $h_d = 1$, and $\|I_d\|_{W_{d,\gamma,1}} = 1$ for all γ.

The Second Sobolev Space. Let $W_{\gamma,2}$ be the Sobolev space of functions which is algebraically the same as the space $W_{\gamma,1}$ but has the different inner product given by

$$\langle f, g \rangle_{W_{\gamma,2}} = f(0)\, g(0) + \gamma^{-1} \int_0^1 f'(t)\, g'(t)\, dt \qquad \forall f, g \in W_{\gamma,2}.$$

The well known reproducing kernel $K_{\gamma,2}$ is given by (as is easily verified)

$$K_{\gamma,2}(x, t) = 1 + \gamma \min(x, t).$$

Univariate integration

$$I_1(f) := \int_0^1 f(t)\, dt = \langle f, h_1 \rangle_{W_{\gamma,2}}$$

is again a continuous linear functional, this time with the representer

$$h_1(x) = \int_0^1 K_{\gamma,2}(x, t)\, dt = 1 + \gamma(x - x^2/2).$$

In this case, we have

$$\|I_1\|_{W_{\gamma,2}} = \|h_1\|_{W_{\gamma,2}} = (1 + \gamma/3)^{1/2}.$$

As before, the d-variate case is given by

$$W_{d,\gamma,2} = W_{\gamma_1,2} \otimes W_{\gamma_2,2} \otimes \cdots \otimes W_{\gamma_d,2}$$

as the tensor product of $W_{\gamma_j,2}$ for possibly different positive γ_j. The Sobolev space $W_{d,\gamma,2}$ of d-variate real functions defined over $[0,1]^d$ is a Hilbert space with the inner product

$$\langle f, g \rangle_{W_{d,\gamma,2}} = \sum_{u \subseteq \{1,2,\ldots,d\}} \prod_{j \in u} \gamma_j^{-1} \int_{[0,1]^{|u|}} \frac{\partial^{|u|} f}{\partial x_u}(x_u, 0)\, \frac{\partial^{|u|} g}{\partial x_u}(x_u, 0)\, dx_u.$$

Here, $(x_u, 0)$ denotes the vector of d components such that $(x_u, 0)_i = x_i$ for all $i \in u$, and $(x_u, 0)_i = 0$ for all $i \notin u$. The reproducing kernel of $W_{d,\gamma,2}$ is simply

$$K_{d,\gamma,2}(x, t) = \prod_{j=1}^{d} K_{\gamma_j,2}(x_j, t_j) \qquad \forall x, t \in [0,1]^d.$$

Again multivariate integration

$$I_d(f) := \int_{[0,1]^d} f(t)\, dt = \langle f, h_d \rangle_{W_{d,\gamma,1}}$$

is a continuous linear functional, now with the representer

$$h_d(x) = \prod_{j=1}^{d} (1 + \gamma_j(x_j - x_j^2/2)),$$

and

$$\|I_d\|_{W_{d,\gamma,2}} = \|h_d\|_{W_{d,\gamma,2}} = \prod_{j=1}^{d} (1+\gamma_j/3)^{1/2}.$$

Note that $\|I_d\|$ is uniformly bounded in d iff $\sum_{j=1}^{\infty} \gamma_j < \infty$.

The Third Sobolev Space. To obtain the third Sobolev space $W_{\gamma,3}$ we take algebraically the same space $W_{\gamma,1}$ and equip it with the inner product

$$\langle f, g\rangle_{W_{\gamma,3}} = \int_0^1 f(t)\,dt \int_0^1 g(t)\,dt + \gamma^{-1} \int_0^1 f'(t)\,g'(t)\,dt \qquad \forall f, g \in W_{\gamma,3}.$$

It is easy to check that the reproducing kernel $K_{\gamma,3}$ is now equal to

$$K_{\gamma,3}(x, t) = 1 + \frac{\gamma}{2}\,(B_2(|x-t|) + 2(x-1/2)(t-1/2)) \qquad \forall x, t \in [0, 1].$$

As for the first Sobolev space we have $\int_0^1 K_{\gamma,3}(x, t)\,dt = 1$ for all $x \in [0, 1]$. Therefore we have for univariate integration

$$I_1(f) := \int_0^1 f(t)\,dt = \langle f, 1\rangle_{W_{\gamma,3}}.$$

Hence, it is again a continuous linear functional with the representer $h_1 = 1$ and $\|I_1\|_{W_{\gamma,3}} = 1$.

The d-variate case is given by the tensor product

$$W_{d,\gamma,3} = W_{\gamma_1,3} \otimes W_{\gamma_2,3} \otimes \cdots \otimes W_{\gamma_d,3}.$$

The Sobolev space $W_{d,\gamma,3}$ of d-variate real functions defined over $[0, 1]^d$ is a Hilbert space with the inner product

$$\langle f, g\rangle_{W_{d,\gamma,3}} = \sum_{u \subseteq \{1,2,\ldots,d\}} \prod_{j\in u} \gamma_j^{-1} \int_{[0,1]^{|u|}} \left(\int_{[0,1]^{d-|u|}} \frac{\partial^{|u|} f}{\partial x_u}(x)\,dx_{-u} \right) \times \left(\int_{[0,1]^{d-|u|}} \frac{\partial^{|u|} g}{\partial x_u}(x)\,dx_{-u} \right) dx_u.$$

Here, x_{-u} denotes the vector $x_{\{1,2,\ldots,d\}-u}$. For $u = \emptyset$ and $u = \{1, 2, \ldots, d\}$, the integral $\int_{[0,1]^0} f(x_u)\,dx_u$ is replaced by 1. The reproducing kernel of $W_{d,\gamma,3}$ is simply

$$K_{d,\gamma,3}(x, t) = \prod_{j=1}^{d} K_{\gamma_j,3}(x_j, t_j) \qquad \forall x, t \in [0, 1]^d.$$

Again multivariate integration

$$I_d(f) := \int_{[0,1]^d} f(t)\,dt = \langle f, 1\rangle_{W_{d,\gamma,3}}$$

is a continuous linear functional with the representer $h_d = 1$ and $\|I_d\|_{W_{d,\gamma,3}} = 1$.

This concludes the definition of the three Sobolev spaces of non-periodic real functions. They are defined by the tensor products of the univariate spaces that are algebraically the same but differ by the choice of the inner products. Obviously, the norms of the three Sobolev norms are equivalent, that is, there are positive numbers $c_{d,\gamma,i}$ and $C_{d,\gamma,i}$ such that

$$c_{d,\gamma,i}\,\|f\|_{W_{d,\gamma,1}} \leqslant \|f\|_{W_{d,\gamma,i}} \leqslant C_{d,\gamma,i}\,\|f\|_{W_{d,\gamma,1}} \qquad \forall f \in W_{d,\gamma,1} \qquad \text{and} \qquad i = 2, 3,$$

but the ratio $C_{d,\gamma,i}/c_{d,\gamma,i}$ may be exponentially large in d. Indeed, for $f = 1$ we have $\|f\|_{W_{d,\gamma,i}} = 1$ for all $i = 1, 2, 3$ showing that $C_{d,\gamma,i} \geqslant 1$. From the Cauchy–Schwarz inequality we have

$$\|f\|_{W_{d,\gamma,3}} \leqslant \|f\|_{W_{d,\gamma,1}} \qquad \forall f \in W_{d,\gamma,1},$$

and hence $C_{d,\gamma,3} = 1$. For $f(x) = x_1 x_2 \ldots x_d$ we have

$$\|f\|_{W_{d,\gamma,1}} = \prod_{j=1}^{d} \left(\frac{1}{3} + \frac{1}{\gamma_j}\right)^{1/2},$$

$$\|f\|_{W_{d,\gamma,2}} = \prod_{j=1}^{d} \frac{1}{\gamma_j},$$

$$\|f\|_{W_{d,\gamma,3}} = \prod_{j=1}^{d} \left(\frac{1}{4} + \frac{1}{\gamma_j}\right)^{1/2}.$$

This shows that

$$c_{d,\gamma,2} \leqslant \prod_{j=1}^{d} \left(1 + \frac{1}{3}\gamma_j\right)^{-1/2},$$

$$c_{d,\gamma,3} \leqslant \prod_{j=1}^{d} \left(\frac{1 + \frac{1}{4}\gamma_j}{1 + \frac{1}{3}\gamma_j}\right)^{-1/2}.$$

Hence, we have an exponential dependence of $C_{d,\gamma,i}/c_{d,\gamma,i}$ on d for $i = 2, 3$ if $\sum_{j=1}^{\infty} \gamma_j = \infty$. This holds for the classical unweighted Sobolev spaces with $\gamma_j = 1$.

We add that the Sobolev spaces $H_d = W_{d,\gamma,i}$ are related to each other when we vary d while keeping i and γ fixed. Indeed, for $s \leqslant d$ we have in each case

$$H_s \subseteq H_d \qquad \text{and} \qquad \|f\|_{H_s} = \|f\|_{H_d} \qquad \forall f \in H_s.$$

That is, a function of s variables from H_s, when treated as a function of d variables with no dependence on the last $d-s$ variables, also belongs to H_d with the same norm as in H_s. This means that we have an increasing sequence of spaces $H_1 \subseteq H_2 \subseteq \cdots \subseteq H_d$, and an increasing sequence of the unit balls B_d of H_d, $B_1 \subseteq B_2 \subseteq \cdots \subseteq B_d$, and $H_s \cap B_d = B_s$.

The first Sobolev space is probably the most classical one and often used in the study of partial differential equations. Tractability issues of multivariate integration have not been so far studied for this space. The second Sobolev space is a typical example of a Hilbert space for which tractability issues for tensor product functionals have been studied; see [3, 4, 11]. The third Sobolev space corresponds to the non-periodic variant of the Korobov space of periodic functions. For the latter, tractability of multivariate integration has been studied in [2, 5]. The exponential dependence between the norms of the Sobolev spaces for some weights does not allow one to conclude tractability of integration in one space in terms of the other space.

We have assumed that all the weights γ_j are positive. It is easy to see that we can also cover the zero weight γ_j by letting a positive γ_j tend to zero. The zero weight γ_j means that the functions f in the space do not depend on the jth variable.

2.2. *Periodic Case*

We now present the variants of the three Sobolev spaces for periodic functions. This will be done by assuming that for the univariate case we only allow functions f for which $f(0) = f(1)$. The construction goes as follows.

Let $H(K)$ be a Hilbert space of univariate functions defined over $[0, 1]$ with the reproducing kernel K. We consider the subspace of $H(K)$ consisting of periodic functions,

$$\widetilde{H(K)} = \{f \in H(K) : f(0) = f(1)\}.$$

It is known that $\widetilde{H(K)} = H(\tilde{K})$ is a Hilbert space with the same inner product as $H(K)$, and that the reproducing kernel for this space is

$$\tilde{K}(x,t) = K(x,t) - \frac{g(x)\, g(t)}{\|g\|^2_{H(K)}},$$

where g is the representer of the linear functional $L(f) = f(1) - f(0) = \langle f, g \rangle_{H(K)}$. This yields $g(x) = K(x, 1) - K(x, 0)$ and

$$\|g\|_{H(K)}^2 = \langle g, g \rangle_{H(K)} = K(1, 1) - 2K(1, 0) + K(0, 0) = g(1) - g(0).$$

It may happen that $g = 0$ if we start with the space $H(K)$ of periodic functions. To deal with this case we adopt the convention that $0/0 = 0$, and then $\tilde{K} = K$.

We can verify that $\tilde{K}$ is indeed the reproducing kernel by noting that $\langle f, g \rangle_{H(K)} = 0$ for all f in $H(\tilde{K})$, and that consequently

$$\langle \tilde{K}(x, \cdot), f \rangle_{H(K)} = \langle K(x, \cdot), f \rangle_{H(K)} = f(x) \qquad \forall f \in H(\tilde{K}).$$

Hence, K and $\tilde{K}$ both reproduce $f(x)$ for functions from $H(\tilde{K})$. However, K is not a reproducing kernel in the space $H(\tilde{K})$ because $K(x, \cdot) \notin H(\tilde{K})$, since in general $K(x, 1) \neq K(x, 0)$. In contrast, $\tilde{K}$ is constructed to satisfy $\tilde{K}(x, 1) = \tilde{K}(x, 0)$ so that $\tilde{K}(x, \cdot) \in H(\tilde{K})$ for $x \in [0, 1]$.

We now consider a general linear continuous functional $L(f) = \langle f, f^* \rangle_{H(K)}$ for $f \in H(K)$. The same functional over $H(\tilde{K})$ takes the form

$$L(f) = \langle f, \tilde{f}^* \rangle_{H(\tilde{K})} \qquad \text{with} \quad \tilde{f}^*(x) = f^*(x) - \frac{\langle f^*, g \rangle_{H(K)}}{\|g\|_{H(K)}^2}\, g(x).$$

In particular, we will use this form of the representer for univariate integration over the three Sobolev spaces.

Using this general construction for the Sobolev spaces $W_{\gamma, i}$ with $i = 1, 2, 3$, we obtain the Sobolev spaces $\tilde{W}_{\gamma, i}$ of periodic functions. It is easy to show that their reproducing kernels are

$$\tilde{K}_{\gamma, 1}(x, t) = K_{\gamma, 1}(x, t) - a(\sinh(b(x - 1/2)) \sinh(b(t - 1/2))),$$
$$\tilde{K}_{\gamma, 2}(x, t) = 1 + \gamma(\min(x, t) - xt),$$
$$\tilde{K}_{\gamma, 3}(x, t) = 1 + \frac{\gamma}{2} B_2(|x - t|),$$

where $a = \sqrt{\gamma}/\sinh \sqrt{\gamma}$ and $b = \sqrt{\gamma}$. For small γ, we have

$$\tilde{K}_{\gamma, 1}(x, t) = 1 + \frac{\gamma}{2}\left((1 - \max(x, t))^2 + (\min(x, t))^2 - \frac{1}{3} - 2(x - 1/2)(t - 1/2)\right) + O(\gamma^2).$$

For $i = 3$, we have the weighted Korobov space $\tilde{W}_{\gamma, 3}$ of periodic functions.

We obtain the d-variate case by taking the tensor products. Hence,

$$\tilde{W}_{d,\gamma,i} = \tilde{W}_{\gamma_1,i} \otimes \tilde{W}_{\gamma_2,i} \otimes \cdots \otimes \tilde{W}_{\gamma_d,i},$$

and the inner product of $\tilde{W}_{d,\gamma,i}$ is the same as the inner product of $W_{d,\gamma,i}$, whereas the reproducing kernel is

$$\tilde{K}_{d,\gamma,i}(x,t) = \prod_{j=1}^{d} \tilde{K}_{\gamma_j,i}(x_j,t_j).$$

The Sobolev spaces $\tilde{W}_{d,\gamma,i}$ consist of periodic functions in the sense that

$$f(x_1, ..., x_{j-1}, 0, x_{j+1}, ..., x_d) = f(x_1, ..., x_{j-1}, 1, x_{j+1}, ..., x_d)$$

for all $x_i \in [0,1]$ with $i \neq j$ and all $j = 1, 2, ..., d$.

We now consider multivariate integration I_d for functions from $\tilde{W}_{d,\gamma,i}$. To obtain its representer, we use the general construction outlined above. For $i=1$ and $i=3$, we have $h_1 = 1$, and

$$\langle h_1, K_{\gamma,i}(\cdot,1)\rangle_{W_{\gamma,i}} = \langle h_1, K_{\gamma,i}(\cdot,0)\rangle_{W_{\gamma,i}} = 1.$$

Therefore $\langle h_1, g\rangle_{W_{\gamma,i}} = 0$, and there is no change of the representer. For $i=2$, there is a change of the representer that is easy to obtain. We thus have

$$I_d(f) = \int_{[0,1]^d} f(t)\,dt = \langle f, \tilde{h}_d\rangle_{\tilde{W}_{d,\gamma,i}} \qquad \forall f \in \tilde{W}_{d,\gamma,i},$$

with

$$\begin{aligned} &\text{for} \quad i=1, \qquad \tilde{h}_d(x) = 1,\\ &\text{for} \quad i=2, \qquad \tilde{h}_d(x) = \prod_{j=1}^{d} (1+\gamma_j(x_j - x_j^2)/2),\\ &\text{for} \quad i=3, \qquad \tilde{h}_d(x) = 1. \end{aligned}$$

Observe that for $i=2$ we now have $\|I_d\|_{W_{d,\gamma,2}} = \prod_{j=1}^{d} (1+\gamma_j/12)^{1/2}$.

3. TRACTABILITY

We analyze tractability of integration in the worst case setting. Let H_d be a reproducing kernel Hilbert space of real functions defined over $[0,1]^d$ with the norm $\|\cdot\|_{H_d}$, and with reproducing kernel K_d. Later in this section we take H_d to be one of the Sobolev spaces $W_{d,\gamma,i}$ or $\widetilde{W}_{d,\gamma,i}$ for $i=1,2,3$, whereas different spaces H_d will be taken in further sections. We assume that

$$I_d(f)=\int_{[0,1]^d} f(t)\,dt \qquad \forall f\in H_d$$

is a well defined continuous linear functional. This holds if $h_d(x)=\int_{[0,1]^d} K_d(x,t)\,dt$ belongs to H_d. Then $I_d(f)=\langle f,h_d\rangle_{H_d}$.

We approximate the multivariate integrals $I_d(f)$ for f from H_d by algorithms of the form

$$Q_{n,d}(f)=\sum_{k=1}^{n} a_k f(t_k)$$

for deterministically chosen $a_k\in\mathbb{R}$ and $t_k\in[0,1]^d$.[1] The worst case error of $Q_{n,d}$ is defined as

$$e(Q_{n,d};H_d)=\sup\{|I_d(f)-Q_{n,d}|\colon f\in H_d,\ \|f\|_{H_d}\leqslant 1\}.$$

For $n=0$, we formally define $Q_{0,d}=0$ and its error $e(Q_{0,d};H_d)=\|I_d\|_{H_d}$ is the initial error that can be achieved without sampling the function. For $\varepsilon\in(0,1)$, let

$$n(\varepsilon,H_d)=\min\{n\colon \exists Q_{n,d} \text{ such that } e(Q_{n,d};H_d)\leqslant\varepsilon\,\|I_d\|_{H_d}\}$$

be the minimal number of function values needed to reduce the initial error by a factor of ε. Multivariate integration is *tractable* in H_d iff there exist non-negative numbers C, p, q such that

$$n(\varepsilon,H_d)\leqslant C\varepsilon^{-p}\,d^q \qquad \forall\varepsilon\in(0,1) \quad\text{and}\quad d\geqslant 1,$$

and is *strongly* tractable if $q=0$ in the last bound.

[1] There is no need to consider more general algorithms since non-linear algorithms and adaptive choice of t_j does not reduce the worst case error; see, e.g., [7].

We are ready to present conditions on strong tractability and tractability for the six Sobolev spaces in terms of the weights γ_j. Some of these conditions have been known as indicated in the proof of the following theorem.

THEOREM 1. *Let H_d be one of the six Sobolev spaces $W_{d,\gamma,i}$ or $\tilde{W}_{d,\gamma,i}$ for $i = 1, 2, 3$.*

(i) *Multivariate integration is strongly tractable in H_d iff*

$$\sum_{j=1}^{\infty} \gamma_j < \infty.$$

If so, then there is a non-negative number C such that

$$n(\varepsilon, H_d) \leqslant C\varepsilon^{-2} \qquad \forall \varepsilon \in (0, 1), \qquad \forall d \geqslant 1.$$

(ii) *Multivariate integration is tractable in H_d iff*

$$a := \limsup_{d \to \infty} \frac{\sum_{j=1}^{d} \gamma_j}{\ln d} < \infty.$$

If so, then for any $q > q^*$ there is a nonnegative number C such that

$$n(\varepsilon, H_d) \leqslant C\varepsilon^{-2}\, d^{qa} \qquad \forall \varepsilon \in (0, 1), \qquad \forall\, d \geqslant 1,$$

where $q^* = 1/6$ for the three non-periodic cases, and $q^* = 1/12$ for the three periodic cases.

Proof. We first prove upper bounds on $n(\varepsilon, H_d)$. It is shown in [4, p. 26] that

$$n(\varepsilon, H_d) \leqslant \lceil \rho_d \varepsilon^{-2} \rceil,$$

where

$$\rho_d = \frac{\int_{[0,1]^d} K_d(x, x)\, dx}{\int_{[0,1]^{2d}} K_d(x, t)\, dx\, dt}.$$

For $i = 3$ we get

$$\rho_d = \prod_{j=1}^{d} (1 + \gamma_j \alpha),$$

where $\alpha = 1/6$ for the non-periodic case, and $\alpha = 1/12$ for the periodic case. Clearly, ρ_d is uniformly bounded in d if $\sum_{j=1}^{\infty} \gamma_j < \infty$ and we have strong tractability. We also have

$$\rho_d = e^{\sum_{j=1}^{d} \ln(1+\gamma_j \alpha)} \leqslant d^{\alpha \sum_{j=1}^{d} \gamma_j / \ln d} \leqslant d^{\alpha a + o(1)}$$

for large d. This proves tractability and bounds on $n(\varepsilon, H_d)$ for $i = 3$.

The case $i = 2$ for the non-periodic case was studied in [4]. The periodic case can be easily done by computing the corresponding ρ_d.

For $i = 1$, recall that $\|f\|_{W_{d,\gamma,3}} \leqslant \|f\|_{W_{d,\gamma,1}}$. Hence the unit ball of $W_{d,\gamma,1}$ is a subset of the unit ball of $W_{d,\gamma,3}$, and $e(Q_{n,d}; W_{d,\gamma,1}) \leqslant e(Q_{n,d}; W_{d,\gamma,3})$ for any algorithm $Q_{n,d}$. Since the initial errors of multivariate integration are 1 in both spaces, this proves that multivariate integration over $W_{d,\gamma,1}$ is no harder than multivariate integration over $W_{d,\gamma,3}$.[2]

We now turn to necessary conditions on strong tractability and tractability. For $i = 2$ these were established in [4] for the non-periodic and periodic cases. For $i = 3$ these were established in [2] for the periodic case. Since the non-periodic case is no easier, the necessary conditions from the periodic case also apply.

Hence, it remains to consider necessary conditions for the case $i = 1$. For the periodic and non-periodic case, the initial error is one. This means that the periodic case is no harder than the non-periodic case, and it is enough to study only necessary conditions for the periodic case. Our approach is to apply the results from [3] which state necessary conditions on strong tractability and tractability of the form of Theorem 1 under the assumptions that the reproducing kernel has a decomposable part and that the corresponding components of the representer of univariate integration are not zero.

Consider the nested subspaces of $\tilde{W}_{\gamma,1}$,

$$A_\gamma = \{ f \in \tilde{W}_{\gamma,1} : f(0) = 0 \},$$
$$B_\gamma = \{ f \in \tilde{W}_{\gamma,1} : f(0) = f(1/2) = 0 \}.$$

[2] It can be checked that the exponent q^* cannot be improved by direct computation of ρ_d. Indeed, it is enough to consider small γ_j and use the asymptotic formulas for the kernels $K_{\gamma_j,1}$ and $\tilde{K}_{\gamma_j,1}$

It can be verified that the reproducing kernels of A_γ and B_γ are

$$\begin{aligned} K_{A_\gamma}(x,t) &= \tilde{K}_{\gamma,1}(x,t) - \frac{\tilde{K}_{\gamma,1}(x,0)\,\tilde{K}_{\gamma,1}(0,t)}{\tilde{K}_{\gamma,1}(0,0)} \\ &= a\sinh(b(1-\max(x,t))\sinh(b\min(x,t)), \\ K_{B_\gamma}(x,t) &= K_{A_\gamma}(x,t) - \frac{K_{A_\gamma}(x,1/2)\,K_{A_\gamma}(1/2,t)}{K_{A_\gamma}(1/2,1/2)}, \end{aligned}$$

where, as before, $a = \sqrt{\gamma}/\sinh(\sqrt{\gamma})$ and $b = \sqrt{\gamma}$. It is easy to check that K_{B_γ} is *decomposable* with $a = 1/2$, see [3], i.e.,

$$K_{B_\gamma}(x,t) = 0 \qquad \forall 0 \leqslant x \leqslant 1/2 \leqslant t \leqslant 1.$$

We split the kernel $\tilde{K}_{\gamma,1}$ of $\tilde{W}_{\gamma,1}$ as

$$\tilde{K}_{\gamma,1} = K_{R_\gamma} + K_{B_\gamma},$$

where

$$K_{R_\gamma} = \frac{\tilde{K}_{\gamma,1}(x,0)\,\tilde{K}_{\gamma,1}(0,t)}{\tilde{K}_{\gamma,1}(0,0)} + \frac{K_{A_\gamma}(x,1/2)\,K_{A_\gamma}(1/2,t)}{K_{A_\gamma}(1/2,1/2)}$$

is the reproducing kernel of the two dimensional space

$$R_\gamma := \operatorname{span}(\tilde{K}_{\gamma,1}(\cdot,0), K_{A_\gamma}(\cdot,1/2)).$$

Obviously, $R_\gamma \cap B_\gamma = \{0\}$, and therefore any element f of $\tilde{W}_{\gamma,1}$ has a unique decomposition, $f = f_{R_\gamma} + f_{B_\gamma}$ with $f_{R_\gamma} \in R_\gamma$, $f_{B_\gamma} \in B_\gamma$. We use this decomposition for the representer $h_1 = 1$ of univariate integration. We have $h_{R_\gamma}(x) = 1 - h_{B_\gamma}(x)$ and $h_{B_\gamma}(x) = \int_0^1 K_{B_\gamma}(x,t)\,dt$. Let $h_{B_\gamma,(0)}(x) = h_{B_\gamma}(x)$ for $x \in [0,1/2]$ and $h_{B_\gamma,(0)}(x) = 0$ for $x \in [1/2,1]$, and $h_{B_\gamma,(1)}(x) = h_{B_\gamma}(x) - h_{B_\gamma,(0)}(x)$ for $x \in [0,1]$. Then we have for $x \in [0,1/2]$,

$$h_{B_\gamma,(0)}(x) = \frac{8as_4\sinh(bx/2)}{b}\,(c_4\sinh(b(1-x)/2) - s_4\cosh(bx/2)),$$

and for $x \in [1/2,1]$,

$$h_{B_\gamma,(1)}(x) = \frac{8as_4\sinh(b(1-x)/2)}{b}\,(c_4\sinh(bx/2) - s_4\cosh(b(1-x)/2)),$$

with $s_4 = \sinh(b/4)$ and $c_4 = \cosh(b/4)$. We have $h_{B_\gamma,(0)}(x) = h_{B_\gamma,(1)}(1-x)$ for $x \in [0, 1]$. Therefore

$$\|h_{B_\gamma,(0)}\|_{\tilde{W}_{\gamma,1}} = \|h_{B_\gamma,(1)}\|_{\tilde{W}_{\gamma,1}}.$$

Let $Q_{n,d}$ be an arbitrary algorithm. Then with the definitions above a slight modification of the proof of Theorem 2 in [3] yields

$$e^2(Q_{n,d}; \tilde{W}_{d,\gamma,1}) \geqslant \sum_{k=0}^{d} (1-n2^{-k})_+ \sum_{u \subseteq \{1,2,\ldots,d\},\,|u|=k} \prod_{j \notin u} \|h_{R_{\gamma_j}}\|^2_{\tilde{W}_{\gamma_j,1}} \prod_{j \in u} \|h_{B_{\gamma_j}}\|^2_{\tilde{W}_{\gamma_j,1}}.$$

Observe that the initial error can be written as

$$e^2(Q_{0,d}; \tilde{W}_{d,\gamma,1}) = \sum_{k=0}^{d} \sum_{u \subseteq \{1,2,\ldots,d\},\,|u|=k} \prod_{j \notin u} \|h_{R_{\gamma_j}}\|^2_{\tilde{W}_{\gamma_j,1}} \prod_{j \in u} \|h_{B_{\gamma_j}}\|^2_{\tilde{W}_{\gamma_j,1}}.$$

Hence, we have

$$\frac{e^2(Q_{n,d}; \tilde{W}_{d,\gamma,1})}{e^2(Q_{0,d}; \tilde{W}_{d,\gamma,1})} \geqslant \frac{\sum_{k=0}^{d} C'_{d,k}(1-n2^{-k})_+}{\sum_{k=0}^{d} C'_{d,k}},$$

with

$$C'_{d,k} = \sum_{u \subseteq \{1,2,\ldots,d\},\,|u|=k} \prod_{j \in u} \frac{\|h_{B_{\gamma_j}}\|^2_{\tilde{W}_{\gamma_j,1}}}{\|h_{R_{\gamma_j}}\|^2_{\tilde{W}_{\gamma_j,1}}}.$$

Observe that we can assume that all γ_j are small, say $\gamma_j \leqslant \gamma^*$ for a sufficiently small γ^*, since the decrease of γ_j makes the unit ball of $\tilde{W}_{\gamma,1}$ smaller, and multivariate integration easier. For small γ, it is easy to check that

$$\|h_{B_\gamma,(i)}\|^2_{\tilde{W}_{\gamma,1}} = \frac{\gamma}{96}(1+o(1))), \qquad i = 0, 1.$$

Therefore

$$\frac{\|h_{B_{\gamma_j}}\|^2_{\tilde{W}_{\gamma_j,1}}}{\|h_{R_{\gamma_j}}\|^2_{\tilde{W}_{\gamma_j,1}}} = \frac{\gamma_j}{48-\gamma_j}(1+o(1)).$$

Hence,

$$C'_{d,k} = \sum_{u \subseteq \{1,2,\ldots,d\},\, |u|=k} \prod_{j \in u} \gamma'_j$$

with $\gamma'_j = \gamma_j/48(1+o(1))$.

This form of $C'_{d,k}$ is exactly as in [3], and therefore we can apply the proof of Theorem 3 of [3] to conclude the necessary conditions on strong tractability and tractability of multivariate integration. This completes the proof. ▮

4. THE PERIODIC CASE MAY BREAK INTRACTABILITY

We now provide an example of a space H_d for which multivariate integration is intractable for the non-periodic case, whereas it is strongly tractable for the periodic case. For such a space, periodicity of functions is a very powerful property.

We define the reproducing kernel space H_d by its kernel. For $d=1$, we take

$$K_{1,\gamma}(x,t) = K_1(x,t) + \gamma K_2(x,t) \qquad \forall x, t \in [0,1],$$

where

$$K_1(x,t) = g_1(x)\, g_1(t) + g_2(x)\, g_2(t), \qquad K_2(x,t) = B_2(|x-t|),$$

where $g_1(x) = 0$ for $x \in [0, 1/2]$, $g_1(x) = \sqrt{2}$ for $x \in (1/2, 1]$, and $g_2(x) = \sqrt{2}$ for $x \in [0, 1/2)$, $g_2(x) = 0$ for $x \in [1/2, 1]$. As before, $B_2(u) = u^2 - u + 1/6$ is the Bernoulli polynomial. Note that the functions g_i have disjoint support, and $g_1(x) = (1/\sqrt{2})\, K_1(x, 1)$, $g_2(x) = (1/\sqrt{2})\, K_1(x, 0)$.

Observe that K_i are reproducing kernels, and they generate Hilbert spaces $H(K_i)$ such that

$$H(K_1) = \operatorname{span}(g_1, g_2), \qquad H(K_2) = \{ f \in \tilde{W}_{2,3} : I_1(f) = 0 \}.$$

The space $H(K_1)$ is two dimensional, and it can easily be checked that g_1 and g_2 are orthonormal. Hence, for $f = c_1 g_1 + c_2 g_2$ we have $\|f\|^2_{H(K_1)} = c_1^2 + c_2^2$. We also have $I_1(f) = (c_1 + c_2)\sqrt{2}/2$, and $I_1(f^2) = c_1^2 + c_2^2 = \|f\|^2_{H(K_1)}$.

The space $H(K_2)$ is a subspace of the periodic space $\tilde{W}_{\gamma,3}$ with $\gamma = 2$. Therefore the inner product in $H(K_2)$ is $\langle f, g \rangle_{H(K_2)} = \int_0^1 f'(t)\, g'(t)\, dt/2$.

Clearly, $H(K_1) \cap H(K_2) = \{0\}$ and for any $f \in H(K_{1,\gamma})$ we have a unique representation $f = f_1 + f_2$ with $f_i \in H(K_i)$, and $\|f\|^2_{H(K_{1,\gamma})} = \|f_1\|^2_{H(K_1)} + \gamma \, \|f_2\|^2_{H(K_2)}$. Since $I_1(f) = I_1(f_1)$ and $f(1) - f(0) = f_1(1) - f_1(0)$, we conclude that

$$f_1(t) = \frac{\sqrt{2}}{2}\left(\int_0^1 f(t)\,dt + \frac{f(1)-f(0)}{2}\right) g_1(t) + \frac{\sqrt{2}}{2}\left(\int_0^1 f(t)\,dt - \frac{f(1)-f(0)}{2}\right) g_2(t),$$

and

$$\|f\|^2_{H(K_{1,\gamma})} = \frac{1}{2}\left(\int_0^1 f(t)\,dt + \frac{f(1)-f(0)}{2}\right)^2 + \frac{1}{2}\left(\int_0^1 f(t)\,dt - \frac{f(1)-f(0)}{2}\right)^2 + \frac{1}{2\gamma}\int_0^1 ((f(t) - f_1(t))')^2\,dt.$$

Consider now the univariate integration $I_1(f) = \int_0^1 f(t)\,dt = \langle f, h \rangle_{H(K_{1,\gamma})}$ with

$$h(x) = \int_0^1 K_{1,\gamma}(x,t)\,dt = \frac{\sqrt{2}}{2}(g_1(x) + g_2(x)) = 1_+.$$

Here $1_+(x) = 1$ for all $x \neq 1/2$, and $1_+(1/2) = 0$. We also have $\|h\|_{H(K_{1,\gamma})} = 1$. We stress that the representer h does not have a component in $H(K_2)$ since $\int_0^1 B_2(|x-t|)\,dt = 0$ for all $x \in [0,1]$.

For arbitrary d, we take as always

$$H_d = H(K_{1,\gamma_1}) \otimes \cdots \otimes H(K_{1,\gamma_d})$$

which has the reproducing kernel

$$K_{d,\gamma}(x,t) = \prod_{j=1}^{d} K_{1,\gamma_j}(x_j, t_j).$$

The multivariate integration $I_d(f) = \langle f, h_d \rangle_{H_d}$ has the representer

$$h_d(x) = 1_+(x_1)\, 1_+(x_2) \cdots 1_+(x_d)$$

which is almost everywhere equal to 1, and has norm 1.

From our construction it follows that multivariate integration over H_d is no easier than multivariate integration over

$$F_d = H(K_1) \otimes H(K_1) \otimes \cdots \otimes H(K_1).$$

This follows from the fact that $\|I_d\|_{H_d} = \|I_d\|_{F_d} = 1$ and $e(Q_{n,d}; H_d) \geqslant e(Q_{n,d}; F_d)$ for any algorithm $Q_{n,d}$.

It is known that multivariate integration over F_d is intractable for deterministic algorithms, see [9]. Hence, it is also intractable over H_d and this holds for arbitrary weights γ_j, including even $\gamma_j = 0$.

We now turn to the periodic case. That is, for $d = 1$ we take

$$H(\tilde{K}_{1,\gamma}) = \{f \in H(K_{1,\gamma}) : f(0) = f(1)\}.$$

It is easy to check that

$$\tilde{K}_{1,\gamma}(x, t) = 1_+ + \gamma B_2(|x-t|).$$

Then $f \in H(\tilde{K}_{1,\gamma})$ has the form $f = f_1 + \gamma f_2$ with $f_1(x) = I_1(f)\, 1_+(x)$ and $f_2 \in H(K_2)$. We now have

$$\|f\|^2_{H(\tilde{K}_{1,\gamma})} = I^2(f) + \frac{1}{2\gamma} \int_0^1 ((f(t) - I_1(f))')^2\, dt.$$

For $d \geqslant 1$, we take

$$\tilde{H}_d = H(\tilde{K}_{1,\gamma_1}) \otimes \cdots \otimes H(\tilde{K}_{1,\gamma_d}).$$

The representer of multivariate integration is still $1_+(x)$, with norm one. Basically, the multivariate integration problem over H_d is the same as over $\tilde{W}_{d,2\gamma,3}$ which is strongly tractable iff $\sum_{j=1}^{\infty} \gamma_j < \infty$. For such weights γ_j, the periodic case breaks intractability.

5. THE PERIODIC CASE MAY INTRODUCE INTRACTABILITY

We now present an example of multivariate integration which is strongly tractable in the nonperiodic case for some weights γ_j and becomes intractable in the periodic case for *arbitrary* positive weights. For this example, periodicity of functions introduces intractability.

For $d=1$, similarly as in [10], we take the kernel $K_{1,\gamma}$ of the space $H(K_{1,\gamma})$ as

$$K_{1,\gamma}(x,t)=h(x)\,h(t)+\gamma(\min(x,t)-xt),$$

where a function h is such that $h(0)=1$, $h(1)=0$ and $I_1(h)\neq 0$. Observe that then $f(1)=0$ for all $f\in H(K_{1,\gamma})$. It is easy to check that the inner product in $H(K_{1,\gamma})$ is of the form

$$\langle f,g\rangle_{H(K_{1,\gamma})}=f(0)\,g(0)+\gamma^{-1}\int_0^1 (f(t)-f(0)\,h(t))'(g(t)-g(0)\,h(t))'\,dt.$$

For $d\geqslant 1$, we take the tensor product

$$H_d=H(K_{1,\gamma_1})\otimes\cdots\otimes H(K_{1,\gamma_d})$$

with the reproducing kernel

$$K_d(x,t)=\prod_{j=1}^{d} K_{1,\gamma_j}(x_j,t_j).$$

It is proven in [10] that multivariate integration for deterministic algorithms is strongly tractable if there exists a number $a\in(0,1)$ such that $\sum_{j=1}^{\infty}\gamma_j^a<\infty$. This holds, in particular, for $\gamma_j=j^{-\alpha}$ with $\alpha>1$.

We now turn to the periodic case. For $d=1$, we already have $f(1)=0$, and therefore we need only to assume that $f(0)=0$. That is, we switch to the subspace $\{f\in H(K_{1,\gamma}):f(0)=0\}$ which has the kernel

$$\begin{aligned}\tilde K_{1,\gamma}(x,t)&=K_{1,\gamma}(x,t)-\frac{K_{1,\gamma}(x,0)\,K_{1,\gamma}(t,0)}{K_{1,\gamma}(0,0)}\\&=K_{1,\gamma}(x,t)-h(x)\,h(t)=\gamma(\min(x,t)-xt).\end{aligned}$$

Note that

$$\langle f,g\rangle_{H(\tilde K_{1,\gamma})}=\gamma^{-1}\int_0^1 f'(t)\,g'(t)\,dt.$$

For $d\geqslant 1$ we have $\tilde H_g=H(\tilde K_{1,\gamma_1})\otimes\cdots\otimes H(\tilde k_{1,\gamma_d})$ with inner product

$$\langle f,g\rangle_{\tilde H_d}=(\gamma_1\ldots\gamma_d)^{-1}\int_{[0,1]^d}\frac{\partial^d f(x)}{\partial x_1\ldots\partial x_d}\,\frac{\partial^d g(x)}{\partial x_1\ldots\partial x_d}\,dt.$$

The reproducing kernel $\tilde{K}_d$ takes the form

$$\tilde{K}_d(x, t) = \gamma_1 \dots \gamma_d \prod_{j=1}^{d} (\min(x_j, t_j) - x_j t_j)$$

and multivariate integration I_d has the norm

$$\|I_d\|_{\tilde{H}_d} = (\gamma_1 \dots \gamma_j)^{1/2} \, 12^{-d/2}.$$

It is easy to see that the weights γ_j do not play any role for this periodic case since they cancel when we consider the ratio $e(Q_{n,d}; \tilde{H}_d)/\|I_d\|$ for any algorithm $Q_{n,d}$. Hence it is enough to consider the unweighted case for which it is known that multivariate integration is intractable; see [3].

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