Homotopy and normalization properties for admissible maps

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Abstract

New normalization and homotopy properties are presented for a subclass of the $B^k$-admissible maps of Park. In addition new random fixed point results are given.

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1. Introduction

This paper presents an essential map approach for the $B^k$-admissible maps of Park. Our results improve considerably those in \cite{1} since we establish a new normalization property. Also in this paper we discuss briefly the existence of random fixed points for random operators of $B^k$ type.

For the remainder of this section we present some definitions and known results. A nonempty subset $W$ of a Hausdorff topological vector space $E$ is said to be \textit{admissible} if for every compact subset $K$ of $W$ and every neighborhood $V$ of 0, there exists a continuous map $h : K \to W$ with $x - h(x) \in V$ for all $x \in K$ and $h(K)$ is contained in a finite dimensional subspace of $E$. $W$ is said to be \textit{$q$-admissible} if any nonempty compact convex subset $\Omega$ of $W$ is admissible.

Let $(E, d)$ be a pseudometric space. For $S \subseteq E$, let $B(S, \epsilon) = \{x \in E : d(x, S) \leq \epsilon\}$, $\epsilon > 0$, where $d(x, S) = \inf_{y \in Y} d(x, y)$. The measure of noncompactness of the set $M \subseteq E$ is defined by

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\[ \alpha(M) = \inf Q(M) \text{ where} \]
\[ Q(M) = \{ \epsilon > 0 : M \subseteq B(A, \epsilon) \text{ for some finite subset } A \text{ of } E \}. \]

Let \( E \) be a locally convex Hausdorff topological vector space, and let \( P \) be a defining system of seminorms on \( E \). Suppose \( F : S \to 2^E \); here \( S \subseteq E \). The map \( F \) is said to be a countably \( P \)-concentrative mapping if \( F(S) \) is bounded, and for \( p \in P \) for each countably bounded subset \( X \) of \( S \) we have \( \alpha_p(F(X)) \leq \alpha_p(X) \), and for \( p \in P \) for each countably bounded non-\( p \)-precompact subset \( X \) of \( S \) (i.e. \( X \) is not precompact in the pseudonormed space \((E, p)\)) we have \( \alpha_p(F(X)) < \alpha_p(X) \); here \( \alpha_p(.) \) denotes the measure of noncompactness in the pseudonormed space \((E, p)\).

**Remark 1.1.** In this paper we can remove the condition that for \( p \in P \) for each countably bounded subset \( X \) of \( S \) we have \( \alpha_p(F(X)) \leq \alpha_p(X) \) in the definition above provided we assume the map \( F \) (in Theorem 2.5 and Definition 2.7) maps relatively compact sets into relatively compact sets.

Let \( X \) be a nonempty, convex subset of a Hausdorff topological vector space \( E \) and \( Y \) a topological space. Recall a *polytope* \( P \) in \( X \) is any convex hull of a nonempty finite subset of \( X \).

**Definition 1.1.** We say \( G \in \mathcal{B}(X, Y) \) if \( G : X \to 2^Y \) (the nonempty subsets of \( Y \)) is such that for any polytope \( P \in X \) and any continuous function \( g : G(P) \to P \), the composition \( g(G|_P) : P \to 2^P \) has a fixed point.

**Definition 1.2.** \( F \in \mathcal{B}^k(X, Y) \) (i.e. \( F \) is \( \mathcal{B}^k \)-admissible) if \( F : X \to 2^Y \) is such that for any compact, convex subset \( K \) of \( X \), there exists a closed map \( G \in \mathcal{B}(K, Y) \) with \( G(x) \subseteq F(x) \) for each \( x \in K \).

We next present the fixed point results we will need in Section 2.

**Theorem 1.1 ([6]).** Let \( E \) be a Hausdorff topological vector space and \( X \) an admissible convex subset of \( E \). Then any compact map \( F \in \mathcal{B}^k(X, X) \) has a fixed point.

**Theorem 1.2 ([1]).** Let \( \Omega \) be a nonempty closed convex bounded subset of a Fréchet space \( E \) (\( P \) a defining system of seminorms). Suppose \( F \in \mathcal{B}^k(\Omega, \Omega) \) is a countably \( P \)-concentrative map. Then \( F \) has a fixed point.

**Theorem 1.3 ([5]).** Let \( \Omega \) be a closed convex \( q \)-admissible subset of a Hausdorff topological vector space \( E \) with \( 0 \in \Omega \). Suppose \( F \in \mathcal{B}^k(\Omega, \Omega) \) satisfies the following condition:
\[ A \subseteq \Omega, \quad A = \overline{\operatorname{co}(\{0\} \cup F(A))} \implies A \text{ is compact}. \] (1.1)

Then \( F \) has a fixed point.

We will also discuss random operators in this paper. Let \((\Omega, \mathcal{A})\) be a measurable space and \( C \) a nonempty subset of a metric space \( X = (X, d) \). Let \( \mathcal{C}(C) \) be the family of all nonempty closed subsets of \( C \). A mapping \( G : \Omega \to 2^C \) is said to be measurable if
\[ G^{-1}(U) = \{ w \in \Omega : G(w) \cap U \neq \emptyset \} \in \mathcal{A} \]
for each open subset \( U \) of \( C \). A mapping \( \xi : \Omega \to C \) is called a measurable selector of the measurable mapping \( G : \Omega \to 2^C \) if \( \xi \) is measurable and \( \xi(w) \in G(w) \) for each \( w \in \Omega \). A mapping \( F : \Omega \times C \to 2^X \) is called a random operator if, for any fixed \( x \in C \), the map \( F(., x) : \Omega \to 2^X \) is measurable. A measurable mapping \( \xi : \Omega \to C \) is said to be a random fixed point of a random operator \( F : \Omega \times C \to 2^X \) if \( \xi(w) \in F(w, \xi(w)) \) for each \( w \in \Omega \). Let \( P_b(X) \) be the bounded subsets of \( X \). The Kuratowskii
measure of noncompactness is the map \( \alpha : P_B(X) \to [0, \infty) \) defined by
\[
\alpha(A) = \inf \left\{ \epsilon > 0 : A \subseteq \bigcup_{i=1}^{n} X_i \text{ and } \text{diam}(X_i) \leq \epsilon \right\};
\]
here \( A \in P_B(X) \). Let \( S \) be a nonempty subset of \( X \), and for each \( x \in X \) define \( d(x, S) = \inf_{y \in S} d(x, y) \), and let \( B(S, r) = \{ x \in X : d(x, S) < r \}, r > 0 \). Let \( H : S \to 2^X \). \( H \) is called (i) countably \( k \)-set contractive \( (k \geq 0) \) if \( H(S) \) is bounded and \( \alpha(H(Y)) \leq k \alpha(Y) \) for all countably bounded sets \( Y \) of \( S \); (ii) countably condensing if \( H(S) \) is bounded and \( \alpha(H(Y)) < \alpha(Y) \) for all countably bounded sets \( Y \) of \( S \) with \( \alpha(Y) \neq 0 \); (iii) hemicompact if each sequence \( (x_n)_{n=1}^{\infty} \) in \( S \) has a convergent subsequence whenever \( d(x_n, H(x_n)) \to 0 \) as \( n \to \infty \).

A random operator \( F : \Omega \times C \to CD(X) \) is said to be continuous (countably \( k \)-set contractive etc.) if for each \( w \in \Omega \), the map \( F(w, .) : C \to CD(X) \) is continuous (countably \( k \)-set contractive etc.).

Next we state a well known result of Tan and Yuan [7].

**Theorem 1.4.** Let \((\Omega, A)\) be a measurable space and \( Z \) a nonempty separable complete subset of a metric space \( X = (X, d) \). Suppose the map \( F : \Omega \times Z \to CD(X) \) is a continuous, hemicompact random operator. If \( F \) has a deterministic fixed point then \( F \) has a random fixed point.

**Remark 1.2.** A single valued map \( \phi : \Omega \to X \) is said to be a deterministic fixed point of \( F \) if \( \phi(w) \in F(w, \phi(w)) \) for each \( w \in \Omega \).

In [2] we established the following convergence result.

**Theorem 1.5.** Let \((X, d)\) be a Fréchet space, \( D \) a closed subset of \( X \) and \( F : D \to 2^X \) a countably condensing map. Then \( F \) is hemicompact.

**Remark 1.3.** It is also possible to discuss in this paper operators random in the sense of Gorniewicz [3, p. 156].

2. Essential maps

Let \( E \) be a Hausdorff topological vector space, \( C \) a closed convex subset of \( E \), \( U \) an open subset of \( C \) with \( 0 \in U \). We will consider a subclass \( A \) of the \( B_k \) maps.

**Definition 2.1.** We let \( F \in D(U, C) \) if \( F \in A(U, C) \) is a closed map with nonempty (closed) values and which satisfies condition \((C)\) (i.e. if \( A \subseteq \overline{U} \) and \( A \subseteq \overline{\partial f (\{0\} \cup F(A))} \) then \( \overline{f} \) is compact).

**Definition 2.2.** We let \( F \in D_{\partial U}(U, C) \) if \( F \in D(U, C) \) with \( x \notin F(x) \) for \( x \in \partial U \); here \( \partial U \) denotes the boundary of \( U \) in \( C \).

**Definition 2.3.** A map \( F \in D_{\partial U}(U, C) \) is essential in \( D_{\partial U}(U, C) \) if for every \( G \in D_{\partial U}(U, C) \) with \( G|_{\partial U} = F|_{\partial U} \) there exists \( x \in U \) with \( x \in G(x) \).

**Theorem 2.1 (Homotopy).** Let \( E \), \( C \) and \( U \) be as above. Suppose \( F \in D(U, C) \) and assume the following conditions hold:

\[
\text{the zero map is essential in } D_{\partial U}(U, C) \quad (2.1)
\]
\[
x \notin \lambda F x \text{ for every } x \in \partial U \text{ and } \lambda \in (0, 1] \quad (2.2)
\]
and
\[
\begin{cases}
\text{for any continuous function } \mu : C \to [0, 1] \text{ and } \\
\text{any map } H \in D(\overline{U}, C) \text{ we have } \mu H \in A(\overline{U}, C).
\end{cases}
\tag{2.3}
\]

Then \( F \) is essential in \( D_{\partial U}(\overline{U}, C) \).

**Proof.** Let \( H \in D_{\partial U}(\overline{U}, C) \) with \( H|_{\partial U} = F|_{\partial U} \). We must show \( H \) has a fixed point in \( U \). Let
\[
B = \{ x \in \overline{U} : x \in \lambda H(x) \text{ for some } \lambda \in [0, 1] \}.
\]

Now \( B \neq \emptyset \) is closed and in fact compact since \( B \subseteq co(\overline{H(B)} \cup \{0\}) \). In addition \( B \cap \partial U = \emptyset \) since (2.2) holds and \( H|_{\partial U} = F|_{\partial U} \) and \( 0 \in U \). Thus there exists a continuous \( \mu : C \to [0, 1] \) with \( \mu(\partial U) = 0 \) and \( \mu(B) = 1 \). Define a map \( R \) by \( R(x) = \mu(x) H(x) \). Now \( R \in A(\overline{U}, C) \) by (2.3) and it is easy to check that \( R \in D(\overline{U}, C) \) (To see \( R \) satisfies condition (C), notice if \( A \subseteq \overline{U} \) and \( A \subseteq \overline{co}(\{0\} \cup R(\Lambda)) \) then \( A \subseteq \overline{co}(\{0\} \cup co(\{0\} \cup H(A))) = \overline{co}(\{0\} \cup H(A)) \)). Also note \( R|_{\partial U} = \{0\} \). Thus \( R \in D_{\partial U}(\overline{U}, C) \), \( R|_{\partial U} = \{0\} \) together with (2.1) implies that there exists \( x \in U \) with \( x \in R x \). Thus \( x \in B \) and so \( \mu(x) = 1 \). As a result \( x \in H(x) \). \( \Box \)

Next we discuss some normalization properties. We first discuss the case when \( E \) is locally convex and \( U \) is convex.

**Theorem 2.2** (Normalization). Let \( E \) be a locally convex Hausdorff topological vector space, \( C \) a closed convex subset of \( E \) and \( U \subseteq C \) an open convex subset of \( E \) with \( 0 \in U \). Also assume the following condition holds:
\[
\begin{cases}
\text{for any continuous map } r : E \to \overline{U} \text{ and any map } \\
\theta \in D(\overline{U}, C), \text{ the map } r|_C \theta \in B^k(\overline{U}, \overline{U}).
\end{cases}
\tag{2.4}
\]

Then the zero map is essential in \( D_{\partial U}(\overline{U}, C) \).

**Proof.** Let \( \theta \in D_{\partial U}(\overline{U}, C) \) with \( \theta|_{\partial U} = \{0\} \). We must show there exists \( x \in U \) with \( x \in \theta(x) \). Let \( \mu \) be the Minkowski functional on \( \overline{U} \) and let \( r : E \to \overline{U} \) be given by
\[
r(x) = \frac{x}{\max\{1, \mu(x)\}} \quad \text{for } x \in E.
\]

Consider \( G = r|_C \theta \). Now (2.4) guarantees that \( G \in B^k(\overline{U}, \overline{U}) \). Next we claim \( G \) satisfies condition (C). To see this let \( A \subseteq \overline{U} \) with \( A \subseteq \overline{co}(\{0\} \cup G(\Lambda)) \). Then since \( r(B) \subseteq co(B \cup \{0\}) \) for any subset \( B \) of \( E \), we have
\[
A \subseteq \overline{co}(\{0\} \cup co(\theta(\Lambda) \cup \{0\})) = \overline{co}(\{0\} \cup \theta(\Lambda)).
\]

Thus \( \overline{A} \) is compact since \( \theta \in D(\overline{U}, C) \). Theorem 1.3 guarantees that there exists \( x \in \overline{U} \) with \( x \in G(x) = r\theta(x) \). Thus \( x = r(y) \) for some \( y \in Fx \); here \( x \in \overline{U} = U \cup \partial U \) (note \( int_C U = U \) since \( U \) is also open in \( C \)). Suppose \( x \in \partial U \). Then \( \mu(x) = 1 \) and so
\[
1 = \mu(x) = \mu(r(y)) = \frac{\mu(y)}{\max\{1, \mu(y)\}} \quad \text{since} \quad r(y) = \frac{y}{\max\{1, \mu(y)\}}.
\]

Thus \( \mu(y) \geq 1 \) and so \( x = r(y) = \frac{y}{\mu(y)} \). This implies
\[
x \in \lambda \theta(x) = \{0\} \quad \text{since} \quad \theta|_{\partial U} = \{0\}; \quad \text{here} \quad \lambda = \frac{1}{\mu(y)}.
\]
This is a contradiction since $0 \in U$. As a result $x \in U$. This implies $\mu(x) < 1$. Consequently

$$1 > \mu(x) = \mu(r(y)) = \frac{\mu(y)}{\max\{1, \mu(y)\}},$$

and so $\mu(y) < 1$. Thus $r(y) = y$, so $x = y \in \theta(x)$ and we are finished. $\square$

Combining Theorems 2.1 and 2.2 yields the following nonlinear alternative of Leray–Schauder type for $D$ maps.

**Theorem 2.3.** Let $E$ be a locally convex Hausdorff topological vector space, $C$ a closed convex subset of $E$ and $U \subseteq C$ an open convex subset of $E$ with $0 \in U$. Suppose $F \in D(U, C)$ and assume (2.2)–(2.4) hold. Then $F$ is essential in $D_{\partial U}(U, C)$.

**Remark 2.1.** It is possible to relax condition (C) in Definition 2.1 if we use the results in [5] in place of Theorem 1.3.

It is also possible to obtain a normalization property when $E$ is not locally convex. To show what is possible we restrict ourselves to compact maps. Let $E$ be a Hausdorff topological vector space, $C$ a convex admissible subset of $E$ and $U$ an open subset of $C$ with $0 \in U$. Also we assume that there exists a retraction $r : C \to \overline{U}$.

**Definition 2.4.** We let $F \in D^*(\overline{U}, C)$ if $F \in A(\overline{U}, C)$ is a closed, compact map with nonempty (closed) values.

**Definition 2.5.** We let $F \in D^*_{\partial U}(\overline{U}, C)$ if $F \in D^*(\overline{U}, C)$ with $x \notin F(x)$ for $x \in \partial U$.

**Definition 2.6.** A map $F \in D^*_{\partial U}(\overline{U}, C)$ is essential in $D^*_{\partial U}(\overline{U}, C)$ if for every $G \in D^*_{\partial U}(\overline{U}, C)$ with $G|_{\partial U} = F|_{\partial U}$ there exists $x \in U$ with $x \in G(x)$.

The analogue of Theorem 2.1 is immediate for $D^*$ maps. Instead we will concentrate on the normalization property.

**Theorem 2.4 (Normalization).** Let $E$ be a Hausdorff topological vector space, $C$ a convex admissible subset of $E$ and $U$ an open subset of $C$ with $0 \in U$. Suppose there exists a retraction (continuous) $r : C \to \overline{U}$. Also assume the following two conditions hold:

$$\begin{align*}
\text{for any continuous map } \mu : C & \to [0, 1] \text{ and any} \\
\text{map } H & \in D^*(C, C), \text{ the map } \mu H \in B^k(C, C)
\end{align*}$$

and

$$\begin{align*}
\text{for any map } \theta & \in D^*(\overline{U}, C), \text{ the map } \theta r \in A(C, C).
\end{align*}$$

Then the zero map is essential in $D^*_{\partial U}(\overline{U}, C)$.

**Proof.** Let $\theta \in D^*_{\partial U}(\overline{U}, C)$ with $\theta|_{\partial U} = \{0\}$. Next let

$$A = \{x \in \overline{U} : x \in \lambda \theta(x) \text{ for some } \lambda \in [0, 1]\}.$$

Now $A \neq \emptyset$ is closed and in fact compact. Also $A \subseteq U$ since $\theta|_{\partial U} = \{0\}$ and $0 \in U$. Thus there exists a continuous function $\mu : C \to [0, 1]$ with $\mu(A) = 1$ and $\mu(C \setminus U) = 0$. Define a map $J$ by

$$J(x) = \mu(x) \theta(r(x))$$

for $x \in C$. 


Now (2.6) implies \( \theta \circ r \in A(C, C) \). In addition it is easy to check that \( \theta \circ r \) is a closed, compact map, so as a result \( \theta \circ r \in D^*(C, C) \). This together with (2.5) yields \( J \in B^k(C, C) \). Theorem 1.1 implies that there exists \( x \in C \) with \( x \in \mu(x) \theta(r(x)) \). If \( x \in C \setminus U \) then \( \mu(x) = 0 \), a contradiction since \( 0 \in U \). Thus \( x \in U \) and so \( x \in \mu(x) \theta(x) \). As a result \( x \in A \), so \( \mu(x) = 1 \). Thus \( x \in \theta(x) \). \( \square \)

Next we discuss countably \( P \)-concentrative maps. Let \( E \) be a Fréchet space (\( P \) a defining system of seminorms), \( C \) a closed convex subset of \( E \) and \( U \) an open subset of \( C \) with \( 0 \in U \).

**Definition 2.7.** We let \( F \in M(\overline{U}, C) \) if \( F \in A(\overline{U}, C) \) is a closed countably \( P \)-concentrative map with nonempty (closed) values.

**Definition 2.8.** We let \( F \in M_{\partial U}(\overline{U}, C) \) if \( F \in M(\overline{U}, C) \) with \( x \notin F(x) \) for \( x \in \partial U \).

**Definition 2.9.** A map \( F \in M_{\partial U}(\overline{U}, C) \) is essential in \( M_{\partial U}(\overline{U}, C) \) if for every \( G \in M_{\partial U}(\overline{U}, C) \) with \( G|_{\partial U} = F|_{\partial U} \) there exists \( x \in U \) with \( x \in G(x) \).

**Theorem 2.5 (Homotopy).** Let \( E, C \) and \( U \) be as above. Suppose \( F \in M(\overline{U}, C) \) satisfies (2.2) and assume the following conditions hold:

- the zero map is essential in \( M_{\partial U}(\overline{U}, C) \) \hspace{1cm} (2.7)
- for any continuous function \( \mu : C \to [0, 1] \) and any map \( H \in M(\overline{U}, C) \) we have \( \mu H \in A(\overline{U}, C) \). \hspace{1cm} (2.8)

Then \( F \) is essential in \( M_{\partial U}(\overline{U}, C) \).

**Proof.** Let \( H \) and \( B \) be as in Theorem 2.1. Now \( B \neq \emptyset \) is closed (in fact \( H \) countably \( P \)-concentrative implies \( B \) is sequentially compact so compact). Essentially the same argument as in Theorem 2.1 establishes the result. \( \square \)

**Theorem 2.6 (Normalization).** Let \( E \) be a Fréchet space (\( P \) a defining system of seminorms), \( C \) a closed convex subset of \( E \) and \( U \subseteq C \) an open bounded convex subset of \( E \) with \( 0 \in U \). In addition assume the following condition is satisfied:

- for any continuous map \( r : E \to \overline{U} \) and any map \( \theta \in M(\overline{U}, C) \), the map \( r|_{\partial U} \in B^k(\overline{U}, C) \). \hspace{1cm} (2.9)

Then the zero map is essential in \( M_{\partial U}(\overline{U}, C) \).

**Proof.** Let \( \theta \in M_{\partial U}(\overline{U}, C) \) with \( \theta|_{\partial U} = [0] \). Let \( \mu \) and \( r \) be as in Theorem 2.2 and \( G = r|_{\partial \theta} \). Notice \( G \in B^k(\overline{U}, C) \) is closed and countably \( P \)-concentrative. To see that \( G \) is countably \( P \)-concentrative consider \( p \in P \) and a countably bounded non-\( p \)-precompact subset \( \Omega \) of \( \overline{U} \). Then since

\[
G(\Omega) \subseteq co (\theta(\Omega) \cup [0]),
\]
we have \( \alpha_p(G(\Omega)) \leq \alpha_p(\theta(\Omega)) < \alpha_p(\Omega) \). Theorem 1.2 guarantees that there exists \( x \in \overline{U} \) with \( x \in G(x) = r \theta(x) \). An argument similar to that in Theorem 2.2 finishes the proof. \( \square \)

Next we present a normalization property without assuming \( U \) is bounded.
Theorem 2.7 (Normalization). Let $E$ be a Fréchet space ($P$ a defining system of seminorms) and $U$ an open convex subset of $E$ with $0 \in U$. In addition assume the following condition is satisfied:

\[
\begin{align*}
&\text{for any continuous map } r : E \to \overline{U} \text{ and any continuous} \\
&\text{map } \mu : E \to [0, 1]\text{ and any map } \theta \in M(\overline{U}, E), \text{ the} \\
&\text{map } \mu \theta r \in \mathcal{B}^k(C, C); \text{ here } C = \overline{\sigma(\theta(U) \cup \{0\})}.
\end{align*}
\]

(2.10)

Then the zero map is essential in $\mathcal{M}_{au}(U, E)$.

Proof. Let $\theta \in \mathcal{M}_{au}(U, E)$ with $\theta |_{au} = \{0\}$. Next let

\[A = \{x \in U : x \in \lambda \theta(x) \text{ for some } \lambda \in [0, 1]\}.
\]

As in Theorem 2.4, $A \subset U$ and there exists a continuous function $\mu : E \to [0, 1]$ with $\mu(A) = 1$ and $\mu(E \setminus U) = 0$. Let $r : E \to \overline{U}$ be given by

\[r(x) = \frac{x}{\max\{1, g(x)\}}
\]

where $g$ is the Minkowski functional on $\overline{U}$. Let $C = \overline{\sigma(\theta(U) \cup \{0\})}$ (which is bounded) and define a map $J$ by $J(x) = \mu(x) \theta(r(x))$. From (2.10) we have $J \in \mathcal{B}^k(C, C)$ and it is also immediate that $J$ is countably P-concentrative. Theorem 1.2 implies that there exists $x \in C$ with $x \in \mu(x) \theta(r(x))$. If $x \notin U$ then $\mu(x) = 0$, a contradiction. Thus $x \in U$ and so $x \in \mu(x) \theta(x)$. As a result $x \in A$, so $\mu(x) = 1$. \[\square\]

In view of Theorems 1.4 and 1.5 it is easy to use our fixed point theory for $\mathcal{B}^k$ maps to establish random fixed point theory. We will present here a new and very general result which includes all well known random fixed point theory in the literature [4,7,8].

Theorem 2.8. Let $(\Omega, A)$ be a measurable space and $X$ a nonempty closed convex bounded subset of a separable Banach space $E$. Suppose $F : \Omega \times X \to CD(X)$ is a random continuous, countably condensing operator with $F(w, \cdot) \in \mathcal{B}^k(X, X)$ for each $w \in \Omega$. Then $F$ has a random fixed point.

Remark 2.2. (i) If $F$ is a random condensing operator then the assumption that $X$ is bounded can be removed in the statement of Theorem 2.7.

(ii) It is possible to consider operators random in the sense of Gorniewicz [3, p. 156]. We leave the details to the reader.

Proof. Theorem 1.5 guarantees that $F : \Omega \times X \to CD(X)$ is hemicompact. Fix $w \in \Omega$. Now Theorem 1.2 guarantees that $F(w, \cdot)$ has a fixed point. As a result $F$ has a deterministic fixed point, so Theorem 1.4 guarantees that $F$ has a random fixed point. \[\square\]

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