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# MONODROMY AND BETTI NUMBERS OF WEIGHTED COMPLETE INTERSECTIONS

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LET $(X, 0)$ be an isolated singularity of complete intersection in $\mathbf{C}^m$ defined by the weighted homogeneous polynomials $f_i$ of degree $d_i$ with respect to the positive integer weights $wt(X_j) = w_j$ for $i = 1, \ldots, p$ and $j = 1, \ldots, m$.

Let $f: (X, 0) \to (\mathbf{C}, 0)$ be a function germ induced by a weighted homogeneous polynomial of degree $d$ with respect to the weights $\mathbf{w} = (w_1, \ldots, w_m)$ such that $(X_0, 0) = (f^{-1}(0), 0)$ is again an isolated singularity of complete intersection with $n = \dim X_0 = \dim X - 1 \geq 1$. If $\bar{X}_0$ denotes the Milnor fiber of the singularity $(X_0, 0)$, then there is a natural (complex) monodromy operator $h: H^n(\bar{X}_0, \mathbf{C}) \to H^n(\bar{X}_0, \mathbf{C})$ associated to the function $f$ [18].

In the first part of this note we show that this monodromy operator is diagonalizable and compute its characteristic polynomial

$$\Delta(\lambda) = \det(\lambda.\,\mathrm{Id} - h)$$

in terms of the weights $\mathbf{w}$ and the degrees $\mathbf{d} = (d_1, \ldots, d_p)$ and $d$.

In the special case of Brieskorn–Pham singularities this result is due to Hamm [9], not to mention the case when $X$ is smooth, treated already by Milnor and Orlik [11] and Brieskorn [2].

Our proof depends on the relation between the monodromy operator $h$ and the Gauss–Manin connection of the function $f$ (as suggested by an example in Looijenga [10], p. 166) and on the knowledge of the Poincaré series of $\Omega^n_{X_0}/\mathrm{d}\Omega^{n-1}_{X_0}$ computed by Greuel and Hamm [7].

In the second part we derive some topological consequences. Namely, there are two spaces naturally associated to the singularity $(X, 0)$: its link $K = X \cap S$, where $S$ is the unit sphere in $\mathbf{C}^m$ and the quasi-smooth weighted complete intersection $Y$ defined by the polynomials $f_i$ in the weighted projective space $\mathbf{P}(\mathbf{w})$ [4]. We show that the results in the first section allow one to compute the (middle) Betti numbers of $K$ and $Y$ in terms of $\mathbf{w}, \mathbf{d}$. Equivalently, we determine the rank of the intersection form of the Milnor lattice of $(X, 0)$.

We also prove that all the quasi-smooth weighted complete intersections of the same type $(\mathbf{w}, \mathbf{d})$ are homeomorphic.

In the final section we determine the mixed Hodge structure on the cohomology of the Milnor fiber of the singularity $(X, 0)$, when $\dim X \leq 3$, using an idea due to Steenbrink [13].

## §1. THE MONODROMY OPERATOR

Let $\mathcal{O}_k$ denote the C-algebra of germs of holomorphic functions at the origin of $\mathbf{C}^k$, $I_X$ the ideal generated by $f_1, \ldots, f_p$ in $\mathcal{O}_m$.

The weights $\mathbf{w}$ give rise to a filtration on the $\mathcal{O}_m$-module $\Omega^k$ of germs of holomorphic $k$-forms at the origin of $\mathbf{C}^m$, such that a monomial form

$$\varphi = x^a \mathrm{d}x_{i_1} \wedge \ldots \wedge \mathrm{d}x_{i_k}$$

has degree $\deg(\varphi) = \deg(x^a) + w_{i_1} + \ldots + w_{i_k}$, where $\deg(x^a) = a_1 w_1 + \ldots + a_m w_m$.

This filtration induces a filtration (compatible with the derivations) on the stalk at the origin of the sheaf of holomorphic $k$-forms relative $f$: $\Omega^k_f = \Omega^k / I_X \cdot \Omega^k + \mathrm{d}f_1 \wedge \Omega^{k-1} + \ldots + \mathrm{d}f_p \wedge \Omega^{k-1} + \mathrm{d}f \wedge \Omega^{k-1}$.

It is known that

(i) $\Omega^n_f/\mathrm{d}\Omega^{n-1}_f$ is (via $f$) a free $\mathcal{O}_1$-module of rank $\mu = \mu(X_0)$, the Milnor number of $(X_0, 0)$ [6], [10].

(ii) $A := \Omega^n_f/\mathrm{d}\Omega^{n-1}_f + (f)\Omega^n_f = \Omega^n_{X_0}/\mathrm{d}\Omega^{n-1}_{X_0}$ is a $\mu$-dimensional vector space over $\mathbf{C}$ with a natural grading $A = \bigoplus_{k\geq 0} A_k$ coming from the above filtration. Moreover, the Poincaré series of $A$

$$P(s) = \sum_{k\geq 0} (\dim A_k)s^k$$

is computed in [7] and in our case is given by

$$P(s) = \mathrm{res}_{t=0} \frac{t^{-m+p}}{1+t}\left[\prod_{i=1}^{m} \frac{1+ts^{w_i}}{1-s^{w_i}} \prod_{j=1}^{p+1} \frac{1-s^{d_j}}{1+ts^{d_j}} + t\right]$$

where $d_{p+1} = d$.

(iii) $H^n(\Omega^{\cdot}_f) \subset \Omega^n_f/\mathrm{d}\Omega^{n-1}_f \subset H^n(\Omega^{\cdot}_f)[u^{-1}]$ where $u$ denotes the coordinate on $\mathbf{C}$ ([10], Proposition 8.24).

Our main result is the following.

THEOREM 1. *The complex monodromy operator $h$ is diagonalizable and its eigenvalues are $d$-roots of the unity. The multiplicity of the root $e^{-2\pi i k/d}$ is*

$$\sum_{j\equiv k (\mathrm{mod}\, d)} \dim A_j = d^{-1} \sum_{s^d=1} P(s)s^{-k}.$$

*Proof.* Choose a homogeneous basis $\varphi_1, \ldots, \varphi_\mu$ for $A$. Then by (i) they form a basis of $\Omega^n_f/\mathrm{d}\Omega^{n-1}_f$ over $\mathcal{O}_1$.

The vector field $\eta = u\dfrac{\mathrm{d}}{\mathrm{d}u}$ on $(\mathbf{C}, 0)$ can be lifted to the vector field $\xi = \mathrm{d}^{-1} \sum_{K=1,m} w_K x_K \dfrac{\partial}{\partial x_K}$ on $(X, 0)$.

The 1-parameter flow generated by $\xi$ is obviously $F_t(x) = (\mathrm{e}^{w_1 t/d}x_1, \ldots, \mathrm{e}^{w_n t/d}x_m)$.

The Lie derivative $L_\xi$ is easy to compute for a homogeneous form $\varphi$

$$L_\xi(\varphi) = \lim_{t\to 0} \frac{F_t^*(\varphi) - \varphi}{t} = \deg(\varphi)\mathrm{d}^{-1}\varphi.$$

Using this and (iii), it follows that a (multivalued) horizontal section of $R^n f_* \mathbf{C} \otimes \mathcal{O}_{\mathbf{C}}$ over $\mathbf{C}\setminus\{0\}$ is given by $u \mapsto u^{-\deg(\varphi)/\mathrm{d}}\varphi$.

Taking $\varphi = \varphi_1, \ldots, \varphi_\mu$ we get a frame in each fiber. Thus, if we put $u = \rho e^{2\pi i\theta}$ and let $\theta$ go from 0 to 1, then we find that the monodromy operator $h$ multiplies $\varphi$ with $\mathrm{e}^{-2\pi i \deg(\varphi)/d}$. This ends the proof of the Theorem. □

*Example* 2. Consider the simple space curve singularity $X_0 = U_7 : g_1 = x^2 + yz = 0$, $g_2 = xy + z^3 = 0$ corresponding to $\mathbf{w} = (4, 5, 3)$ and $\mathbf{d} = (8, 9)$ [5].

Then a direct computation using the formula for $P(s)$ given in (ii) shows that

$$P(s) = s^{14} + s^{13} + s^{11} + s^{10} + s^9 + s^8 + s^7.$$

Let $\Delta_i(\lambda)$ be the characteristic polynomial of the monodromy operator of the function germ $g_i : (\{g_j = 0\}, 0) \to (\mathbf{C}, 0)$ for $i \neq j$. Then Theorem 1 gives us

$$\Delta_1(\lambda) = (\lambda^8 - 1)(\lambda + 1)^{-1}, \quad \Delta_2(\lambda) = (\lambda^9 - 1)(\lambda^2 + \lambda + 1)^{-1}.$$

## §2. THE BETTI NUMBERS OF THE LINK K AND OF THE VARIETY Y

Recall from the introduction the definition of the spaces $K$ and $Y$ associated to the singularity $(X, 0)$. Let $K_0$ and $Y_0$ be the similar spaces associated to the singularity $(X_0, 0)$.

Note first that $K$ is a smooth compact oriented $(2n+1)$-dimensional manifold which is $(n-1)$-connected [8]. In particular, we have to determine only the middle Betti numbers $b_n(K) = b_{n+1}(K)$. On the other hand, it is known that

$$b_n(K) = \mu(X) - \text{rang } L$$

where $L$ is the intersection form of the Milnor lattice of $(X, 0)$ [10]. Hence we will get a procedure to compute rank $L$ in terms of $(\mathbf{w}, \mathbf{d})$. One of the applications of the computation of rank $L$ is the estimation of the number of singularities which may occur on a fiber in a deformation of $(X, 0)$ [3].

As to the projective variety $Y$, it is a $V$-variety and hence a $2n$-dimensional $\mathbf{Q}$-manifold [4]. The action of $S^1$ on $S$ given by

$$t\cdot(x_1, \ldots, x_m) = (t^{w_1}x_1, \ldots, t^{w_m}x_m)$$

leaves $K$ invariant and $K/S^1 = Y$. For a point $y = \hat{x} = (x_1 : \ldots : x_m) \in Y$ we define

$$w(y) = \text{g.c.d.}\{w_i;\, x_i \neq 0\}.$$

It follows easily that the isotropy group $S^1_x$ of a point $x \in K$ is precisely the group of $w(\hat{x})$-roots of the unit. In particular, if $w(y)$ is constant for $y \in Y$, then $Y$ is in a natural way a smooth manifold [1]. We will say in this case that $Y$ is *strongly smooth*. Note that $Y$ can be a smooth algebraic variety without being strongly smooth!

First we show that the topology of a quasi-smooth complete intersection depends only on its type.

PROPOSITION 3. *Two quasi-smooth complete intersections $Y_1$ and $Y_2$ of the same type $(\mathbf{w}, \boldsymbol{d})$ are homeomorphic. Moreover, if one of them is strongly smooth then so is the other and they are diffeomorphic.*

*Proof.* Let $Q(\mathbf{w}, d)$ be the complex vector space of homogeneous polynomials of degree $d$ with respect to $\mathbf{w}$ and $P = Q(\mathbf{w}, d_1)x \ldots xQ(\mathbf{w}, d_p)$.

The set

$$B = \left\{(x,f) \in (\mathbf{C}^m \setminus \{0\})xP;\; f = (f_1, \ldots, f_p),\; rk\left(\frac{\partial f_i}{\partial x_j}(x)\right) < p\right\}$$

where $i = 1, \ldots, p; j = 1, \ldots, m$ is an algebraic subset in $(\mathbf{C}^m \setminus \{0\})xP$. Let $U = P \setminus pr_2(B)$ and note that $U$ is a Zariski open subset in $P$. Hence either $U = \emptyset$ or $U$ is a dense connected subset, which is what we assume from now on.

The set

$$Z = \{(x,f) \in SxU;\, f(x) = 0\}$$

is a smooth manifold and the map induced by the second projection $\pi: Z \to U$ is a proper submersion. There is a $S^1$-action on $Z$ coming from the action on $S$ defined above.

Next we need the following.

LEMMA 4. (Equivariant Ehresmann fibration theorem). *Let $p: E \to B$ be a proper submersion. If $G$ is a compact Lie group acting on E such that all the orbits are contained in the fibers of $p$, then $p$ is a locally trivial $G$-fibration.*

[This means: for any $b \in B$ there is an open set $U \subset B$ with $b \in U$ and an equivariant diffeomorphism $f: p^{-1}(U) \to UxF$, where $F = p^{-1}(b)$ and $G$ acts on $UxF$ by the formula $g\cdot(x, y) = (x, gy)$, such that $pr_1 \circ f = p$.]

*Proof.* The usual proof of the Ehresmann fibration theorem applies if we show that any vector field $\eta$ on $B$ can be lifted to an equivariant vector field $\xi$ on $E$ (i.e. $d_x L_g(\xi(x)) = \xi(L_g(x))$ for any $x \in E$, $g \in G$, where $L_g(x) = g\cdot x$).

Let $\xi_0$ be any lifting of $\eta$. Then

$$\xi(x) = \int_{g \in G} (d_x L_g)^{-1}(\xi_0(L_g(x)))\mathrm{d}g$$

where $dg$ is a normalized invariant Haar measure on $G$, is an equivariant lifting of $\eta$. □

From this lemma we obtain that the fibers of $\pi$ are equivariantly diffeomorphic and this ends the proof of Proposition 3. □

COROLLARY 5. *If a two-dimensional quasi-smooth complete intersection is nonsingular, then any other quasi-smooth complete intersection of the same type is also nonsingular.*

*Proof.* Use the fact that the local fundamental group is a topological invariant and that the singular points on a normal surface are precisely those with nontrivial local fundamental group [12]. □

Now we give the basic result for the computation of the Betti numbers of $K$ and $Y$. Let $\mathbf{P}^n$ be the usual projective n-space.

PROPOSITION 6. (i) *One has* $b_k(Y) = b_k(P^n)$ *for* $k \neq n$ *and* $b_n(Y) = b_n(K) + b_n(P^n)$.

(ii) *If $Y$ is strongly smooth, then all the integer homology groups of $Y$ are torsion free.*

(iii) *For $n \geq 2$ one has*

$$b_n(K) + b_{n-1}(K_0) = \dim \ker(h - \mathrm{Id}).$$

*Proof.* The Smith–Gysin exact sequence in homology with **C**-coefficients [1] associated to the action of $S^1$ on $K$ give the result (i).

When $Y$ is strongly smooth we can use the Gysin sequence with **Z**-coefficients and Poincaré duality over **Z** to get (ii).

Comparing the Smith–Gysin exact sequences associated to the $S^1$ actions on $K$ and $K_0$, we find out that the morphism $H_n(K_0) \to H_n(K)$ induced by inclusion is trivial for $n \geq 2$. The exact sequence of the pair $(K, K_0)$ then gives

$$b_{n+1}(K) + b_n(K_0) = \dim H_{n+1}(K, K_0)$$

Finally, the exact sequence (1.8) in [8] shows that

$$\dim H_{n+1}(K, K_0) = \dim \ker(h - \mathrm{Id}). \qquad \square$$

Since $\dim \ker(h - \mathrm{Id})$ is equal to the multiplicity of 1 as a root of $\Delta(\lambda)$, this number can be computed using Theorem 1. Then one can compute $b_n(K)$, $b_n(Y)$ by descending induction on $n = \dim Y$ as follows.

When $n = 0$, $K$ is a disjoint union of circles (and $Y$ a finite set of points), one for each irreducible branch of the curve $X$. The number of branches of $X$ is computable in terms of the type $(\mathbf{w}, \mathbf{d})$ as shown by Giusti [5], Chap. II (see also Remark 8 below).

When $n = 1$, $Y$ is a smooth curve and there is a simple formula for its geometric genus $p_g(Y)$ in terms of $(\mathbf{w}, \mathbf{d})$ [4] (3.4.4). Hence $b_1(K) = b_1(Y) = 2p_g(Y)$ is known in this case.

For $n > 1$, there exists a weighted homogeneous function $f$ of degree $d$, where $d$ is any common multiple of $(w_1, \ldots, w_m)$ such that $X_0 = X \cap f^{-1}(0)$ is an isolated singularity of complete intersection (see for instance [5], (2.4), Chap. II). Then, using (iii) we can compute $b_n(K)$ from the previously computed number $b_{n-1}(K_0)$.

When the defining equations $f_i$ of the variety $Y$ can be chosen such that the weighted complete intersections

$$Y_k\colon f_1(x) = \ldots = f_k(x) = 0$$

are quasi-smooth for $k = 1, \ldots, p$, then one can use (and sometimes is simpler) increasing induction on $n$ to compute $b_n(K)$.

## §3. THE HODGE NUMBERS OF THE SINGULARITY $(X, 0)$

We keep the main notations from above, except that we let $n$ denote $\dim X$. The equations $f_1 = a_1, \ldots, f_p = a_p$ (for suitable constants $a_i$) define a non-singular affine variety $V$ in $\mathbf{C}^m$ which is diffeomorphic to the Milnor fibre of $(X, 0)$.

The middle cohomology group $H^n(V, \mathbf{C})$ has a canonical mixed Hodge structure and our aim now it to explain how the corresponding Hodge numbers $h^{pq}(V)$ can be computed in terms of $(\mathbf{w}, \mathbf{d})$ when $n \leq 3$. There is also a canonical *pure* Hodge structure on the cohomology of the projective variety $Y$ associated to $X$ and its Hodge numbers can also be computed when $\dim Y \leq 3$.

The method for these computations is the same as in Steenbrink [13] and we recall it briefly here. Let $\bar{V}$ be the closure of $V$ in the weighted projective space $\mathbf{P}(\mathbf{w}, 1)$. Then $\bar{V}$ and $Y = \bar{V}\setminus V$ are quasi-smooth complete intersections and hence their canonical Hodge structures are pure. Moreover

(i) $$h^{no}(\bar{V}) = \dim H^n(\bar{V}, \mathcal{O}_{\bar{V}}) = p_g(\bar{V})$$

is computable in terms of $(\mathbf{w}, \mathbf{d})$ as in [4] (3.4.4) and similarly for $Y$. The exact sequence

(ii) $$0 \to P^n(\bar{V}) \to H^n(V) \to P^{n-1}(Y)(-1) \to 0$$

where $P^n$ denotes the primitive part of $H^n$, shows that $Gr_k^W H^n(V) = 0$ for $k \neq n, n+1$ and $Gr_n^W H^n(V) = P^n(\bar{V})$, $Gr_{n+1}^W H^n(V) = P^{n-1}(Y)(-1)$.

Hence to determine the Hodge numbers $h^{pq}(V)$ it is enough to compute those of $\bar{V}$ and $Y$. For $n \leq 3$, all these numbers can be determined using (i), the formula for $\mu(X)$ [7] and, when $n = 3$, the Betti number $b_2(Y)$ computed as explained above in the paper. Let now $n = 2$ and $(\mu_0, \mu_+, \mu_-)$ denote the invariants of the intersection form on $H_2(V, \mathbf{R})$. Then, using the above and an argument as in [13] to relate $\mu_0$, $\mu_+$ and $\mu_-$ to the Hodge numbers of $V$ we get the following nice equalities

$$\begin{aligned}
\mu_0 &= 2h^{21}(V) = 2p_g(Y) \\
\mu_+ &= 2h^{20}(V) = 2p_g(\bar{V}) \\
\mu_- &= h^{11}(V) = \mu(X) - \mu_0 - \mu_+.
\end{aligned}$$

*Example* 7. Consider the triangle singularity $X: xy + z^2 = 0$, $w^2 + xz + y^3 = 0$. In this case $\mathbf{w} = (10, 6, 8, 9)$ and $\mathbf{d} = (16, 18)$. Using [4] (3.4.4) it follows

$$h^{21}(V) = p_g(Y) = 0,\ h^{20}(V) = p_g(\bar{V}) = 1,\ h^{11}(V) = 7.$$

Hence in this case $V$ has a pure Hodge structure of weight 2.

*Remark* 8. The exact sequence (ii) for $n = 1$ and the formulas for $\mu(X)$ and $p_g(\bar{V})$ give a simple way to compute the number of irreducible branches of the curve singularity $(X, 0)$ in terms of $(\mathbf{w}, \mathbf{d})$. (Compare to [5] Chap. II).

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**NOTE ADDED IN PROOF**

(i) Professor H. A. Hamm recently informed us about his computations for invariants of a quasi-smooth complete intersection $Y$. In particular, he has determined the mixed Hodge numbers $h^{pq}(Y)$ in all the cases. He has also obtained detailed information on the integral homology of the link $K$.

(ii) We learned also from Professor Hamm about the paper: R. C. Randell: The homology of generalized Brieskorn manifolds, *Topology*, **14** (1975), 347–355, which is closely related to the subject of our paper.

(iii) In our forthcoming paper "Singularities and coverings of weighted complete intersections" we analyse (among other things) the singular locus of a quasi-smooth complete intersection $Y$ and bring more light on the concept of strongly smoothness introduced in section 2 above.