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On generalized Hamacher families of triangular operators

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Abstract

A general functional connective generator is proposed for the Hamacher family in this short study. The connective generator generates a variety of different families in which the original Hamacher family can be observed as a special case. Although the original additive connective generator that was proposed by Dombi is also capable of generating Hamacher norms, we show that there is yet another connective generator that uses simple, monotonic, and continuous functions that are both bounded in the domain and range of [0,1] in contrast to the properties of the functions that used in the additive connective generator.

Keywords: Triangular operators; T-norms; T-conorms; Connective generator; Hamacher family; Generalized Hamacher family

1. Introduction

Fuzzy systems have essential components called triangular operators, often called *t-operators* [14]. The t-operators are in fact Union, Intersection, and Complement operators in fuzzy set theory which are symbolized by T-conorm (T^*), T-norm (T), and Negation (N), respectively. They have the same operations as their counterparts in the Cantorian sets. It has been noticed that these operators play a dominant role both in the formal theory and in applications.

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The theory of t-operators has been under investigation for a considerable amount of time, even before the onset of fuzzy theory. Although it originated from the field of statistical metric spaces, a considerable effort in its development took place after the advent of fuzzy logic [22–25]. It is certainly an important issue of discussion particularly in the framework of fuzzy inferencing and fuzzy decision making. It has been observed that a suitable choice of triangular operators on different applications can considerably enhance or deteriorate the system's performance.

Many t-operators have been proposed in literature [7,8,11,29] and have been used in variety of fuzzy systems. Among them there is a class of t-operators which Hamacher investigated in [12]. Hamacher showed that any representation like $P(x,y)/Q(x,y)$ where $P(x,y)$ and $Q(x,y)$ were polynomials of x and y would satisfy the properties of fuzzy connectives. Following those lines the functional connectives he derived are usually represented by the next pair of equations:

$$T(x,y) = \frac{\lambda xy}{1-(1-\lambda)(x+y-xy)}, \tag{1}$$

$$T^*(x,y) = \frac{\lambda(x+y)+xy(1-2\lambda)}{\lambda+xy(1-\lambda)}, \tag{2}$$

where $\lambda \geqslant 0$.

Later, Dombi [6] was able to generalize Hamacher's connectives by using an additive connective generator, which we will discuss in Section 2. Fodor and Keresztfalvi [10] provided a simple characterization of the Hamacher family of t-norms with positive parameters. However, we did not find any significant improvement in those directions. In the third section we will examine the properties of the proposed connective generator for the generalized Hamacher family which is given by

$$\begin{aligned} &h(x,y:\mu) \\ &= \phi\left(\frac{\mu\lambda\phi^{-1}(x)\phi^{-1}(y)}{(\mu+1)\lambda-\zeta-(\zeta-\lambda)(\phi^{-1}(x)+\phi^{-1}(y)-\phi^{-1}(x)\phi^{-1}(y))}\right), \end{aligned} \tag{3}$$

for any $\mu, \lambda, \zeta \neq 0$. In short, we will refer to the families generated by Eq. (3) as μ-families, because μ is the factor that would mainly decide the category of the family. Also, it has been realized that Hamacher norms is a special case of the above functional connective generator provided $\mu = -1$ and $\zeta = 1$. Along with this, we will also see some of the examples of t-operators of different families and can be thought as an extensions of the Hamacher class with respect to the proposed generator. We would like to refer to these families as generalized Hamacher families which are characterized by the choice of μ.

2. T-operators and functional equations

2.1. Definitions of t-operators

Zadeh used max and min operators in his seminal paper as T^* and T [30]. Apart from Zadeh's max–min operator there are quite a few t-operators such as max–product etc. See Refs. [8,11], for a comprehensive survey on the topic. Klir and Folgers [14] have delineated four fundamental axioms related to T^*, T, and N. The axioms are often called *axiomatic skeleton of fuzzy operation*. For the sake of completeness we will add the definitions here in this study.

Definition 1. $T^* : [0,1] \times [0,1] \to [0,1]$ satisfies the following axioms and conditions:
Axiom 1.1. $T^*(0,0) = 0$ and $T^*(1,0) = T^*(0,1) = T^*(1,1) = 1$, (Boundary conditions).
Axiom 1.2. $T^*(x,y) = T^*(y,x)$ (commutativity).
Axiom 1.3. $T^*(T^*(x,y),z) = T^*(x,T^*(y,z))$ (Associativity).
Axiom 1.4. $T^*(x,y) \leqslant T^*(x,z)$ if $y \leqslant z$ (Monotonicity).
Condition 1.1. $T^*(x,0) = x$.
A T^* is said to be Archimedean if and only if the following two conditions are satisfied:
Condition 1.2. $T^*(x,y)$ is continuous, (Continuity).
Condition 1.3. $T^*(x,x) > x$, for all $x \in [0,1]$ (Condition of strictness).

Definition 2. $T : [0,1] \times [0,1] \to [0,1]$ satisfies the following axioms and conditions:
Axiom 2.1. $T(0,0) = T(1,0) = T(0,1) = 0$ and $T(1,1) = 1$, (Boundary conditions).
Axiom 2.2. $T(x,y) = T(y,x)$ (commutativity).
Axiom 2.3. $T(T(x,y),z) = T(x,T(y,z))$ (Associativity).
Axiom 2.4. $T(x,y) \leqslant T(x,z)$ if $y \leqslant z$ (Monotonicity).
Condition 2.1. $T(x,1) = x$.
A T is said to be Archimedean if and only if the following two conditions are satisfied:
Condition 2.2. $T(x,y)$ is continuous, (Continuity).
Condition 2.3. $T(x,x) < x$, for all $x \in [0,1]$ (Condition of strictness).

Definition 3. $N : [0,1] \to [0,1]$ satisfies the following axioms and conditions:
Axiom 3.1. $N(0) = 0, N(1) = 1$ (Boundary conditions).
Axiom 3.2. $N(x) \leqslant N(y)$ for $x \geqslant y$ (Monotonicity).
Axiom 3.3. $N(N(x)) = x$ (Involuteness).
Condition 3.1. $N(x)$ is continuous, (Continuity).

Condition 3.2. $N(x) < N(y)$ given $x > y$ for all $x, y \in [0, 1]$ (Condition of strictness).

2.2. The nexus between functional equations and t-operators

It was realized that these operators could be derived systematically by using connective generators which are in fact functional equations [1]. A connection between functional equations and multivalued logic was established on more concrete grounds by Schweizer and Sklar [23], Dombi [6], Weber [27] and others [18–20]. The approach of generating operators from functional equations was initially suggested by Bellman and Giertz [3] and they were able to further justify the link with Zadeh's max–min operators.

As mentioned earlier t-operators originated from statistical metric spaces that were conceived by Menger [17]. Schweizer and Sklar [22–25] explained with clarity the role of probabilitic and statistical metric spaces and their connection with Menger's work. However, originally in [17] Menger proposed new distance measures satisfing the four basic properties that later formed the definitions of triangular operators. Schweizer and Sklar's work on associative functions and transformation of abstract semigroups [22,23], and Ling's research on associative representation of the semigroups [15] gave a new impetus to the current research on t-operators. The details of classical semigroups are discussed in [5,16]. They also emphasized on a class of t-operators called strict operators, which satisfied the constraints of strictness that are reflected in the definitions. It is easily understood from the definitions that T and T^* are in fact semigroup operators and also has been shown in [8].

Alsina et al. [2], Dombi [6], Klement [13], Silvert [26], Weber [27], Yager [28] and Zimmerman and Zysno [31], and Roychowdhury [18–20] has contributed and extended the connection between functionality and multivalued logic. Dubois and Prade [8] have presented a clear review on this topic of fuzzy aggregation connectives. Their paper contains an immense amount of information about the properties, like idempotency, nilpotency of these connectives apart from the skeleton axioms.

The conventional generator found in Aczel [1], is popularly known as *additive generator* and is expressed as

$$h_{\mathrm{A}}(x,y) = \phi^{(-1)}(\phi(x) + \phi(y)), \tag{4}$$

where

$$\phi^{(-1)}(x) = \begin{cases} \phi^{-1}(x) & x \in [0,1], \\ 0 & x \in [1,\infty]. \end{cases}$$

The above generator was also used by Schweizer and Sklar [23] and Weber [27] in their research.

Moreover, Schweizer and Sklar were able to show the existence of another connective generator called *multiplicative generator*. The multiplicative generator is given below:

$$h_{\mathrm{M}}(x,y) = \phi^{(-1)}(\phi(x) \times \phi(y)), \tag{5}$$

where $\phi^{(-1)}(x)$ is the pseudoinverse.

In the context of associativity of abstract semigroups, Frank [9] was able to point out an important functional association which is given as:

$$T_{\mathrm{F}}^{*}(x,y) = x + y - T_{\mathrm{F}}(x,y), \tag{6}$$

where T_{F} and T_{F}^{*} are Frank's T-norm and T-conorm. However, he used multiplicative generator (5) to generate his T-norms.

Dombi [6] proposed another additive functional generator similar to the one given by Aczel [1]. However, in his case, the complete inverse of a function is valid. He proposed the following generator:

$$h_{\mathrm{D}}(x,y) = \phi(\phi^{-1}(x) + \phi^{-1}(y)). \tag{7}$$

He showed that his connective generator could generate the T and T^{*} when ϕ is a monotonically decreasing and a monotonically increasing function, respectively. Moreover, $\phi(x) : \mathbb{R}^{+} \to (0,1]$, and $\phi^{-1}(x) : (0,1] \to \mathbb{R}^{+}$ including the basic limit properties: (1) $\lim_{x\to\infty} \phi(x) = 0$ and $\phi(0) = 1$ for a T and $\lim_{x\to\infty} \phi(x) = 1$ and (2) $\phi(0) = 0$ for a T^{*}. These were the necessary and sufficient conditions to generate a class of norms.

Dombi [6] clearly showed that his additive generator could generate Hamacher norms for a specific function. When $\phi = \mathrm{e}^{x}/(\lambda + (1-\lambda)\mathrm{e}^{-x})$, it generates

$$T(x,y) = \frac{\lambda xy}{1 - (1-\lambda)(x + y - xy)}. \tag{8}$$

When $\phi = \lambda(1 - \mathrm{e}^{x})/(\lambda + (1-\lambda)\mathrm{e}^{-x})$, it generates

$$T^{*}(x,y) = \frac{\lambda(x+y) + xy(1-2\lambda)}{\lambda + xy(1-\lambda)}. \tag{9}$$

Using the similar properties of ϕ and its inverse as in the Dombi's generator, Roychowdhury and Wang [18] proposed another connective generator called *additive-product generator* which is given as follows:

$$h_{\mathrm{RW}}(x,y) = \phi(\phi^{-1}(x) + \phi^{-1}(y) + \phi^{-1}(x)\phi^{-1}(y)) \tag{10}$$

and they were able to show the existence of exponential norms.

In [19,20] Roychowdhury was able to show the existence of a few more generators that generate triangular operators satisfying the skeleton axioms. It is surprising and yet motivating that one can still find many different operators and their generators. In the next section, we will discuss another proposed connective generator. This generator generalizes the Hamacher triangular operators.

3. The generalized Hamacher connective generator

We will show an alternative connective generator that can generate t-operators in the generalized Hamacher class. The generator is as follows:

$$h(x,y:\mu) = \phi\left(\frac{\mu\lambda\phi^{-1}(x)\phi^{-1}(y)}{(\mu+1)\lambda-\zeta+(\zeta-\lambda)(\phi^{-1}(x)+\phi^{-1}(y)-\phi^{-1}(x)\phi^{-1}(y))}\right) \quad (11)$$

for any $\mu,\lambda,\zeta \neq 0$. We observe that in the above connective generator, ϕ can either monotonically increasing or monotonically decreasing function. Depending on the properties and with a suitable choice of ϕ it is possible to generate either a T or a T^*. The properties of ϕ are listed below:

1. $\phi:\ [0,1] \to [0,1]$ is a continuous, and strictly monotonically increasing function. Also, $\phi^{-1}:\ [0,1] \to [0,1]$ exists, and is continuous, strictly monotonically increasing function.
2. $\phi(0) = \phi^{-1}(0) = 0$ and $\phi(1) = \phi^{-1}(1) = 1$.
3. An increasing function generates a T.

When ϕ is monotonically decreasing then the properties of the function are:

1. $\phi:\ [0,1] \to [0,1]$ is a continuous. Also, $\phi^{-1}:\ [0,1] \to [0,1]$ exists, and is continuous, strictly monotonically decreasing function.
2. $\phi(0) = \phi^{-1}(0) = 1$ and $\phi(1) = \phi^{-1}(1) = 0$.
3. Such function generates a T^*.

Theorem 3.1. *The t-operators generated by the proposed connective generator $h(x,y:\mu)$ satisfy the skeletal axioms.*

Proof.

1. Commutativity:

$$h(x,y:\mu) = \phi\left(\frac{\mu\lambda\phi^{-1}(x)\phi^{-1}(y)}{(\mu+1)\lambda-\zeta+(\zeta-\lambda)\big(\phi^{-1}(x)+\phi^{-1}(y)-\phi^{-1}(x)\phi^{-1}(y)\big)}\right), \quad (12)$$

$$= h(y,x:\mu). \quad (13)$$

2. Associativity: Let

$$\begin{aligned}\psi(x,y,z) =&((\mu+1)\lambda-\zeta)^2+((\mu+1)\lambda-\zeta)(\zeta-\lambda)\Big(\phi^{-1}(x)+\phi^{-1}(y)\\ &+\phi^{-1}(z)+(\zeta-\lambda)^2(\phi^{-1}(x)\phi^{-1}(y)+\phi^{-1}(x)\phi^{-1}(z)\phi^{-1}(y)\phi^{-1}(z))\\ &-(\zeta-\lambda)(\zeta-\lambda+\mu\lambda)\phi^{-1}(x)\phi^{-1}(y)\phi^{-1}(z)\Big). \quad (14)\end{aligned}$$

We can show after some computation that

$$h(h(x,y:\mu),z:\mu)=\phi\left(\frac{(\lambda\mu)^2\phi^{-1}(x)\phi^{-1}(y)\phi^{-1}(z)}{\psi(x,y,z)}\right), \tag{15}$$

$$=h(x,h(y,z:\mu):\mu). \tag{16}$$

3. Monotonicity: Let us consider $y\leqslant z$ and ϕ^{-1} be a monotonically decreasing function such that

$$\phi^{-1}(y)\geqslant\phi^{-1}(z). \tag{17}$$

From the above inequality it is clear that the reciprocal of Eq. (17) is,

$$\frac{1}{\phi^{-1}(y)}\leqslant\frac{1}{\phi^{-1}(z)}. \tag{18}$$

Multiply $((\mu+1)\lambda-\zeta)/(\mu\lambda\phi(x))$ to both sides of Eq. (18) and we have,

$$\frac{(\mu+1)\lambda-\zeta}{\mu\lambda\phi^{-1}(y)\phi^{-1}(x)}\leqslant\frac{(\mu+1)\lambda-\zeta}{\mu\lambda\phi^{-1}(z)\phi^{-1}(x)}. \tag{19}$$

Add $1/\phi(x)$ again to both the sides of Eq. (18) and subtract 1 and followed by the multiplication of the factor $(\zeta-\lambda)/\mu\lambda$. We get,

$$\frac{\zeta-\lambda}{\mu\lambda}\left(\frac{1}{\phi^{-1}(y)}+\frac{1}{\phi^{-1}(x)}-1\right)\leqslant\frac{\zeta-\lambda}{\mu\lambda}\left(\frac{1}{\phi^{-1}(z)}+\frac{1}{\phi^{-1}(x)}-1\right). \tag{20}$$

Addition of Eq. (21) and Eq. (19) leads to the following:

$$\begin{aligned}&\frac{(\mu+1)\lambda-\zeta}{\mu\lambda\phi^{-1}(x)\phi^{-1}(y)}+\frac{1-\lambda}{\mu\lambda}\left(\frac{1}{\phi^{-1}(y)}+\frac{1}{\phi^{-1}(x)}-1\right)\\&\leqslant\frac{(\mu+1)\lambda-\zeta}{\mu\lambda\phi^{-1}(x)\phi^{-1}(z)}+\frac{1-\lambda}{\mu\lambda}\left(\frac{1}{\phi^{-1}(z)}+\frac{1}{\phi^{-1}(x)}-1\right).\end{aligned} \tag{21}$$

Reciprocate Eq. (21) in order to have

$$\begin{aligned}&\frac{\mu\lambda\phi^{-1}(x)\phi^{-1}(y)}{(\mu+1)\lambda-\zeta+(\zeta-\lambda)\phi^{-1}(x)+\phi^{-1}(y)-\phi^{-1}(x)\phi^{-1}(y)}\\&\geqslant\frac{\mu\lambda\phi^{-1}(x)\phi^{-1}(z)}{(\mu+1)\lambda-\zeta+(\zeta-\lambda)\phi^{-1}(x)+\phi^{-1}(z)-\phi^{-1}(x)\phi^{-1}(y)}.\end{aligned} \tag{22}$$

Since ϕ is decreasing we have

$$\begin{aligned}&\phi\left(\frac{\mu\lambda\phi^{-1}(x)\phi^{-1}(y)}{(\mu+1)\lambda-\zeta+(\zeta-\lambda)\phi^{-1}(x)+\phi^{-1}(y)-\phi^{-1}(x)\phi^{-1}(y)}\right)\\&\leqslant\phi\left(\frac{\mu\lambda\phi^{-1}(x)\phi^{-1}(z)}{(\mu+1)\lambda-\zeta+(\zeta-\lambda)\phi^{-1}(x)+\phi^{-1}(z)-\phi^{-1}(x)\phi^{-1}(y)}\right).\end{aligned}\tag{23}$$

Thus, we have

$$h(x,y:\mu)\leqslant h(x,z:\mu).\tag{24}$$

Following similar steps we can prove the rest for an increasing function which is rather trival.

4. Boundedness: It is sufficient to show the following two cases. At $y=1$, we have

$$h(x,y:\mu)$$

$$=\phi\left(\frac{\mu\lambda\phi^{-1}(x)\phi^{-1}(1)}{(\mu+1)\lambda-\zeta+(\zeta-\lambda)\left(\phi^{-1}(x)+\phi^{-1}(1)-\phi^{-1}(x)\phi^{-1}(1)\right)}\right),\tag{25}$$

$$=\phi(\phi^{-1}(x)),\tag{26}$$

$$=x.\tag{27}$$

When ϕ is monotonically increasing then we know $\phi^{-1}(1)=1$.

At $y=0$, we have $\phi^{-1}(1)=0$ when ϕ is monotonically decreasing function.

$$h(x,1:\mu)$$

$$=\phi\left(\frac{\mu\lambda\phi^{-1}(x)\phi^{-1}(1)}{(\mu+1)\lambda-\zeta+(\zeta-\lambda)\left(\phi^{-1}(x)+\phi^{-1}(1)-\phi^{-1}(x)\phi^{-1}(1)\right)}\right),\tag{28}$$

$$=\phi(0),\tag{29}$$

$$=0.\qquad\square\tag{30}$$

One of the fundamental properties is De-Morgan's operation in the context of t-operators,

$$N(T^*(x,y))=T(N(x),N(y)),\tag{31}$$

when $N(x)=\phi^{(-1)}(\phi(1)-\phi(x))$, where $\phi_{\mathrm{i}}(x)$ and $\phi_{\mathrm{d}}(x)$ are increasing and decreasing functions respectively.

Theorem 3.2. *De-Morgan's identity is satisfied even by this family provided we have* $\phi_{\mathrm{i}}(x)=N(\phi_{\mathrm{d}}(x))$.

Proof. The proof is obvious. We will only mention that the class provides $\phi_{\mathrm{i}}^{-1}(x)=N(\phi_{\mathrm{d}}^{-1}(x))$.

$$h_{\mathrm{i}}(x,y:\ \mu)$$
$$= \phi_{\mathrm{d}}\left(\frac{\mu\lambda\phi_{\mathrm{i}}^{-1}(x)\phi_{\mathrm{i}}^{-1}(y)}{(\mu+1)\lambda-\zeta+(\zeta-\lambda)\phi_{\mathrm{i}}^{-1}(x)+\phi_{\mathrm{i}}^{-1}(y)-\phi_{\mathrm{i}}^{-1}(x)\phi_{\mathrm{i}}^{-1}(y)}\right). \tag{32}$$

$$h_{\mathrm{d}}(x,y:\ \mu)$$
$$= N\left(\phi_{\mathrm{i}}\left(\frac{\mu\lambda\phi_{\mathrm{i}}^{-1}(x)\phi_{\mathrm{i}}^{-1}(y)}{(\mu+1)\lambda-\zeta+(\zeta-\lambda)\phi_{\mathrm{i}}^{-1}(x)+\phi_{\mathrm{i}}^{-1}(y)-\phi_{\mathrm{i}}^{-1}(x)\phi_{\mathrm{i}}^{-1}(y)}\right)\right), \tag{33}$$
$$= N(h_{\mathrm{i}}(x,y:\ \mu)). \tag{34}$$

4. The new families

In the previous section we have seen that the proposed connective generator is capable of generating different families of connectives generators, depending on the different choices of μ. Here we will look at the Hamacher family, as well as other families. It should be noted that μ is the only significant factor that differentiates between the classes, whereas the other parameters do not render substanstial effects.

4.1. The μ_{-1}-family: Hamacher operators

When $\mu = -1$, the connective generator generates the Hamacher family of t-operators. Using $\zeta = 1$ and $\lambda \geqslant 0$, the connective generator (3) reduces to

$$h(x,y,\ :\ -1) = \phi\left(\frac{\lambda\phi^{-1}(x)\phi^{-1}(y)}{1-(1-\lambda)\phi^{-1}(x)+\phi^{-1}(y)-\phi^{-1}(x)\phi^{-1}(y)}\right). \tag{35}$$

Example 1. Consider the following functions, let $\phi(x) = x^{\alpha}$ be an increasing function and the decreasing function be $\phi(x) = 1 - x^{1/\alpha}$ in the domain of [0,1]. Thus we get the following t-operators by using Eq. (37):

$$N(x) = 1 - x, \tag{36}$$

$$T(x,y) = \left(\frac{\lambda(xy)^{1/\alpha}}{1-(1-\lambda)\left(x^{1/\alpha}+y^{1/\alpha}-(xy)^{1/\alpha}\right)}\right)^{\alpha}, \tag{37}$$

$$T^{*}(x,y)$$
$$= 1 - \left(\frac{\lambda((1-x)(1-y))^{1/\alpha}}{1-(1-\lambda)((1-x)^{1/\alpha}+(1-y)^{1/\alpha}-((1-x)(1-y))^{1/\alpha})}\right)^{\alpha}, \tag{38}$$

where $\alpha \in \mathbb{R}^+$. If $\alpha = 1$ and $\lim \lambda \to \infty$ then the operators reduce to

$$T(x,y) = \frac{xy}{x+y-xy}, \tag{39}$$

$$T^*(x,y) = \frac{x+y-2xy}{1-xy}. \tag{40}$$

On the other hand, if we allow $\alpha = 1$ and $\lambda = 1$ then the operators get modified to (see Figs. 1 and 2)

$$T(x,y) = xy, \tag{41}$$

$$T^*(x,y) = x+y-xy. \tag{42}$$

4.2. *The* μ_1*-family*

Let $\mu = 1$, then the connective generator generates another class of t-operators. For our convenience, we fix $\zeta = 1$ and $\lambda \geqslant 0$. With these parameters the connective generator (3) reduces to

$$h(x,y:\,1) = \phi\left(\frac{\lambda\phi^{-1}(x)\phi^{-1}(y)}{2\lambda - 1 + (1-\lambda)\left(\phi^{-1}(x) + \phi^{-1}(y) - \phi^{-1}(x)\phi^{-1}(y)\right)}\right). \tag{43}$$

Example 2. Again consider the same functions that we considered in our previous example; let $\phi(x) = x^\alpha$ be an increasing function and the deceasing

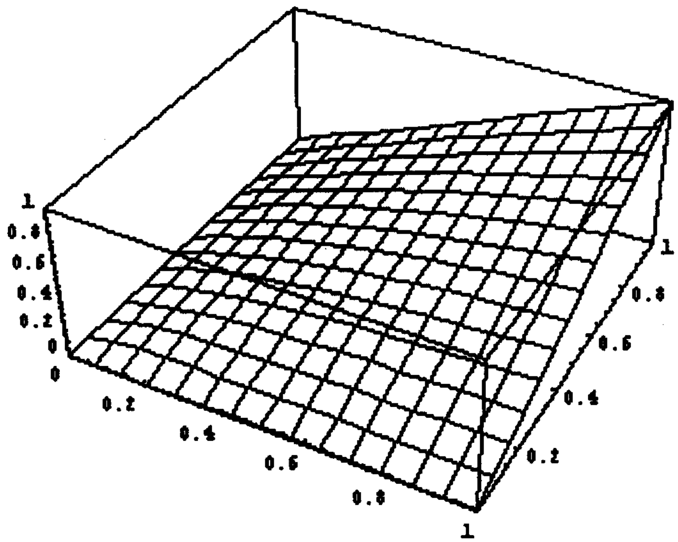

Fig. 1. (μ_{-1}-family): T-norm generated from Eq. (39) which is $xy/(x+y-xy)$.

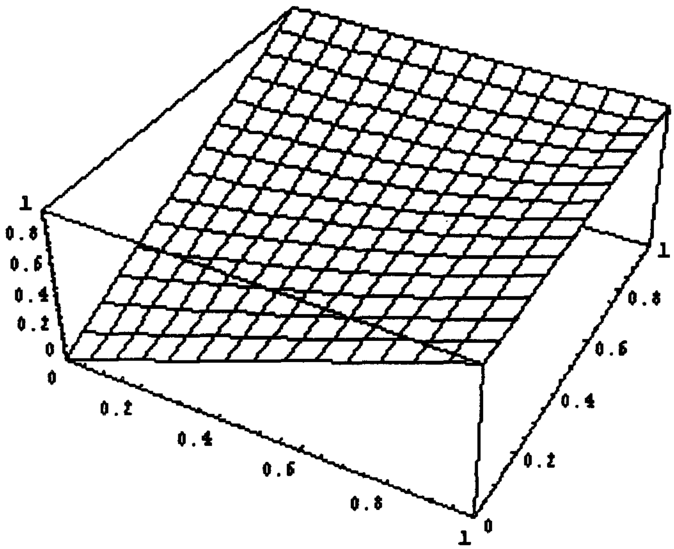

Fig. 2. (μ_{-1}-family): T-conorm generated from Eq. (40) which is $(x+y-2xy)/(1-xy)$.

function be $\phi(x) = 1 - x^{1/\alpha}$ in the domain of [0, 1]. Thus we get the following norms:

$$N(x) = 1 - x, \tag{44}$$

$$T(x,y) = \left(\frac{\lambda(xy)^{1/\alpha}}{2\lambda - 1 + (1-\lambda)\left(x^{1/\alpha} + y^{1/\alpha} - (xy)^{1/\alpha}\right)}\right)^{\alpha}, \tag{45}$$

$$T^*(x,y) = 1 - \left(\frac{\lambda((1-x)(1-y))^{1/\alpha}}{2\lambda - 1 + (1-\lambda)\left((1-x)^{1/\alpha} + (1-y)^{1/\alpha} - ((1-x)(1-y))^{1/\alpha}\right)}\right)^{\alpha}, \tag{46}$$

where $\alpha \in \mathbb{R}^+$. If $\alpha = 1$ and $\lim \lambda \to \infty$ then the operators reduce to

$$T(x,y) = \frac{xy}{2-x-y+xy}, \tag{47}$$

$$T^*(x,y) = \frac{x+y}{1+xy}. \tag{48}$$

Also if we allow $\alpha = 1$ and $\lambda = 1$ then we observe Bandler and Kohout's norms [4] (see Figs. 3 and 4).

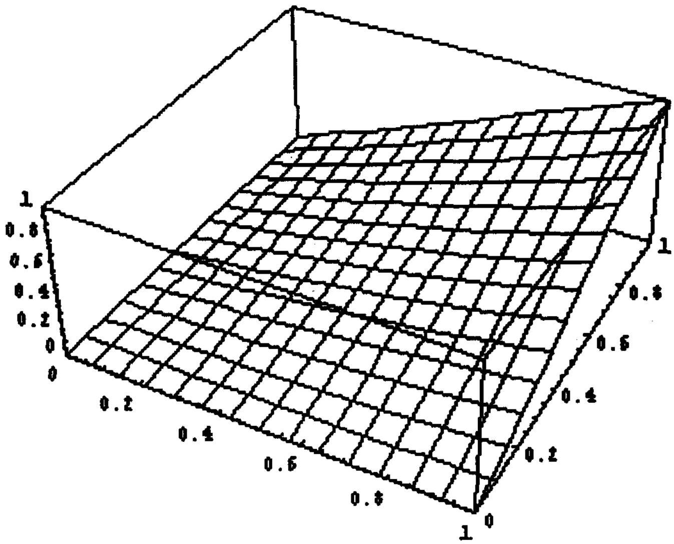

Fig. 3. (μ_1-family): T-norm generated from Eq. (47) which is $(xy)/(2-x-y+xy)$.

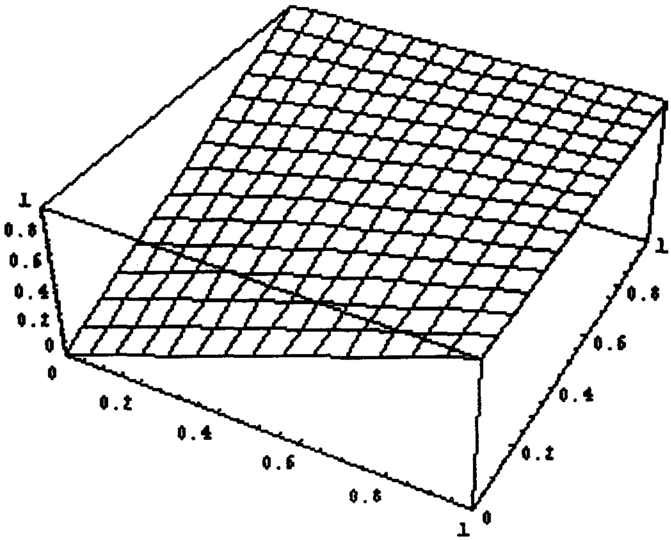

Fig. 4. (μ_1-family): T-conorm generated from Eq. (48) which is $(x+y)/(1+xy)$.

$$T(x,y) = xy, \tag{49}$$

$$T^*(x,y) = x + y - xy. \tag{50}$$

4.3. The μ_n-family

When $\mu = n \geqslant 0$, the connective generator generates the Hamacher family of t-operators. We note that $\zeta = 1$ and $\lambda \geqslant 0$. Thus the connective generator (3) reduces to

$$h(x,y : n) = \phi\left(\frac{n\lambda\phi^{-1}(x)\phi^{-1}(y)}{(n+1)\lambda - 1 + (1-\lambda)\phi^{-1}(x) + \phi^{-1}(y) - \phi^{-1}(x)\phi^{-1}(y)}\right). \tag{51}$$

Example 3. Consider the following functions, let $\phi(x) = x^\alpha$ be an increasing function and the decreasing function be $\phi(x) = 1 - x^{1/\alpha}$ in the domain of [0,1]. Thus we get the following norms:

$$N(x) = 1 - x, \tag{52}$$

$$T(x,y) = \left(\frac{n\lambda(xy)^{1/\alpha}}{(n+1)\lambda - 1 + (1-\lambda)\left(x^{1/\alpha} + y^{1/\alpha} - (xy)^{1/\alpha}\right)}\right)^\alpha, \tag{53}$$

$$T^*(x,y) = 1 - \left(\frac{n\lambda((1-x)(1-y))^{1/\alpha}}{(n+1)\lambda - 1 + (1-\lambda)\left((1-x)^{1/\alpha} + (1-y)^{1/\alpha} - ((1-x)(1-y))^{1/\alpha}\right)}\right)^\alpha, \tag{54}$$

where $\alpha \in \mathbb{R}^+$. If $\alpha = 1$ and $\lim \lambda \to \infty$ then the operators reduce to

$$T(x,y) = \frac{nxy}{n + 1 - (x + y - xy)}, \tag{55}$$

$$T^*(x,y) = \frac{nx + ny + (1-n)xy}{n + xy}. \tag{56}$$

Also if we allow $\alpha = 1$ and $\lambda = 1$ then the operators get reduced to Eq. (51) and Eq. (52), respectively.

Example 4. Here we will use some log functions and their inverses to generate another set of t-operators. Let us consider a decreasing function, $\phi(x) = \log_a(1 + (a-1)x)$ and its inverse $\phi^{-1}(x) = (a^x - 1)/(a-1)$. The decreasing function generates T, whereas the increasing function which is $\phi(x) = a/(\log_a(1 + (a-1)x))$ and its inverse $\phi^{-1}(x) = (a^{(1-x)} - 1)/(a-1)$ generates a T^* (see Figs. 5–8).

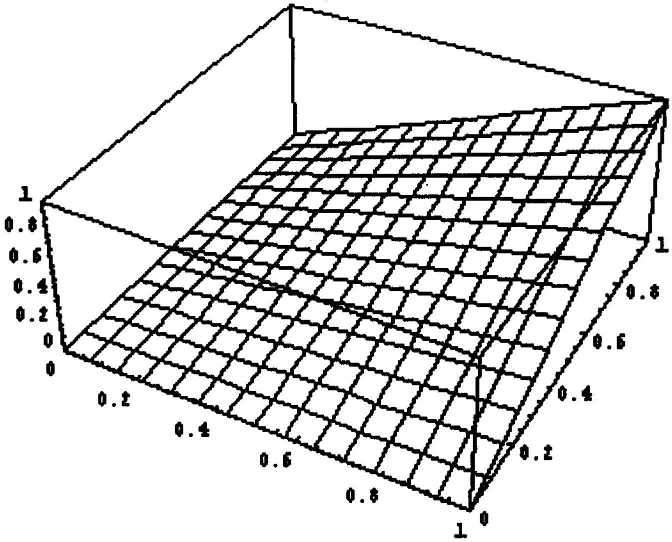

Fig. 5. (μ_n-family): T-norm generated from Eq. (55) which is $(nxy)/(n+1-x-y+xy)$; $n=20$.

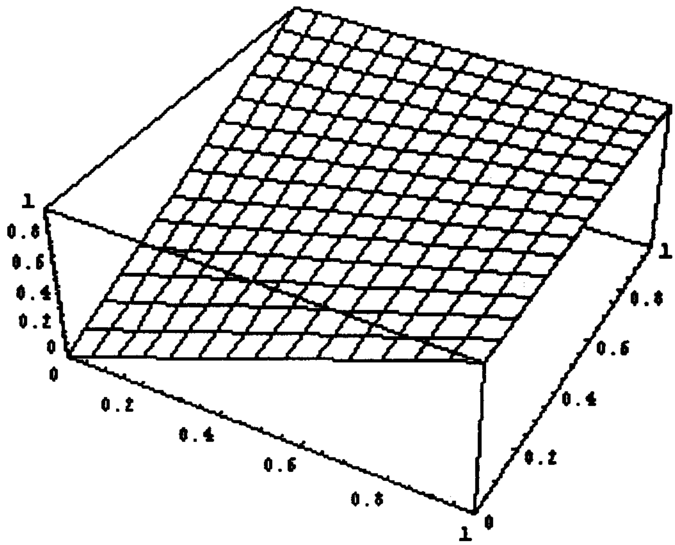

Fig. 6. (μ_n-family): T-conorm generated from Eq. (56) which is $(nx+ny+(1-n)xy)/(n+xy)$; $n=20$.

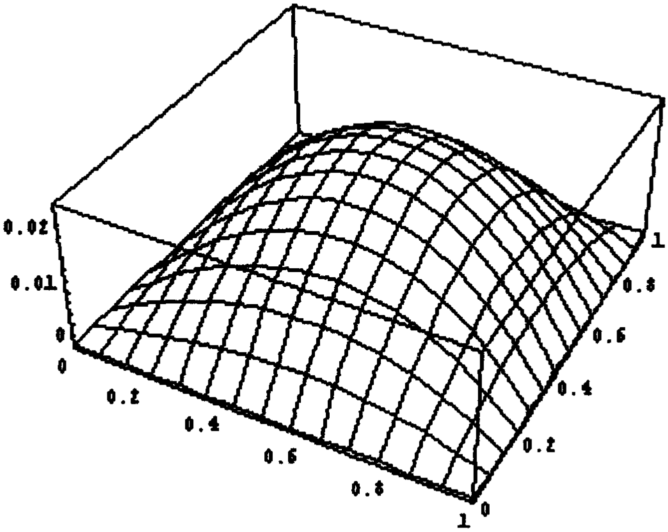

Fig. 7. Difference of T-conorms given by Eq. (56) generated with $n = 20$ and $n = 2$.

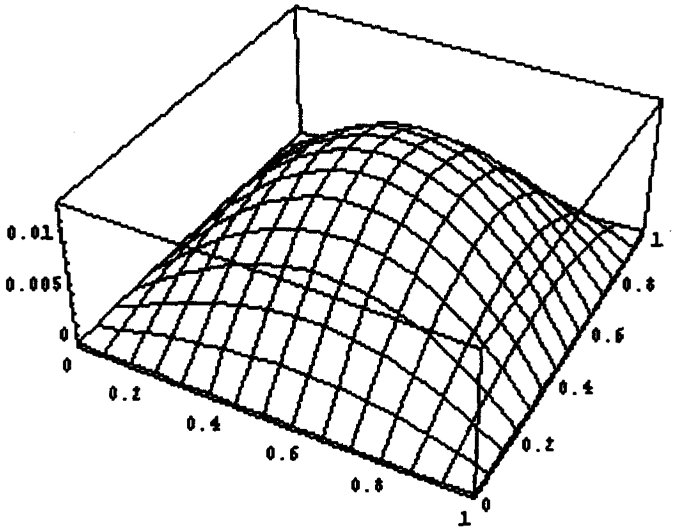

Fig. 8. Difference of T-conorms given by Eq. (56) generated with $n = 2$, $\lambda = 2$, and $\alpha = 20$ and $n = 2$, $\lambda = 2$, and $\alpha = 1$.

$$N(x) = 1 - x, \tag{57}$$

$$T(x,y) = \log_a\left(1+\frac{n\lambda(a-1)(a^x-1)(a^y-1)}{((n+1)\lambda-1)(a-1)^2+(1-\lambda)[a^x+a^y-2-(a^x-1)(a^y-1)]}\right), \tag{58}$$

$$T^*(x,y) = \log_a\left(\frac{a}{1+\frac{n\lambda(a^{1-x}-1)(a^{1-y}-1)}{((n+1)\lambda-1)(a-1)^2+(1-\lambda)[(a^{1-x}+a^{1-y}-2)(a-1)-(a^{1-x}-1)(a^{1-y}-1)]}}\right). \tag{59}$$

5. A comparative evaluation of new t-operators in fuzzy inference

An extensive research has been done on the topic of fuzzy inference, and numerous comments and criticisms have appeared from time to time [14,21]. Yet, it still remains an important topic to contemplate and research, as we need fuzzy inference engines in many fuzzy systems. Here we study the effect of generalized Hamacher triangular operators on fuzzy inference. Our focus is to perform a comparative evaluation of various fuzzy set operations with respect to generalized Hamacher triangular operators on the Compositional Rule of Inference (CRI).

The CRI is given by the following equation:

$$Y'(y) = T^*(X'(x) \circ T(x,y))), \tag{60}$$

where $Y'(y)$ and $X'(x)$ are the inferred fuzzy set and the input fuzzy set respectively. Other fuzzy inference schemes have been proposed in [21], however we will limit the scope of our discussion to CRI. Alternatively, the CRI can also be rewritten in terms of an increasing function ϕ_{i} and a decreasing function ϕ_{d} using the proposed operator generator, and is given below:

$$\begin{aligned} Y'(y) \\ = \phi_{\mathrm{d}}\left(\frac{\mu\lambda\phi_{\mathrm{d}}^{-1}(x)\phi_{\mathrm{d}}^{-1}(T(x,y))}{(\mu+1)\lambda-\zeta+(\zeta-\lambda)\left(\phi_{\mathrm{d}}^{-1}(x)+\phi_{\mathrm{d}}^{-1}(T(x,y))-\phi_{\mathrm{d}}^{-1}(x)\phi_{\mathrm{d}}^{-1}(T(x,y))\right)}\right), \end{aligned} \tag{61}$$

where,

$$\begin{aligned} T(x,y) \\ = \phi_{\mathrm{i}}\left(\frac{\mu\lambda\phi_{\mathrm{i}}^{-1}(x)\phi_{\mathrm{i}}^{-1}(y)}{(\mu+1)\lambda-\zeta+(\zeta-\lambda)\left(\phi_{\mathrm{i}}^{-1}(x)+\phi_{\mathrm{i}}^{-1}(y)-\phi_{\mathrm{i}}^{-1}(x)\phi_{\mathrm{i}}^{-1}(y)\right)}\right). \end{aligned} \tag{62}$$

Let us consider a hypothetical fuzzy SISO rule for linear expansion of heat: IF $\delta Temp = (-20, 0, 20)$°C THEN $\delta Len = (-20, 0, 20)$ μm. The antecedent variable $\delta Temp$ denotes the temperature that is equal to a triangular fuzzy set (−20, 0, 20)°C. And on the other hand, the consequent variable δLen has triangular fuzzy set (−20, 0, 20) μm. The universe of discourse of both variables is (−50, 50). The quantization of fuzzy sets in this case is equal to 20. The rule is encoded using the T-norms. Specifically, in Fig. 9 we used the *min* operator to encode the above rule. However, during the other simulation runs we used

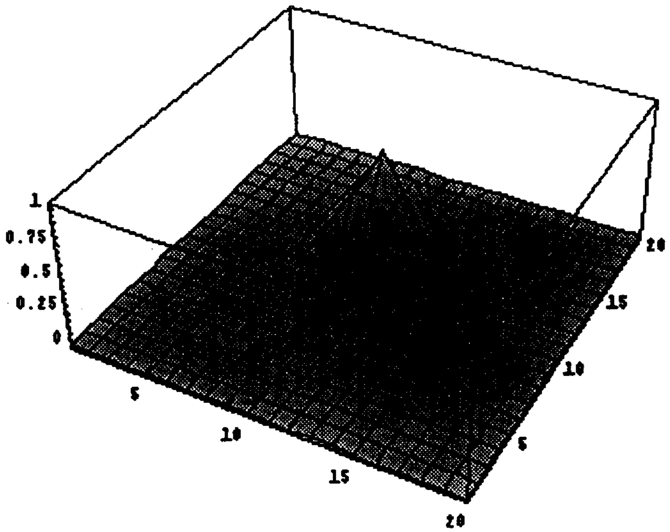

Fig. 9. Rule encoding with min operator.

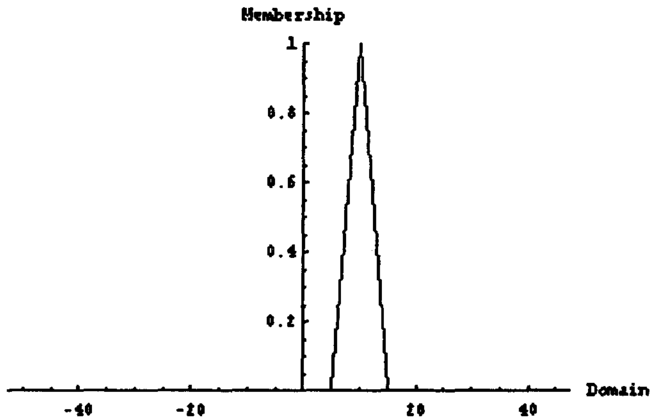

Fig. 10. Input fuzzy set to the rule; triangular fuzzy set (5, 10, 20).

different T-norms. The input $\delta Temp$ fuzzy set is given by a triangular membership function (5, 10, 15)°C as shown in Fig. 10.

Figs. 11–14 show different output fuzzy sets that are inferred from the above rule by various combinations of T-norm–T-conorms. Fig. 11 shows the output fuzzy set derived by using max–min operators. Fig. 12, the output fuzzy set is an output of the CRI which used xy and $x + y - xy$ t-operators. Among the new triangular operators we have used $T(x,y) = \exp(1 - (1 - \ln(x))(1 - \ln(y)))$ and $T^*(x,y) = 1 - \exp(1 - (1 - \ln(1 - x))(1 - \ln(1 - y)))$ which is discussed in [18], and is shown Fig. 13. The following figure, Fig. 14, is due to $T(x,y)$ given by Eq. (55) and $T^*(x,y)$ given by Eq. (56) of the μ_n-family with $n = 2$, $\lambda = 2$, and $\alpha = 2$. Note the differences of the T-conorms with the change of free parameters.

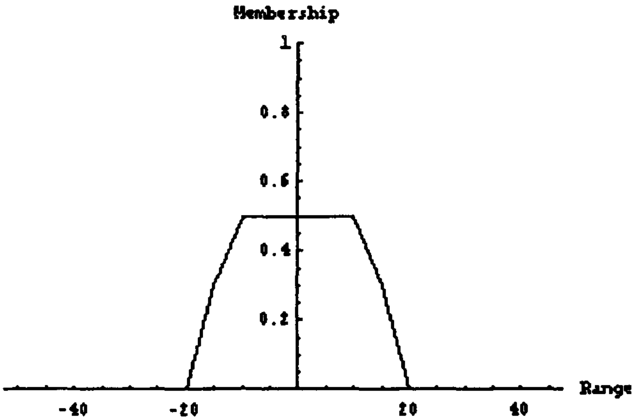

Fig. 11. Output fuzzy set; $T(x,y)$ = min and $T^*(x,y)$ = max.

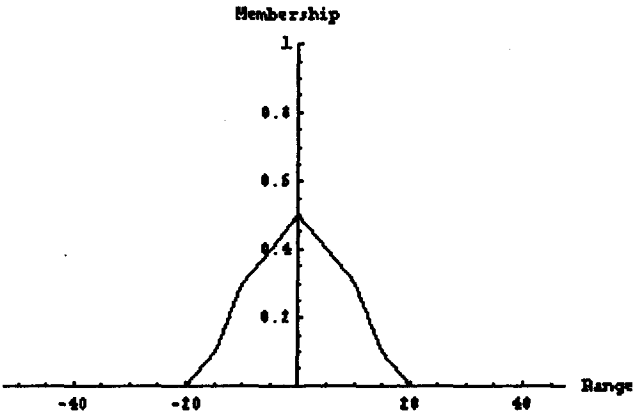

Fig. 12. Output fuzzy set; $T(x,y) = xy$ and $T^*(x,y) = x + y - xy$.

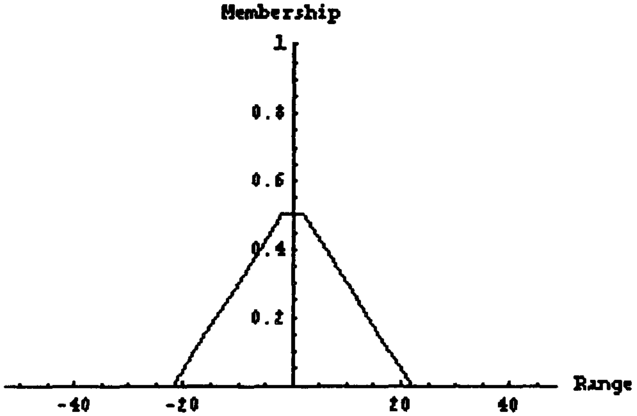

Fig. 13. Output fuzzy set; $T(x,y) = \exp(1-(1-\ln(x))(1-\ln(y)))$ and $T^*(x,y) = 1-\exp(1-(1-\ln(1-x))(1-\ln(1-y)))$.

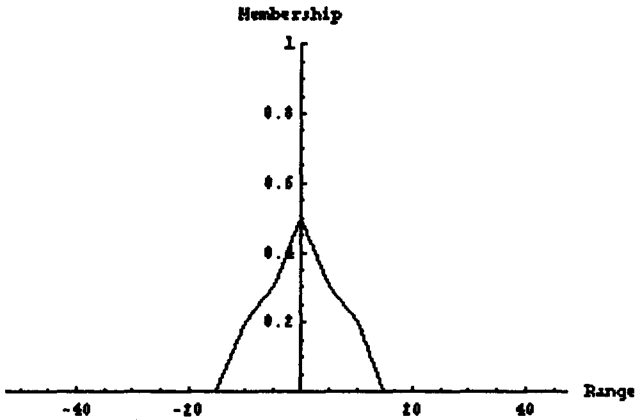

Fig. 14. Output fuzzy set generated from generalized Hamacher triangular operator; T-conorm (56) and T-norm (55).

It should be noted that we have compared the output results of fuzzy inference based on CRI that used a diverse set of families of triangular operators. Furthermore, we believe that these four triangular operator combinations provide a sufficient representation of most of the known classes of t-operators from a practical viewpoint.

6. Concluding remarks

We have proposed an alternative connective generator that generalizes the Hamacher family of triangular operators. In the proposed generator, the suitable choice of μ decides the category of a family. Interestingly, when $\mu = -1$ we get the original Hamacher norms.

Given $\mu \neq 0$, it is easy to create new classes of norms for further studies. We have also shown some of logarithmic operators that belong to the generalized Hamacher class. A brief comparative study of triangular operators was reported in this study.

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