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Hochschild cohomology and Atiyah classes ☆

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Abstract

In this paper we prove that on a smooth algebraic variety the HKR-morphism twisted by the square root of the Todd genus gives an isomorphism between the sheaf of poly-vector fields and the sheaf of poly-differential operators, both considered as derived Gerstenhaber algebras. In particular we obtain an isomorphism between Hochschild cohomology and the cohomology of poly-vector fields which is compatible with the Lie bracket and the cupproduct. The latter compatibility is an unpublished result by Kontsevich.

Our proof is set in the framework of Lie algebroids and so applies without modification in much more general settings as well.

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Contents

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1. Introduction

In the body of the paper we will work over a ringed site over a field of characteristic zero k. Thus our results are for example applicable to stacks. However in this introduction we will state our results for a commutatively ringed space $(X, \mathcal{O}_X)$.

By definition a *Lie algebroid* on X is a sheaf of Lie algebras $\mathcal{L}$ which is an $\mathcal{O}_X$-module and is equipped with an action $\mathcal{L} \times \mathcal{O}_X \to \mathcal{O}_X$ with properties mimicking those of the tangent bundle (see Section 4.2 for a more precise definition). Throughout $\mathcal{L}$ will be a locally free Lie algebroid over $(X, \mathcal{O}_X)$ of constant rank d.

Lie algebroids are a means of algebraizing differential geometry. For example they allow us to treat the algebraic/complex analytic and C^∞-case in a uniform way.

Example 1.1. The following are examples of (locally free) Lie algebroids.

(1) The sheaf of vector fields on a C^∞-manifold.
(2) The sheaf of holomorphic vector fields on a complex analytic variety.
(3) The sheaf of algebraic vector fields on a smooth algebraic variety.
(4) $\mathcal{O}_X \otimes \mathfrak{g}$ where $\mathfrak{g}$ is the Lie algebra of an algebraic group acting on a smooth algebraic variety X.

Example 1.2. Assume that X is an affine integral singular algebraic variety. Then $\mathcal{T}_X$ is not locally free. However it is always possible to construct a locally free sub Lie algebroid $\mathcal{L} \subset \mathcal{T}_X$. So our setting applies to some extent to the singular case as well.

The Atiyah class $A(\mathcal{L})$ of $\mathcal{L}$ is the element of $\mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{L}, \mathcal{L}^* \otimes_{\mathcal{O}_X} \mathcal{L})$ which is the obstruction against the existence of an $\mathcal{L}$-connection on $\mathcal{L}$. The ith ($i > 0$) scalar Atiyah class $a_i(\mathcal{L})$ is defined as

$$\mathrm{Alt}\,\mathrm{Tr}\big(A(\mathcal{L})^i\big) \in H^i\Big(X, \Big(\bigwedge^i \mathcal{L}\Big)^*\Big).$$

In the C^∞ or affine case we have $a_i(\mathcal{L}) = 0$ as the cohomology groups $H^i(X, (\bigwedge^i \mathcal{L})^*)$ vanish. If X is a Kahler manifold and $\mathcal{L}$ is the sheaf of holomorphic vector fields then $a_i(\mathcal{T}_X)$ coincides with the ith Chern class of $\mathcal{T}_X$ (see, e.g., [20, (1.4)]).

The Todd class of $\mathcal{L}$ is defined as

$$\mathrm{td}(\mathcal{L}) = \det\big(q\big(A(\mathcal{L})\big)\big)$$

where

$$q(x) = \frac{x}{1 - e^{-x}}.$$

One sees without difficulty that $\mathrm{td}(\mathcal{L})$ can be expanded formally in terms of $a_i(\mathcal{L})$.

The sheaf of $\mathcal{L}$-poly-vector fields on X is defined as $T_{\mathrm{poly}}^{\mathcal{L}}(\mathcal{O}_X) = \bigoplus_i \bigwedge^i \mathcal{L}$. This agrees with the standard definitions in case $\mathcal{L}$ is one of the variants of the tangent bundle described in the examples above. It is easy to prove that $T_{\mathrm{poly}}^{\mathcal{L}}(\mathcal{O}_X)$ is a sheaf of Gerstenhaber algebras on X.

In the case that X is a C^∞-manifold Kontsevich introduced the sheaf of so-called poly-differential operators on X. This is basically a localized version of the Hochschild complex.[2] It is straightforward to construct a Lie algebroid generalization $D_{\mathrm{poly}}^{\mathcal{L}}(\mathcal{O}_X)$ of this concept as well (see [4] or Section 4.2.2). Like $T_{\mathrm{poly}}^{\mathcal{L}}(\mathcal{O}_X)$, $D_{\mathrm{poly}}^{\mathcal{L}}(\mathcal{O}_X)$ is equipped with a Lie bracket and an associative cupproduct but these operations satisfy the Gerstenhaber axioms only up to globally defined homotopies (see, e.g., [17]).

The so-called Hochschild–Kostant–Rosenberg map is a quasi-isomorphism between $T_{\mathrm{poly}}^{\mathcal{L}}(\mathcal{O}_X)$ and $D_{\mathrm{poly}}^{\mathcal{L}}(\mathcal{O}_X)$ [4,41]. This paper is concerned with the failure of the HKR-map to be compatible with the Lie brackets and cupproducts on $T_{\mathrm{poly}}^{\mathcal{L}}(\mathcal{O}_X)$ and $D_{\mathrm{poly}}^{\mathcal{L}}(\mathcal{O}_X)$.

Let $\mathsf{D}(X)$ be the derived category of sheaves of k-vector spaces. This category is equipped with a symmetric monoidal structure given by the derived tensor product. As indicated above $D_{\mathrm{poly}}^{\mathcal{L}}(\mathcal{O}_X)$ is a Gerstenhaber algebra in $\mathsf{D}(X)$. We have the following result.

Theorem 1.3. *(See Section 10.) The map in* $\mathsf{D}(X)$

$$T_{\mathrm{poly}}^{\mathcal{L}}(\mathcal{O}_X) \xrightarrow{\mathrm{HKR}\circ(\mathrm{td}(\mathcal{L})^{1/2}\wedge -)} D_{\mathrm{poly}}^{\mathcal{L}}(\mathcal{O}_X) \tag{1.1}$$

is an isomorphism of Gerstenhaber algebras in $\mathsf{D}(X)$.

Applying the hypercohomology functor $\mathbb{H}^*(X,-)$ we get

Corollary 1.4. *The map*

$$\bigoplus_{i,j} H^j\Big(X, \bigwedge^i \mathcal{L}\Big) \xrightarrow{\mathrm{HKR}\circ(\mathrm{td}(\mathcal{L})^{1/2}\wedge -)} \mathbb{H}^*\big(X, D_{\mathrm{poly}}^{\mathcal{L}}(\mathcal{O}_X)\big) \tag{1.2}$$

is an isomorphism of Gerstenhaber algebras.

Let us restrict to the setting where X is a smooth algebraic variety and $\mathcal{L} = \mathcal{T}_X$. In that case it follows from the proof of [36, Thm. 3.1(1)] together with [23, Thm. 7.5.1] that the right-hand side of (1.2) can be viewed as the Hochschild cohomology $\mathrm{HH}^*(X)$ of X (in the sense that it controls for example the deformation theory of $\mathrm{Mod}(\mathcal{O}_X)$). So we may rephrase Corollary 1.4 as

[2] It is not entirely trivial to make the Hochschild complex into a (pre)sheaf as the assignment $U \mapsto C^*(\Gamma(U, \mathcal{O}_U))$ is not compatible with restriction.

Corollary 1.5. *There is an isomorphism of Gerstenhaber algebras*

$$\bigoplus_{i,j} H^j\Big(X, \bigwedge^i \mathcal{T}_X\Big) \xrightarrow{\mathrm{HKR}\circ(\mathrm{td}(\mathcal{L})^{1/2}\wedge -)} \mathrm{HH}^*(X). \tag{1.3}$$

A version of this result which refers only to the cupproduct was proved by Kontsevich (see [9, Thm. 5.1]). For the cupproduct one can use Swan's definition of Hochschild cohomology [32]

$$\mathrm{HH}^i(X) = \mathrm{Ext}^i_{X\times X}(\mathcal{O}_X, \mathcal{O}_X) \tag{1.4}$$

as Yekutieli [40,41] shows that there is an isomorphism

$$\mathbb{H}^i\big(X, D_{\mathrm{poly}}(\mathcal{O}_X)\big) \to \mathrm{Ext}^i_{X\times X}(\mathcal{O}_X, \mathcal{O}_X) \tag{1.5}$$

which is compatible with the cupproduct on the left and the Yoneda product on the right.

Remark 1.6. The algebra isomorphism (1.3) is part of a more general conjecture by Caldararu [9] which involves also the Hochschild *homology* of X. Other parts of this conjecture were proved by Markarian and Ramadoss [25,28].

Remark 1.7. The cupproduct on $T^{\mathcal{L}}_{\mathrm{poly}}(\mathcal{O}_X)$ and $D^{\mathcal{L}}_{\mathrm{poly}}(\mathcal{O}_X)$ is $\mathcal{O}_X$-linear and hence these objects can also be considered as algebras in $\mathsf{D}(\mathrm{Mod}(\mathcal{O}_X))$. Likewise the map (1.1) can be viewed as an isomorphism in $\mathsf{D}(\mathrm{Mod}(\mathcal{O}_X))$.

The cupproduct on $T^{\mathcal{L}}_{\mathrm{poly}}(\mathcal{O}_X)$ is commutative and the cupproduct on $D^{\mathcal{L}}_{\mathrm{poly}}(\mathcal{O}_X)$ is commutative up to a homotopy given by the bullet product [17]. However the latter is *not* $\mathcal{O}_X$-linear. Hence $D^{\mathcal{L}}_{\mathrm{poly}}(\mathcal{O}_X)$ is not commutative in $\mathsf{D}(\mathrm{Mod}(\mathcal{O}_X))$ and thus Theorem 1.3 does *not* hold in $\mathsf{D}(\mathrm{Mod}(\mathcal{O}_X))$ even if we consider only the cupproduct (we thank Andrei Caldararu for bringing this point to our attention).

This situation is reminiscent of the Duflo isomorphism $S\mathfrak{g} \to U\mathfrak{g}$ which only becomes an algebra isomorphism after taking invariants. The analogue of taking invariants in our setting is taking global sections. For an extensive discussion on how the results in this paper relate to Duflo's theorem, see [6].

If we look only at the Lie algebra structure we actually prove a result which is somewhat stronger than Theorem 1.3. Let $\mathrm{HoLieAlg}(X)$ be the category of sheaves of DG-Lie algebras on X with quasi-isomorphisms inverted.

Theorem 1.8. *(See Section 7.4.) The isomorphism* (1.1) *between* $T^{\mathcal{L}}_{\mathrm{poly}}(\mathcal{O}_X)$ *and* $D^{\mathcal{L}}_{\mathrm{poly}}(\mathcal{O}_X)$ *is obtained from an isomorphism in* $\mathrm{HoLieAlg}(X)$.

This theorem can be considered as a generalization of global formality results in [8,13,15,22, 36,42]. Global formality on the sheaf level is important for deformation theory. See for example [5,21,44,36,42].

The isomorphism in $\mathrm{HoLieAlg}(X)$ is obtained by globalizing a local formality isomorphism [22,33]. If we take Kontsevich's local formality isomorphism then we obtain compatibility with cupproduct in Theorem 1.3 from the compatibility of the local formality isomorphism with tangent cohomology [22,24,26]. Kontsevich's local formality isomorphism is only defined when the ground field contains $\mathbb{R}$ but this is not a problem since we show that it is sufficient to prove Theorem 1.3 over a suitable extension of the base field.

An alternative approach to Theorem 1.3 could be to work directly in the setting of G_∞-algebras. Unfortunately it is unknown if Kontsevich's L_∞-morphism can be lifted to a G_∞-morphism. In [7] we will use Tamarkin's local G_∞-formality isomorphism to construct a G_∞-quasi-isomorphism between $T^{\mathcal{L}}_{\mathrm{poly}}(\mathcal{O}_X)$ and $D^{\mathcal{L}}_{\mathrm{poly}}(\mathcal{O}_X)$. In the case that $\mathcal{L}$ is a tangent bundle this was proved recently in [16] using very different methods. Like in [16] we are unfortunately not able to write down the resulting isomorphism on hypercohomology. Thus in this way we obtain a result which is less precise than Corollary 1.5. This is why we have decided to publish the current paper separately.

2. Acknowledgments

This paper is hugely in debt to Kontsevich's fundamental work on formality. In particular without the many deep results and insights contained in [22] this paper could not have been written.

Our proof of the global formality result Theorem 1.8 follows the general outline of [36] which in turn was heavily inspired by [42]. We use in an essential way an algebraic version of formal geometry. Algebraic versions of formal geometry were introduced independently and around the same time by Bezrukavnikov and Kaledin in [3] and Yekutieli in [42]. The language we use is closer to [42]. As a result various technical statements can be traced back in some form to [42].

We wish to thank Andrei Caldararu, Vasiliy Dolgushev, Charles Torossian and Amnon Yekutieli for useful conversations and comments.

3. Notation and conventions

Throughout this paper k is a field of characteristic zero. Unadorned tensor products are over k.

Many of the objects we use are equipped with some kind of topology, but if an object is introduced without a specified topology we assume that it is equipped with the discrete topology.

If an object carries a natural grading then all constructions associated to it are implicitly performed in the graded context. This applies in particular to completions.

Since all our constructions are natural in the sense that they do not depend on any choices we work mostly with rings and modules instead of with sheaves since this often simplifies the notation. However we freely sheafify such constructions if needed.

We work throughout in a "super" context. I.e. notions like symmetric product and commutativity should be interpreted using the standard sign conventions.

On a double (or higher) complex we use the Koszul sign convention with respect to total degree.

4. Preliminaries

4.1. Categories of vector spaces

Below we will use algebras and modules over various enhanced symmetric monoidal categories of k-vector spaces. Which category we work in will usually be clear from the context but in order to be precise we list here the possibilities.

4.1.1. Complete topological vector spaces

For us a complete topological vector space V will be a topological vector space whose topology is generated by a separated, exhaustive descending filtration $V = F_0 V \supset F_1 V \supset \cdots$. This filtration is however not considered as part of the structure. Note that a vector space equipped with the discrete topology is complete (put $F^0 V = V$, $F^n V = 0$ for $n > 0$).

The completed tensor product

$$V \hat{\otimes} W = \operatorname{proj}\lim_{p} \big(V \otimes W / (F_p V \otimes W + V \otimes F_p W) \big)$$

makes the category of complete topological vector spaces into a symmetric monoidal category. Completed symmetric and exterior products can be defined by similar formulas.

4.1.2. Filtered complete topological vector spaces

A filtered complete topological vector space is by definition a topological vector space V, equipped with an ascending separated, exhaustive filtration $F^m V$ (which is considered part of the structure) such that each $F^m V$ is a complete topological vector space and the inclusion maps $F^m V \hookrightarrow F^{m+1} V$ are continuous.

The (completed) tensor product of two filtered complete topological vector space V and W is defined by

$$F^m (V \hat{\otimes} W) = \sum_{p+q=m} F^p V \hat{\otimes} F^q W$$

(where the summation sign refers of course to convergent sums). Completed symmetric and exterior products can be defined by similar formulas.

A (bounded) morphism $f : V \to W$ between filtered complete topological vector spaces is a morphism of vector spaces such that $f(F^n V) \subset F^{n+l} W$ for some l such that the induced map $F^n V \to F^{n+l} W$ is continuous. If we may choose $l = 0$ then we say that f is filtered. It is clear that the completed tensor product is compatible with bounded and filtered morphisms.

4.1.3. Complexes and quasi-isomorphisms

Complete and filtered complete topological vector spaces form additive categories so we can take complexes over them. In all the filtered complexes we will encounter the differential will be filtered (instead of just bounded).

To define quasi-isomorphisms we note that complete and filtered complete topological vector spaces are equipped with an obvious forgetful functor to ordinary vector spaces. We say that a complex is acyclic if it is acyclic after applying this functor. Quasi-isomorphisms are defined similarly.

Of course in general this would not be a satisfactory definition, and in order to develop homological algebra one would need to define suitable exact structures.

But our ultimate aim is to construct quasi-isomorphisms between objects carrying the *discrete topology* (the local versions of $T^{\mathcal{L}}_{\mathrm{poly}}$ and $D^{\mathcal{L}}_{\mathrm{poly}}$), and complete objects appear only at intermediate stages. So the naive definition is sufficient for our applications.

4.2. *Lie algebroids*

Below R is a commutative k-algebra and L is a *Lie algebroid* over R which is free[3] of rank d. Namely, L is a Lie k-algebra equipped with an R-module structure and a Lie algebra map $\rho : L \to \mathrm{Der}(R)$ such that $[l_1, rl_2] = r[l_1, l_2] + \rho(l_1)(r)l_2$ for $l_1, l_2 \in L$ and $r \in R$. ρ is called the *anchor map* and we usually suppress it from the notation writing $l(r)$ instead of $\rho(l)(r)$ ($l \in L$, $r \in R$). In particular, $R \oplus L$ becomes a Lie algebra with bracket given by $[(r,l),(r',l')] = (l(r') - l'(r), [l,l'])$.

Associated to L there are various constructions which are analogous to constructions occurring for enveloping algebras and rings of differential operators. In the next few paragraphs we fix some notation for them and recall the properties we need. For more information the reader is referred to [4,8,27,39].

4.2.1. *The enveloping algebra of a Lie algebroid*

Let UL be the enveloping algebra associated to L. It is the quotient of the enveloping algebra associated to the Lie algebra $R \oplus L$ by the following relations: $r \otimes l = rl$ ($r \in R$, $l \in R \oplus L$). If we want to emphasize R then we write $U_R L$. UL has a canonical filtration obtained by respectively assigning length 0 and 1 to elements of R and L. We equip UL with the left R-module structure given by the natural embedding $R \to UL$ and we view UL as an R-bimodule *with the same left and right structure*. For this bimodule structure UL is a cocommutative R-coring in the sense that there is a natural cocommutative coassociative comultiplication $\Delta : UL \to UL \otimes_R UL$ and counit $\epsilon : UL \to R$. Assuming the Sweedler convention the comultiplication is defined by

$$\Delta(f) = f \otimes 1 \quad \text{for } f \in R,$$
$$\Delta(l) = l \otimes 1 + 1 \otimes l \quad \text{for } l \in L,$$
$$\Delta(DE) = D_{(1)}E_{(1)} \otimes D_{(2)}E_{(2)} \quad \text{for } D, E \in UL.$$

Note that it requires some verification to show that this is well defined. To do this note that $UL \otimes_R UL$ is a right $UL \otimes UL$-module in the obvious way. One proves inductively on the length of D, expressed as a product of elements of L, that in $UL \otimes_R UL$ one has

$$(D_{(1)} \otimes D_{(2)})(f \otimes 1 - 1 \otimes f) = 0$$

for f in R. It follows immediately that if $E' \otimes E'' \in UL \otimes_R UL$ then

$$(D_{(1)} \otimes D_{(2)}) \cdot \big(E' \otimes E''\big) \stackrel{\text{def}}{=} D_{(1)}E' \otimes D_{(2)}E''$$

is well defined.

There is a unique way to extend the anchor map into an algebra morphism $\rho : U(L) \to \mathrm{End}(R)$. As before we write $D(r)$ instead of $\rho(D)(r)$ ($D \in UL$, $r \in R$), and then the counit on UL is given by

$$\epsilon(D) = D(1).$$

[3] What we say remains valid if we only assume L to be finitely generated projective of constant rank d. But since our intent is to describe the local situation we may as well assume that L is free.

UL is a so-called "Hopf algebroid with anchor" [39]. As we are in the cocommutative case this is expressed by the property

$$D_{(1)}(f)D_{(2)} = Df \quad (f \in R,\ D \in UL).$$

4.2.2. L-poly-vector fields and L-poly-differential operators

$T^L_{\text{poly}}(R)$ is the Lie algebra of L-poly-vector fields [4]. I.e. it is the graded vector space $\bigwedge_R(L)[1]$ equipped with the graded Lie bracket obtained by extending the Lie bracket on L. We equip $T^L_{\text{poly}}(R)$ with the standard cupproduct (which is of degree one with our shifted grading). In this way $T^L_{\text{poly}}(R)$ becomes a (shifted) Gerstenhaber algebra. In fact we will regard $T^L_{\text{poly}}(R)$ as a DG-Gerstenhaber algebra with zero differential. Furthermore we consider $T^L_{\text{poly}}(R)$ as being filtered by L-power (in each homological degree this filtration has only one non-trivial quotient).

$D^L_{\text{poly}}(R)$ is the DG-Lie algebra of L-poly-differential operators [4]. I.e. it is the graded filtered vector space $T_R(UL)[1]$ equipped with the natural structure of a DG-Lie algebra [4]. The Lie bracket on $D^L_{\text{poly}}(R)$ given by $[D_1, D_2]_G = D_1 \bullet D_2 - (-1)^{|D_1||D_2|} D_2 \bullet D_1$, where

$$D_1 \bullet D_2 = \sum_{i=0}^{|D_1|} (-1)^{i|D_2|} \left(\mathrm{id}^{\otimes i} \otimes \Delta^{|D_2|} \otimes \mathrm{id}^{\otimes |D_1|-i}\right)(D_1) \cdot \left(1^{\otimes i} \otimes D_2 \otimes 1^{\otimes |D_1|-i}\right).$$

Let $m = 1 \otimes 1 \in D^L_{\text{poly}}(R)_1$. Then the differential d on $D^L_{\text{poly}}(R)$ is given by

$$d(D) = [m, -].$$

For the cupproduct we use the sign-modification by Gerstenhaber and Voronov [17]. This sign-modification is necessary to make the cohomology of $D^L_{\text{poly}}(R)$ into a (shifted) Gerstenhaber algebra. We put

$$D_1 \cup D_2 = (-1)^{(|D_1|+1)(|D_2|+1)} D_1 \otimes D_2.$$

There is a HKR-theorem relating $T^L_{\text{poly}}(R)$ and $D^L_{\text{poly}}(R)$ [4]. Namely the map

$$\mu : l_1 \wedge \cdots \wedge l_n \mapsto (-1)^{n(n-1)/2} \frac{1}{n!} \sum_{\sigma \in S_n} \epsilon(\sigma) l_{\sigma(1)} \otimes \cdots \otimes l_{\sigma(n)} \tag{4.1}$$

defines a quasi-isomorphism between $(T^L_{\text{poly}}(R), 0)$ and $(D^L_{\text{poly}}(R), d)$ which induces an isomorphism of shifted Gerstenhaber algebras on cohomology. Note that for this last fact to be true one needs the unconventional sign in (4.1).

4.2.3. Algebraic functoriality

The formation of UL, $T^L_{\text{poly}}(R)$ and $D^L_{\text{poly}}(R)$ depends functorially on L in a suitable sense.

Definition 4.2.1. An *algebraic morphism* of Lie algebroids

$$(R, L) \to (T, M)$$

is a pair (ϱ, ℓ) of an algebra morphism $\varrho : R \to T$ and a Lie algebra morphism $\ell : L \to M$ such that for any $r \in R$ and any $l \in L$,

$$\varrho\big(l(r)\big) = \ell(l)\big(\varrho(r)\big) \quad \text{and} \quad \ell(rl) = \varrho(r)\ell(l).$$

For any algebraic morphism $(R, L) \to (T, M)$ there are obvious associated maps

$$U_R L \to U_T M,$$
$$T^L_{\mathrm{poly}}(R) \to T^M_{\mathrm{poly}}(T),$$
$$D^L_{\mathrm{poly}}(R) \to D^M_{\mathrm{poly}}(T),$$

that are compatible with all algebraic structures.

4.2.4. Pairings, the De Rham complex and L-connections

Put $L^* = \mathrm{Hom}_R(L, R)$. We identify $\bigwedge^n_R L^*$ with the R-dual of $\bigwedge^n_R L$ via the pairing

$$\langle \sigma_1 \wedge \cdots \wedge \sigma_n, l_1 \wedge \cdots \wedge l_n\rangle = \det \sigma_i(l_j). \tag{4.2}$$

If $\tau \in L^*$ then we denote contraction by τ acting on $\bigwedge^n_R L$ by $\tau \wedge -$. I.e.

$$\tau \wedge (l_1 \wedge \cdots \wedge l_n) = \sum_i (-1)^{i-1} \tau(l_i)(l_1 \wedge \cdots \wedge \hat{l}_i \wedge \cdots \wedge l_n).$$

We make $\bigwedge_R L$ into a $\bigwedge_R L^*$-module by extending the $- \wedge -$-action. I.e.

$$(\tau_1 \wedge \cdots \wedge \tau_m) \wedge (l_1 \wedge \cdots \wedge l_n) = \tau_1 \wedge \big(\tau_2 \wedge \big(\cdots \wedge \big(\tau_m \wedge (l_1 \wedge \cdots \wedge l_n)\big) \cdots\big)\big).$$

An easy verification shows

$$\langle \sigma_1 \wedge \cdots \wedge \sigma_n, l_1 \wedge \cdots \wedge l_n\rangle = \big\langle \sigma_{m+1} \wedge \cdots \wedge \sigma_n, (\sigma_m \wedge \cdots \wedge \sigma_1) \wedge (l_1 \wedge \cdots \wedge l_n)\big\rangle. \tag{4.3}$$

The Lie algebroid analogue for the De Rham complex is a DG-algebra which as graded algebra is equal to $\bigwedge_R L^*$. With the identification (4.2) the differential on $\bigwedge_R L^*$ is given by the usual formula for differential forms [37, Prop. 2.25(f)]:

$$\begin{aligned} d\omega(l_0, \ldots, l_n) = {} & \sum_{i=0}^{n} (-1)^i l_i\big(\omega(l_0, \ldots, \hat{l}_i, \ldots, l_p)\big) \\ & + \sum_{i<j} (-1)^{i+j} \omega\big([l_i, l_j], l_0, \ldots, \hat{l}_i, \ldots, \hat{l}_j, \ldots, l_n\big). \end{aligned} \tag{4.4}$$

In other words the anchor map $\rho : L \to \mathrm{Der}_k(R) = \mathrm{Hom}_R(\Omega^1_R, R)$ dualizes to a morphism of DG-algebras

$$\rho^* : \Omega_R \to \bigwedge\nolimits_R L^*. \tag{4.5}$$

From (4.4) we deduce in particular $(df)(l) = l(f)$ for $f \in R$, $l \in L$, and if $(l_i)_i$ is an R-basis of L then

$$d\big(l_k^*\big)(l_i, l_j) = -l_k^*\big([l_i, l_j]\big).$$

In other words $(\bigwedge_R L^*, d)$ completely encodes the Lie algebroid structure of L.

Remark 4.2.2. In the literature a morphism between Lie algebroids $(R, L) \to (T, M)$ is usually defined as a morphism of DG-algebras $\eta : \bigwedge_T M^* \to \bigwedge_R L^*$. See, e.g., [10]. One could call such morphisms "geometric" to differentiate them from the algebraic ones we use. We have already encountered one geometric morphism, namely (4.5).

If M is an R-module then an L-connection on M is a map $L \otimes M \to M : l \otimes m \mapsto \nabla_l(m)$ with the following properties: for $l, l_1, l_2 \in L$, $m \in M$, $f \in R$ we have

$$\nabla_l(fm) = l(f)m + f\nabla_l(m),$$
$$\nabla_{fl}(m) = f\nabla_l(m).$$

The connection is flat if in addition we have

$$[\nabla_{l_1}, \nabla_{l_2}] = \nabla_{[l_1,l_2]}.$$

In that case M automatically becomes a left UL-module. Moreover, if $(l_i)_i$ is a basis of L then we put a left $\bigwedge_R L^*$-DG-module structure on $\bigwedge_R L^* \otimes_R M$ by defining the differential as

$$\nabla(\omega \otimes m) = d\omega \otimes m + \sum_i l_i^* \omega \otimes \nabla_{l_i}(m). \tag{4.6}$$

Recall that if C is a commutative R-DG-algebra then a *flat connection* on M is a derivation of square zero on $C \otimes_R M$ of degree one which makes $C \otimes_R M$ into a DG-C-module.[4] Thus ∇ is a flat $\bigwedge_R L^*$-connection on M.

4.2.5. L-jets

Let $(UL)_{\leqslant n}$ be the elements of degree $\leqslant n$ for the canonical filtration on UL introduced in Section 4.2.1. The L-n-jets are defined as

$$J^n L = \operatorname{Hom}_R\big((UL)_{\leqslant n}, R\big)$$

(this is unambiguous, as the left and right R-modules structures on UL are the same, see Section 4.2.1). We view JL as a complete topological vector space in the sense of Section 4.1.1.

We also put

$$JL = \operatorname{Hom}_R(UL, R) = \operatorname{proj}\lim_n J^n L \quad \Big(\text{as } UL = \operatorname{inj}\lim_n (UL)_{\leqslant n}\Big). \tag{4.7}$$

[4] We recall that if M is in a category of complete topological vector spaces then one has to write $C \hat{\otimes} M$ instead.

We now formulate some properties of JL. Most of these properties hold for J^nL as well. JL has a natural commutative algebra structure obtained from the comultiplication on UL. Thus for $\phi_1, \phi_2 \in JL$, $D \in UL$ we have

$$(\phi_1\phi_2)(D) = \phi_1(D_{(1)})\phi_2(D_{(2)}),$$

and the unit in JL is given by the counit on UL. It is well known that JL has a lot of extra structure which we now elucidate. First of all there are two distinct monomorphisms of k-algebras

$$\alpha_1 : R \to JL : r \mapsto \bigl(D \mapsto r\epsilon(D)\bigr),$$
$$\alpha_2 : R \to JL : r \mapsto \bigl(D \mapsto D(r)\bigr).$$

It will be convenient to write $R_i = \alpha_i(R)$ and to view JL as an R_1–R_2-bimodule.

Define $\epsilon : JL \to R$ by $\epsilon(\phi) = \phi(1)$ and put $J^cL = \ker \epsilon$. It is easy to see that $\epsilon \circ \alpha_1 = \epsilon \circ \alpha_2 = \mathrm{id}_R$. We conclude that

$$JL = R_1 \oplus J^cL = R_2 \oplus J^cL. \tag{4.8}$$

The filtration on JL induced by (4.7) coincides with the J^cL-adic filtration. If we filter JL with the J^cL-adic filtration then we obtain

$$\operatorname{gr} JL = \mathrm{S}_R L^* \tag{4.9}$$

and the R_1- and R_2-action on the r.h.s. of this equation coincide (here and below the letter S stands for "symmetric algebra").

Since we have assumed that L is free we may lift a basis of L^* to JL and then from (4.8), (4.9) we obtain

$$JL \cong R_1[\![x_1, \ldots, x_d]\!]. \tag{4.10}$$

There are also two different commuting actions by derivations of L on JL. Let $l \in L$, $\phi \in JL$, $D \in UL$.

$${}^1\nabla_l(\phi)(D) = l\bigl(\phi(D)\bigr) - \phi(lD),$$
$${}^2\nabla_l(\phi)(D) = \phi(Dl).$$

Again it will be convenient to write L_i for L acting by ${}^i\nabla$. Then ${}^i\nabla$ defines a flat L_i-connection on JL, considered as an R_i-module. Thus JL becomes a UL_1–UL_2-bimodule (with both UL_1 and UL_2 acting on the left). For some of the verifications below we note that the UL_2-action on JL takes the simple form

$$(D \cdot \phi)(E) = \phi(ED)$$

(for $D, E \in UL_2$, $\phi \in JL$).

The induced actions on $\operatorname{gr} JL = \mathrm{S}_R L^*$ of $l \in L$, considered as an element of L_1 and L_2, are given by the contractions i_{-l} and i_l.

Example 4.2.3. In case R is the coordinate ring of a smooth affine algebraic variety and $L = \mathrm{Der}_k(R)$ then we may identify JL with the completion of $R \otimes R$ at the kernel of the multiplication map $R \otimes R \to R$. The two actions of R on JL are respectively $R \otimes 1$ and $1 \otimes R$.

Similarly a derivation on R can be extended to $R \otimes R$ in two ways by letting it act respectively on the first and second factor. Since derivations are continuous they act on adic completions and hence in particular on JL. This provides the two actions of L on JL.

As ${}^1\nabla_l$ acts by derivation on JL it is easy to see that the resulting $\bigwedge_{R_1} L_1^*$-DG-module $(\bigwedge_{R_1} L_1^* \otimes_{R_1} JL, {}^1\nabla)$ (see (4.6)) is actually a commutative $\bigwedge_{R_1} L_1^*$-DG-algebra. The following result is well known.

Proposition 4.2.4. *The inclusion $\alpha_2 : R \hookrightarrow JL$ defines a quasi-isomorphism*

$$R \to \bigwedge\nolimits_{R_1} L_1^* \otimes_{R_1} JL; \quad r \mapsto 1 \otimes \alpha_2(r).$$

Proof. It is easy to see that if $r \in R_2$ then ${}^1\nabla(1 \otimes r) = 0$. To prove that we obtain a quasi-isomorphism we filter JL by the $J^c L$-adic filtration. We obtain the following associated graded complex

$$0 \to R \to \mathrm{S}_R L^* \to L^* \otimes_R \mathrm{S}_R L^* \to \cdots \to \bigwedge\nolimits_R^d L^* \otimes \mathrm{S}_R L^* \to 0 \tag{4.11}$$

where the differential is given by $-\sum_j l_j^* \otimes i_{l_j}$ for a basis $(l_j)_j$ of L. It is easy to see that (4.11) is exact. □

4.3. Relative poly-vector fields, poly-differential operators and forms

4.3.1. Definitions

We need relative poly-differential operators and poly-vector fields. So assume that $A \to B$ is a morphism of commutative DG-k-algebras. Then

$$T_{\mathrm{poly},A}(B) = \bigoplus_n T^n_{\mathrm{poly},A}(B),$$

$$D_{\mathrm{poly},A}(B) = \bigoplus_n D^n_{\mathrm{poly},A}(B)$$

where $T^n_{\mathrm{poly},A}(B) = \bigwedge_B^{n+1} \mathrm{Der}_A(B)$. Similarly $D^n_{\mathrm{poly},A}(B)$ is the set of maps with $n+1$ arguments $B \otimes_A \cdots \otimes_A B \to B$ which are differential operators when we equip B with the diagonal $B \otimes_A \cdots \otimes_A B$-algebra structure.

If we consider A and B just as algebras then $T_{\mathrm{poly},A}(B)$ and $D_{\mathrm{poly},A}(B)$ are DG-Lie algebras in the usual way. The differential on $T_{\mathrm{poly},A}(B)$ is trivial and the differential on $D_{\mathrm{poly},A}(B)$ is the restriction of the Hochschild differential. We denote it by d_{Hoch}. The differential d_B on B induces a differential $[d_B, -]$ on $T^n_{\mathrm{poly},A}(B)$, $D^n_{\mathrm{poly},A}(B)$ which commutes with d_{Hoch} on the latter. The total differentials on $T^n_{\mathrm{poly},A}(B)$ and $D^n_{\mathrm{poly},A}(B)$ are respectively $[d_B, -]$ and $[d_B, -] + d_{\mathrm{Hoch}}$.

Recall that one can also consider the DG-algebra $\Omega_{B/A}$ of relative differentials.[5] The differential is $d_B + d_{\mathrm{DR}}$. There is a contraction map between relative one-differentials and relative vector fields: it is defined as the B-linear map

$$\Omega^1_{B/A} \otimes \mathrm{Der}_A(B) \to B : f\,dg \otimes \xi \mapsto f\xi(g).$$

4.3.2. Relation with L-jets

Let us begin by the observation that

$$L_2 \to \mathrm{Der}_{R_1}(JL) : l \mapsto \big(\theta \mapsto {}^2\nabla_l(\theta)\big) \tag{4.12}$$

is a Lie algebra morphism.[6]

Lemma 4.3.1. (4.12) *yields a well-defined isomorphism of JL-modules*

$$JL \otimes_{R_2} L_2 \to \mathrm{Der}_{R_1}(JL) : \phi \otimes l \mapsto \big(\theta \mapsto \phi \cdot {}^2\nabla_l(\theta)\big). \tag{4.13}$$

Proof. It suffices to check that this is the case for the associated graded modules for the J^cL-adic filtration, which is easy. □

Let $D_{R_1}(JL)$ be the ring of differential operators of JL relative to R_1, considered as an R_2-module. Since the L_2-action on JL commutes with the R_1-action we obtain a ring homomorphism

$$UL_2 \to D_{R_1}(JL) : D \mapsto \big(\theta \mapsto D(\theta)\big). \tag{4.14}$$

It is easy to check that together with $R_2 \to JL$ this gives a Hopf algebroid homomorphism $(R_2, UL_2) \to (JL, D_{R_1}(JL))$.

Lemma 4.3.2. (4.14) *yields a well-defined isomorphism of JL-modules.*

$$JL \hat{\otimes}_{R_2} UL_2 \to D_{R_1}(JL) : \phi \otimes D \mapsto \big(\theta \mapsto \phi D(\theta)\big). \tag{4.15}$$

Proof. It is easily verified that this map is well defined. To prove that it is an isomorphism we extend the natural filtration on UL_2 to a filtration on the l.h.s. of (4.15) and we filter the r.h.s. by order of differential operators. We then obtain a map

$$JL \hat{\otimes}_{R_2} \mathrm{S}_{R_2}(L_2) = \mathrm{S}_{JL}(JL \otimes_{R_2} L_2) \to \mathrm{S}_{JL}\big(\mathrm{Der}_{R_1}(JL)\big)$$

which is induced from the natural map $JL \otimes_{R_2} L_2 \to \mathrm{Der}_{R_1}(JL)$. This map is an isomorphism by Lemma 4.3.1. □

[5] Here and in the rest of the paper the symbol "Ω" is used in the sense of continuous differentials (since we usually deal with complete topological vector spaces). In other words the topology on B induces a topology on ordinary (relative) differentials and then we complete. Continuous differentials are discussed in [36, §5.4].

[6] Together with $R_2 \to JL$ this actually is an algebraic morphism of Lie algebroids.

Using again that the L_2-action on JL commutes with the R_1-action we obtain natural DG-Lie algebra morphisms

$$\begin{aligned} T_{\text{poly}}^{L_2}(R_2) &\to T_{\text{poly},R_1}(JL), \\ D_{\text{poly}}^{L_2}(R_2) &\to D_{\text{poly},R_1}(JL). \end{aligned} \tag{4.16}$$

We obtain the following (see [42, Lemma 5.1(22)]):

Lemma 4.3.3. *The maps* (4.16) *induce well-defined isomorphisms of JL-modules*

$$\begin{aligned} JL \mathbin{\hat{\otimes}}_{R_2} T_{\text{poly}}^{L_2}(R_2) &\to T_{\text{poly},R_1}(JL), \\ JL \mathbin{\hat{\otimes}}_{R_2} D_{\text{poly}}^{L_2}(R_2) &\to D_{\text{poly},R_1}(JL). \end{aligned}$$

Proof. It easy to check that these maps are well defined. As an example we prove that the second map is an isomorphism. We have isomorphisms of vector spaces

$$\begin{aligned} D_{\text{poly},R_1}(JL) &= T_{JL}\big(D_{R_1}(JL)\big)[1] = T_{JL}(JL \mathbin{\hat{\otimes}}_{R_2} UL_2)[1] \\ &= JL \mathbin{\hat{\otimes}}_{R_2} T_{R_2}(UL_2)[1] = JL \mathbin{\hat{\otimes}}_{R_2} D_{\text{poly}}^{L_2}(R_2). \end{aligned}$$

In the first line we have used Lemma 4.3.2. One now easily shows that the resulting isomorphism $JL \mathbin{\hat{\otimes}}_{R_2} D_{\text{poly}}^{L_2}(R_2) \cong D_{\text{poly},R_1}(JL)$ is indeed the morphism given in the statement of the lemma. □

We also have:

Lemma 4.3.4. *Let C be a commutative R_1-DG-algebra. The canonical maps*

$$\begin{aligned} (C \mathbin{\hat{\otimes}}_{R_1} JL) \mathbin{\hat{\otimes}}_{R_2} T_{\text{poly}}^{L_2}(R_2) &\to T_{\text{poly},C}(C \mathbin{\hat{\otimes}}_{R_1} JL), \\ (C \mathbin{\hat{\otimes}}_{R_1} JL) \mathbin{\hat{\otimes}}_{R_2} D_{\text{poly}}^{L_2}(R_2) &\to D_{\text{poly},C}(C \mathbin{\hat{\otimes}}_{R_1} JL) \end{aligned} \tag{4.17}$$

obtained by linearly extending the canonical maps

$$\begin{aligned} T_{\text{poly}}^{L_2}(R_2) &\to T_{\text{poly},C}(C \mathbin{\hat{\otimes}}_{R_1} JL), \\ D_{\text{poly}}^{L_2}(R_2) &\to D_{\text{poly},C}(C \mathbin{\hat{\otimes}}_{R_1} JL) \end{aligned}$$

are well-defined isomorphisms. If JL carries a flat C-connection ∇ commuting with the (R_2, L_2)-actions on $C \mathbin{\hat{\otimes}}_{R_1} JL$ then $\nabla \otimes \mathrm{id}$ on the left of (4.17) *corresponds to* $[\nabla, -]$ *on the right.*

Proof. We restrict ourselves to the case of poly-differential operators. The case of poly-vector fields is similar. Using the fact that JL is of the form $R_1[\![x_1, \ldots, x_d]\!]$ (see (4.10)) we easily deduce that the canonical map

$$C \mathbin{\hat{\otimes}}_{R_1} D_{\text{poly},R_1}(JL) \to D_{\text{poly},C}(C \mathbin{\hat{\otimes}}_{R_1} JL)$$

is an isomorphism. Combining this with Lemma 4.3.4 yields the required isomorphism. It is easily seen that this isomorphism yields the asserted compatibility for flat C-connections. □

4.3.3. Differentials and L-jets

Let us introduce a JL-linear map

$$\Omega^1_{JL/R_1} \to JL \mathbin{\hat{\otimes}}_{R_2} L_2^* : \phi d\theta \mapsto \phi \mathbin{\hat{\otimes}} \tilde{\theta} \tag{4.18}$$

with $\tilde{\theta}(l) \overset{\text{def}}{=} {}^2\nabla_l(\theta)$ for any $l \in L_2$. If we respectively denote (4.18) and (4.12) by u and v then by definition we have

$$u(\xi)(l) = \xi\big(v(l)\big) \quad \big(\forall \xi \in \Omega^1_{JL/R_1}\big).$$

It then follows from taking the R-dual of (4.13) that (4.18) is an isomorphism, and that the restriction $L_2^* \to \Omega^1_{JL/R_1}$ to L_2^* of its inverse fits into the commutative diagram

$$\begin{array}{ccc} L_2^* \otimes L_2 & \xrightarrow{\text{contraction}} & R_2 \\ \downarrow & & \downarrow{\scriptstyle \alpha_2} \\ \Omega^1_{JL/R_1} \mathbin{\hat{\otimes}} \operatorname{Der}_{R_1}(JL) & \xrightarrow[\text{contraction}]{} & JL. \end{array} \tag{4.19}$$

The next result follows by applying $\bigwedge(-)$ to the inverse of (4.18) and is parallel to Lemma 4.3.1:

Lemma 4.3.5. *The maps $R_2 \to JL$ and $L_2^* \to \Omega^1_{JL/R_1}$ induce a DG-algebra morphism*

$$\bigwedge\nolimits_{R_2} L_2^* \to \Omega_{JL/R_1}$$

that extends to an isomorphism $JL \mathbin{\hat{\otimes}}_{R_2} \bigwedge_{R_2} L_2^* \to \Omega_{JL/R_1}$ of JL-modules. This isomorphism is compatible with differentials if we put on $JL \mathbin{\hat{\otimes}}_{R_2} \bigwedge_{R_2} L_2^*$ the canonical differential obtained from the L_2-connection on JL.

Proof. We may view (4.18) as an isomorphism between the Lie algebroids over JL given by $JL \otimes_{R_2} L_2$ and $\operatorname{Der}_{R_1}(JL)$ where the first Lie algebroid structure is deduced from the L_2-connection on JL.

As a result we obtain an isomorphism between the corresponding DG-algebras:

$$JL \mathbin{\hat{\otimes}}_{R_2} \bigwedge\nolimits_{R_2} L_2^* = \bigwedge\nolimits_{JL}\big(JL \mathbin{\hat{\otimes}}_{R_2} L^*\big) \cong \bigwedge\nolimits_{JL} \operatorname{Der}_{R_1}(JL)^* = \Omega_{JL/R_1}.$$

The statement of the lemma follows easily from this. □

We also have the following analogue of Lemma 4.3.4:

Lemma 4.3.6. *Let C be a commutative R_1-DG-algebra. The canonical map*

$$(C \mathbin{\hat{\otimes}}_{R_1} JL) \mathbin{\hat{\otimes}}_{R_2} \Big(\bigwedge\nolimits_{R_2} L_2^*\Big) \to \Omega_{C \hat{\otimes}_{R_1} JL/C} \tag{4.20}$$

is an isomorphism. If JL carries a flat C-connection ∇ commuting with the (R_2, L_2)-actions on $C \hat{\otimes}_{R_1} JL$ then $\nabla \otimes \mathrm{id}$ on the left corresponds to the differential d^∇ on the right.

Here d^∇ is characterized by the properties that it coincides with ∇ on $C \hat{\otimes}_{R_1} JL = \Omega^0_{C\hat{\otimes}_{R_1} JL/C}$ and that it commutes with the De Rham differential.

5. Coordinate spaces

We keep the notation of the previous section.

5.1. The coordinate space of a Lie algebroid

The following definition is inspired by [42]. We define $R^{\mathrm{coord},L}$ as the commutative R_1-algebra which trivializes JL, i.e. which is universal for the property that there is an isomorphism of $R^{\mathrm{coord},L}$-algebras

$$t : R^{\mathrm{coord},L} \hat{\otimes}_{R_1} JL \cong R^{\mathrm{coord},L}[\![t_1, \dots, t_d]\!] \tag{5.1}$$

such that $R^{\mathrm{coord},L} \hat{\otimes}_{R_1} J^c L$ is mapped to $(t_1, \dots, t_d)$.

For use below we give an explicit description of $R^{\mathrm{coord},L}$. Since we have assumed L to be free of rank d we may assume that

$$JL \cong R_1[\![x_1, \dots, x_d]\!]$$

where $(x_i)_i$ is mapped to a basis of $J^c L/(J^c L)^2 \cong L^*$. Let T be the polynomial ring over R_1 in the variables $y_{i,a_1\cdots a_d}$ where $i = 1, \dots, d$, $a_j \in \mathbb{N}$ and $(a_1, \dots, a_d) \neq (0, \dots, 0)$. Then $R^{\mathrm{coord},L}$ is the localization of T at $\det(y_{i,e_j})$ where $e_j = (0, \dots, 1, \dots, 0)$ with the 1 occurring in the jth place and t is given by

$$t(x_i) = \sum_{a_1\cdots a_d} y_{i,a_1\cdots a_d} t_1^{a_1} \cdots t_d^{a_d}. \tag{5.2}$$

Since $R^{\mathrm{coord},L}$ is universal any k-linear automorphism α of $k[\![t_1, \dots, t_d]\!]$ yields a corresponding *unique* R_1-linear automorphism $\bar{\alpha}$ of $R^{\mathrm{coord},L}$ such that the combined automorphism $(\bar{\alpha}, \alpha)$ of $R^{\mathrm{coord},L}[\![t_1, \dots, t_d]\!]$ leaves the image under t of JL pointwise invariant. Since $\mathrm{Gl}_d(k)$ acts on $k[\![t_1, \dots, t_d]\!]$ we obtain a corresponding R_1-linear action of $\mathrm{Gl}_d(k)$ on $R^{\mathrm{coord},L}$.

In fact if we write

$$R^{\mathrm{coord},L} \cong R_1 \otimes S^{\mathrm{coord}} \tag{5.3}$$

with $S^{\mathrm{coord}} = k[(y_{i,a_1,\dots,a_d})]_{\det(y_{i,e_j})}$ then the $\mathrm{Gl}_d(k)$-action on $R^{\mathrm{coord},L}$ is obtained from a $\mathrm{Gl}_d(k)$-action on S^{coord} which preserves the finite dimensional vector spaces with basis $\{y_{i,a_1,\dots,a_d} \mid \sum_j a_j = s\}$. Needless to say that the decomposition (5.3) depends on the choice of the generators $(x_i)_i$ of $J^c L$.

It follows in particular that the $\mathrm{Gl}_d(k)$-action is rational. Hence we may consider the derived action of $\mathfrak{gl}_d(k)$ on $R^{\mathrm{coord},L}$. The action of $\mathrm{Gl}_d(k)$ on $k[\![t_1, \dots, t_d]\!]$ also yields a derived action of $\mathfrak{gl}_d(k)$. The two actions of $\mathfrak{gl}_d(k)$ satisfy the following compatibility.

Lemma 5.1.1. *For $v \in \mathfrak{gl}_d(k)$ let L_v be the action of v on $R^{\mathrm{coord},L}[\![t_1,\dots,t_d]\!]$ obtained by linearly extending the action of v on $k[\![t_1,\dots,t_d]\!]$.*

Let $L_{\bar{v}}$ be the action of v on $R^{\mathrm{coord},L}[\![t_1,\dots,t_d]\!]$ obtained by linearly extending the action of v on $R^{\mathrm{coord},L}$. Then $L_{\bar{v}} + L_v$ is zero on $t(JL)$.

Proof. This is the derived version of the fact that $\mathrm{Gl}_d(k)$ leaves $t(JL)$ pointwise fixed. □

5.2. On some DG-algebras associated to coordinate spaces

5.2.1. The DG-algebra $C^{\mathrm{coord},L}$

Put $C^{\mathrm{coord},L} = \Omega_{R^{\mathrm{coord},L}} \otimes_{\Omega_{R_1}} \bigwedge_{R_1} L_1^*$. Using the DG-algebra structures on $\bigwedge_{R_1} L_1^*$ we see that $C^{\mathrm{coord},L}$ is naturally a commutative $\bigwedge_{R_1} L_1^*$-DG-algebra.

As we have not put any restrictions on R, the DG-algebras Ω_{R_1} and $\Omega_{R^{\mathrm{coord},L}}$ may be enormous objects. However only their "difference" matters and this is controlled by (5.3). In fact from (5.3) we obtain a coordinate dependent isomorphism of DG-algebras

$$C^{\mathrm{coord},L} = \Omega_{S^{\mathrm{coord}}} \otimes \bigwedge\nolimits_{R_1} L_1^*.$$

We will now form the completed tensor product over $R^{\mathrm{coord},L}$ of the domain and codomain of the map (5.1) with $C^{\mathrm{coord},L}$ ignoring the differentials. For the domain we get

$$\begin{aligned} C^{\mathrm{coord},L} \hat{\otimes}_{R^{\mathrm{coord},L}} R^{\mathrm{coord},L} \hat{\otimes}_{R_1} JL &= C^{\mathrm{coord},L} \hat{\otimes}_{R_1} JL \\ &= \Omega_{R^{\mathrm{coord},L}} \hat{\otimes}_{\Omega_{R_1}} \Big(\bigwedge\nolimits_{R_1} L_1^* \otimes_{R_1} JL\Big). \end{aligned} \tag{5.4}$$

For the codomain we get

$$C^{\mathrm{coord},L} \hat{\otimes}_{R^{\mathrm{coord},L}} R^{\mathrm{coord},L}[\![t_1,\dots,t_d]\!] = C^{\mathrm{coord},L}[\![t_1,\dots,t_d]\!].$$

So we obtain an isomorphism of graded algebras

$$\tilde{t} : C^{\mathrm{coord},L} \hat{\otimes}_{R_1} JL \to C^{\mathrm{coord},L}[\![t_1,\dots,t_d]\!]. \tag{5.5}$$

Both domain and codomain of $\tilde{t}$ carry a natural differential. The differential on $C^{\mathrm{coord},L} \hat{\otimes}_{R_1} JL$ is obtained by combining the ordinary differential on $\Omega_{R^{\mathrm{coord},L}}$ and the differential ${}^1\nabla$ on $\bigwedge_{R_1} L_1^* \otimes_R JL$ using the right-hand side of (5.4). The differential on $C^{\mathrm{coord},L}[\![t_1,\dots,t_d]\!]$ is obtained from extending the differential on $C^{\mathrm{coord},L}$. Let us denote the resulting differentials by ${}^1\nabla^{\mathrm{coord}}$ and d respectively.

The constructed differentials transform the obvious morphisms of graded algebras

$$\begin{aligned} &C^{\mathrm{coord},L} \to \Omega_{R^{\mathrm{coord},L}} \hat{\otimes}_{\Omega_R} \Big(\bigwedge\nolimits_{R_1} L_1^* \otimes_{R_1} JL\Big), \\ &C^{\mathrm{coord},L} \to C^{\mathrm{coord},L}[\![t_1,\dots,t_d]\!] \end{aligned}$$

into morphisms of DG-algebras.

By the middle equality in (5.4) ${}^1\nabla^{\mathrm{coord}}$ may be viewed as a flat $C^{\mathrm{coord},L}$-connection on JL. The map

$$\left(\tilde{t} \circ {}^1\nabla^{\mathrm{coord}} \circ \tilde{t}^{-1} - d\right) : C^{\mathrm{coord},L}[\![t_1,\dots,t_d]\!] \to C^{\mathrm{coord},L}[\![t_1,\dots,t_d]\!] \tag{5.6}$$

is now a $C^{\mathrm{coord},L}$-linear derivation.

From [36, §6.4] we obtain the existence of elements $\omega^i \in C^{\mathrm{coord},L}[\![t_1,\dots,t_d]\!]_1$ such that for

$$\omega = \sum_i \omega^i \frac{\partial}{\partial t_i} \in C^{\mathrm{coord},L} \hat{\otimes} \mathrm{Der}_k\left(k[\![t_1,\dots,t_d]\!]\right) \tag{5.7}$$

we have $\tilde{t} \circ {}^1\nabla^{\mathrm{coord}} \circ \tilde{t}^{-1} = d + \omega$ and furthermore ω satisfies the Maurer–Cartan equation

$$d\omega + \frac{1}{2}[\omega,\omega] = 0$$

in $C^{\mathrm{coord},L} \hat{\otimes} \mathrm{Der}_k(k[\![t_1,\dots,t_d]\!])$.

5.2.2. The $\mathfrak{gl}_d(k)$-action on $C^{\mathrm{coord},L}$ and the Maurer–Cartan form

We need the following result.

Lemma 5.2.1. *For $v \in \mathfrak{gl}_d(k)$ let $i_{\bar{v}}$ be the derivation on $C^{\mathrm{coord},L} = \Omega_{R^{\mathrm{coord},L}} \otimes_{\Omega_{R_1}} \bigwedge_{R_1} L_1^*$ obtained by linearly extending the contraction of $\Omega_{R^{\mathrm{coord},L}}$ with the action as R_1-derivation of v on $R^{\mathrm{coord},L}$ (cf. Section* 5.1*). Extend $i_{\bar{v}}$ to a map of degree -1 from $C^{\mathrm{coord},L} \hat{\otimes} \mathrm{Der}_k(k[\![t_1,\dots,t_d]\!])$ to itself. Then we have*

$$i_{\bar{v}}\omega = 1 \otimes v \tag{5.8}$$

where both sides are considered as elements of $R^{\mathrm{coord},L} \hat{\otimes} \mathrm{Der}_k(k[\![t_1,\dots,t_d]\!])$.

Proof. We may prove (5.8) by evaluation on an arbitrary element $g \in R^{\mathrm{coord},L}[\![t_1,\dots,t_d]\!]$. Since $\omega = \sum_i \omega^i(\partial/\partial t_i)$ we have $i_{\bar{v}}(\omega) = \sum_i i_{\bar{v}}(\omega^i)(\partial/\partial t_i)$ and hence $(i_{\bar{v}}\omega)(g) = i_{\bar{v}}(\omega(g))$. Thus we need to show

$$i_{\bar{v}} \circ \omega = L_v \tag{5.9}$$

as operators on $R^{\mathrm{coord},L}[\![t_1,\dots,t_d]\!]$ (as in Lemma 5.1.1 L_v is the extension of the v-action on $k[\![t_1,\dots,t_d]\!]$).

It is clear that $R^{\mathrm{coord},L}[\![t_1,\dots,t_d]\!]$ is topologically generated by $R^{\mathrm{coord},L}$ and $t(JL)$. Since the operators occuring are $R^{\mathrm{coord},L}$-linear it is sufficient to prove the identity

$$i_{\bar{v}} \circ \omega \circ t = L_v \circ t \tag{5.10}$$

as operators on JL. We may rewrite the l.h.s. of (5.10) as follows

$$i_{\bar{v}} \circ \omega \circ t = i_{\bar{v}} \circ (d+\omega) \circ t - i_{\bar{v}} \circ d \circ t = i_{\bar{v}} \circ \tilde{t} \circ {}^1\nabla^{\mathrm{coord}} - L_{\bar{v}} \circ t = L_v \circ t. \tag{5.11}$$

In the second equality we have used the Cartan relation $L_{\bar{v}} = d \circ i_{\bar{v}} + i_{\bar{v}} \circ d$ and the fact that the term $d \circ i_{\bar{v}}$ acts as zero on $R^{\mathrm{coord},L}[\![t_1, \dots, t_d]\!]$ (for degree reasons). We have also used (5.6).

In the third equality we have used the fact that JL is mapped to $\bigwedge_{R_1} L_1^* \otimes_{R_1} JL$ under ${}^1\nabla^{\mathrm{coord}}$ and the image of $\bigwedge_{R_1} L_1^* \otimes_{R_1} JL$ in $C^{\mathrm{coord},L} \hat{\otimes} \operatorname{Der}_k(k[\![t_1, \dots, t_d]\!])$ under t lies in the part generated by $R^{\mathrm{coord},L}$ and $\operatorname{Der}_k(k[\![t_1, \dots, t_d]\!])$ and on this part $i_{\bar{v}}$ is zero. □

5.3. The affine coordinate space of a Lie algebroid

We put $R^{\mathrm{aff},L} = (R^{\mathrm{coord},L})^{\mathrm{Gl}_d(k)}$. We easily verify from (5.3) that $R^{\mathrm{aff},L}$ is of the form $R_1 \otimes S^{\mathrm{aff}}$ with $S^{\mathrm{aff}} = (S^{\mathrm{coord}})^{\mathrm{Gl}_d(k)}$ and furthermore $S^{\mathrm{aff}} = k[(z_i)_i]$ for a set of (infinitely many) variables $(z_i)_i$. To verify this last claim note that a collection of power series $(\sum_{a_1 \cdots a_d} y_{i,a_1 \cdots a_d} t_1^{a_1} \cdots t_d^{a_d})_{i=1,\dots,d}$ (cf. (5.2)) such that $\det(y_{i,e_j}) \neq 0$ may be brought uniquely into the form $(t_i + \phi_i(t_1, \dots, t_d))_i$, with quadratic $(\phi_i)_i$, using a linear transformation in the t's. The coefficients of $(\phi_i)_i$ (as formal power series in $(t_i)_i$), written as rational functions in $y_{i,a_1 \cdots a_d}$, are the z_i's.

Below we will also use the DG-algebra $C^{\mathrm{aff},L} = \Omega_{R^{\mathrm{aff},L}} \otimes_{\Omega_{R_1}} \bigwedge_{R_1} L_1^*$. Exactly as for $C^{\mathrm{coord},L}$ one produces a flat $C^{\mathrm{aff},R}$-connection on JL which we denote by ${}^1\nabla^{\mathrm{aff}}$. Furthermore we have a coordinate dependent isomorphism of DG-algebras

$$C^{\mathrm{aff},L} = \Omega_{S^{\mathrm{aff}}} \otimes \bigwedge\nolimits_{R_1} L_1^*. \tag{5.12}$$

Lemma 5.3.1. *For any free R_2-module M we have that*

$$M \to \left(C^{\mathrm{aff},L} \hat{\otimes}_{R_1} JL, {}^1\nabla^{\mathrm{aff}}\right) \hat{\otimes}_{R_2} M$$

is a quasi-isomorphism.

Proof. Using the decomposition (5.12) we need to show that

$$M \to \Omega_{S^{\mathrm{aff}}} \otimes \bigwedge\nolimits_{R_1} L_1^* \hat{\otimes}_{R_1} JL \hat{\otimes}_{R_2} M \tag{5.13}$$

is a quasi-isomorphism. We filter the r.h.s. of (5.13) with the J^cL-adic filtration. This means we have to show that

$$M \to \Omega_{S^{\mathrm{aff}}} \otimes \bigwedge\nolimits_R L^* \otimes_R \mathrm{S}_R\left(L^*\right) \otimes_R M$$

is a quasi-isomorphism.

Using the proof of Proposition 4.2.4 we see that we may replace $\bigwedge_R L^* \otimes_R \mathrm{S}_R L^*$ by R. Furthermore since S^{aff} is a polynomial ring we also find that $\Omega_{S^{\mathrm{aff}}}$ is quasi-isomorphic to k. Thus we are done. □

5.4. The universal property of the affine coordinate space

The affine coordinate space has a universal property, similar to (5.1).

Proposition 5.4.1. *There is a filtered isomorphism of R-algebras*

$$R^{\mathrm{aff},L} \mathbin{\hat{\otimes}}_{R_1} JL \cong R^{\mathrm{aff},L} \mathbin{\hat{\otimes}}_R \widehat{\mathrm{S}_R(L^*)} \tag{5.14}$$

which induces the canonical isomorphism

$$R^{\mathrm{aff},L} \otimes_R \mathrm{gr}(JL) \cong R^{\mathrm{aff},L} \otimes_R \mathrm{S}_R(L^*)$$

obtained by extending (4.9). *Moreover $R^{\mathrm{aff},L}$ is universal for the existence of such an isomorphism.*

Proof. We start with the isomorphism

$$R^{\mathrm{coord},L} \mathbin{\hat{\otimes}}_{R_1} JL \cong R^{\mathrm{coord},L}[\![t_1,\ldots,t_d]\!]. \tag{5.15}$$

It follows from the discussion after (5.2) that this isomorphism if Gl_d-equivariant is we equip the right-hand side with a Gl_d-action which is a combination of the linear action on the $(t_i)_i$'s and the extension of the Gl_d-action on $R^{\mathrm{coord},L}$.

Put

$$\tilde{L}^* = R^{\mathrm{coord},L} t_1 + \cdots + R^{\mathrm{coord},L} t_d$$

considered as a Gl_d-module. Then (tautologically) we have a Gl_d-equivariant isomorphism

$$R^{\mathrm{coord},L}[\![t_1,\ldots,t_d]\!] \cong \mathrm{S}_{R^{\mathrm{coord},L}}(\tilde{L}^*)^\wedge.$$

Combining this with (5.15) we get a Gl_d-equivariant isomorphism

$$R^{\mathrm{coord},L} \mathbin{\hat{\otimes}}_R JL \cong \mathrm{S}_{R^{\mathrm{coord},L}}(\tilde{L}^*)^\wedge$$

and looking at degree one of the associated graded rings:

$$R^{\mathrm{coord},L} \mathbin{\hat{\otimes}}_R L^* \cong \tilde{L}^*.$$

Thus

$$R^{\mathrm{coord},L} \mathbin{\hat{\otimes}}_R JL \cong \mathrm{S}_{R^{\mathrm{coord},L}}(R^{\mathrm{coord},L} \mathbin{\hat{\otimes}}_R L^*)^\wedge \cong R^{\mathrm{coord},L} \mathbin{\hat{\otimes}} \widehat{\mathrm{S}_R(L^*)}. \tag{5.16}$$

It now suffices to take Gl_d-invariants to get the isomorphism (5.14).

We will only sketch the proof of universality since we will not need it in the sequel. Assume that we have an isomorphism

$$W \mathbin{\hat{\otimes}}_{R_1} JL \cong W \mathbin{\hat{\otimes}}_R \widehat{\mathrm{S}_R(L^*)}$$

which induces the canonical isomorphism

$$W \otimes_R \mathrm{gr}(JL) \cong W \otimes_R \mathrm{S}_R(L^*).$$

We need to construct a corresponding morphism $R^{\mathrm{aff},L} \to W$.

We let $\tilde{W}$ be the commutative W-algebra which is universal for the existence of an isomorphism

$$\tilde{W} \otimes_W (W \otimes_R L) \cong \tilde{W} t_1 + \cdots + \tilde{W} t_d$$

of $\tilde{W}$-modules.

Thus $\operatorname{Spec} \tilde{W} / \operatorname{Spec} W$ is a Gl_d-torsor and in particular $\tilde{W}^{\mathrm{Gl}_d} \cong W$. We then have

$$\begin{aligned}
\tilde{W} \hat{\otimes}_R JL &= \tilde{W} \hat{\otimes}_W (W \hat{\otimes}_R JL) \\
&= \tilde{W} \hat{\otimes}_W \big(W \hat{\otimes}_R \widehat{\mathrm{S}_R(L^*)}\big) \\
&= \tilde{W} \hat{\otimes}_R \widehat{\mathrm{S}_R(L^*)} \\
&= \mathrm{S}_{\tilde{W}}(\tilde{W} t_1 + \cdots + \tilde{W} t_d)\hat{} \\
&= \tilde{W}[\![t_1, \ldots, t_d]\!].
\end{aligned}$$

Hence there exists a corresponding morphism $R^{\mathrm{coord},L} \to \tilde{W}$. Taking Gl_d-invariants yields the requested morphism $R^{\mathrm{aff},L} \to W$. One easily checks that this morphism satisfies the appropriate uniqueness properties. □

6. L_∞-algebras

In this section we recall some properties of L_∞-algebras and we fix some notation.

6.1. L_∞-algebras and morphisms

An *L_∞-structure* on a vector space $\mathfrak{g}$ is a coderivation Q of degree one on $\mathrm{S}(\mathfrak{g}[1])$ which has square zero. Such a coderivation is fully determined its "Taylor coefficients" which are the coefficients

$$Q_i : \mathrm{S}^i\big(\mathfrak{g}[1]\big) \xrightarrow{\text{inclusion}} \mathrm{S}\big(\mathfrak{g}[1]\big) \xrightarrow{Q} \mathrm{S}\big(\mathfrak{g}[1]\big) \xrightarrow{\text{projection}} \mathfrak{g}[1].$$

If for $a, b \in \mathfrak{g}$ one puts

$$da = -Q_1(a) \quad \text{and} \quad [a,b] = (-1)^{|a|} Q_2(a,b), \tag{6.1}$$

then $d^2 = 0$ and d is a derivation of degree one of $\mathfrak{g}$ with respect to the binary operation of degree zero $[-,-]$. If $\partial^i Q = 0$ for $i > 2$ then $\mathfrak{g}$ is a DG-Lie algebra. Conversely any DG-Lie algebra can be made into an L_∞-algebra by defining Q_1, Q_2 according to (6.1) and by putting $Q_i = 0$ for $i > 2$.

A morphism of L_∞-algebras $\mathfrak{g} \to \mathfrak{h}$, or *$L_\infty$-morphism* is by definition an augmented coalgebra map of degree zero $\mathrm{S}(\mathfrak{g}[1]) \to \mathrm{S}(\mathfrak{h}[1])$ commuting with Q. A morphism of L_∞-algebras is again determined by its Taylor coefficients

$$\psi_i : \mathrm{S}^i\big(\mathfrak{g}[1]\big) \xrightarrow{\text{inclusion}} \mathrm{S}\big(\mathfrak{g}[1]\big) \xrightarrow{\psi} \mathrm{S}\big(\mathfrak{h}[1]\big) \xrightarrow{\text{projection}} \mathfrak{h}[1].$$

One has $d\psi_1 = \psi_1 d$ and hence ψ_1 defines a morphism of complexes.

We will use these notions in the case that the underlying symmetric monoidal category consists of filtered complete topological vector spaces. Of course in that case the symmetric products have to be replaced by completed symmetric products.

6.2. Twisting

If $\mathfrak{g}$ is an L_∞-algebra then its L_∞-Maurer–Cartan equation is written formally as

$$\sum_{i=1}^{\infty} \frac{1}{i!} Q_i\left(\omega^i\right) = 0$$

and for a solution ω the corresponding twisted L_∞-structure is formally defined by

$$Q_{\omega,i}(\gamma) = \sum_{j \geqslant 0} \frac{1}{j!} Q_{i+j}\left(\omega^j \gamma\right) \quad (\text{for } i > 0). \tag{6.2}$$

These sums are in general infinite so one must deal with issues of convergence. However for DG-Lie algebras the sums are finite and there is no problem. For DG-Lie algebras the L_∞-Maurer–Cartan equation translates into the usual Maurer–Cartan equation

$$d\omega + \frac{1}{2}[\omega, \omega] = 0 \tag{6.3}$$

and we obtain $Q_{\omega,1}(\gamma) = Q_1(\gamma) + Q_2(\omega\gamma)$, $Q_{\omega,2}(\gamma) = Q_2(\gamma)$ and $Q_{\omega,i}(\gamma) = 0$ for $i \geqslant 3$. Translated into differentials and Lie brackets we get

$$d_\omega = d + [\omega, -] \quad \text{and} \quad [-,-]_\omega = [-,-]. \tag{6.4}$$

Assume now that $\psi : \mathfrak{g} \to \mathfrak{h}$ is an L_∞-morphism between DG-Lie-algebras over the category of complete filtered topological vector spaces and let $\omega \in \mathfrak{g}_1$ be a solution of the Maurer–Cartan equation in $\mathfrak{g}$. Assume that the following condition holds:

(*) for each i, i' there exists j_0 such that for $\gamma \in \widehat{\mathrm{S}}^i(\mathfrak{g}[1])_{i'}$ and $j \geqslant j_0$ we have $\psi_{i+j}(\omega^j\gamma) = 0$.

Define ψ_ω and ω' by

$$\psi_{\omega,i}(\gamma) = \sum_{j \geqslant 0} \frac{1}{j!} \psi_{i+j}\left(\omega^j \gamma\right) \quad (\text{for } i > 0), \tag{6.5}$$

$$\omega' = \sum_{j \geqslant 1} \frac{1}{j!} \psi_j\left(\omega^j\right) \tag{6.6}$$

for $\gamma \in S^i(\mathfrak{g}[1])_{i'}$. It follows from (*) that the sums are finite. Then by [43, Thms. 3.21, 3.27] ω' is a solution of the Maurer–Cartan equation in $\mathfrak{h}$ and furthermore $\mathfrak{g}$, $\mathfrak{h}$, when equipped with Q_ω, $Q_{\omega'}$ are again DG-Lie-algebras. If we denote these by $\mathfrak{g}_\omega$ and $\mathfrak{h}_{\omega'}$ then ψ_ω defines an L_∞-map $\mathfrak{g}_\omega \to \mathfrak{h}_{\omega'}$. The condition (*) implies that the maps defining ψ_ω are compatible with the topology (i.e. they are "bounded" in the sense of Section 4.1.2).

Remark 6.2.1. Formulas similar to (6.1), (6.5), (6.6) also appear at other places in the literature, e.g., [14, Eqs. (60), (61)] and [34, Eq. (4.4)]. They are implicit in the language of formal Q-manifolds employed by Kontsevich in [22].

6.3. Descent for L_∞-morphisms

Assume that $\mathfrak{g}$ is an algebra over a DG-operad $\mathcal{O}$ with underlying graded operad $\tilde{O}$ and consider a set of $\tilde{O}$-derivations of degree -1 $(i_v)_{v\in S}$ on $\mathfrak{g}$. Put $L_v = di_v + i_v d$. This is a derivation of $\mathfrak{g}$ of degree zero which commutes with d.

Put

$$\mathfrak{g}^S = \{w \in \mathfrak{g} \mid \forall v \in S\colon\ i_v w = L_v w = 0\}. \tag{6.7}$$

It is easy to see that $\mathfrak{g}^S$ is an algebra over $\mathcal{O}$ as well. Informally we will call such a set of derivations $(i_v)_{v\in S}$ an S-action.

Remark 6.3.1. By definition the notion of an S-action only depends on the graded structure of $\mathfrak{g}$. However the construction of $\mathfrak{g}^S$ also depends on the differential.

The following is a slightly strengthened version of [36, Prop. 7.6.3].

Proposition 6.3.2. *Assume that ψ is an L_∞-morphism $\mathfrak{g} \to \mathfrak{h}$ between L_∞-algebras equipped with an S-action as above. Assume that ψ commutes with the S-action in the sense that for all $v \in S$,*

$$i_v\psi_n(w_1,\dots,w_n) = \sum (-1)^{|w_1|+\cdots+|w_{l-1}|-(l-1)}\psi_n\big(w_1,\dots,i_v(w_l),\dots,w_n\big).$$

Then ψ descends to an L_∞-morphism $\psi^S : \mathfrak{g}^S \to \mathfrak{h}^S$.

6.4. Compatibility with twisting

We work over the category of filtered complete topological vector spaces. Assume that $\mathfrak{g}$, $\mathfrak{h}$ are DG-Lie algebras and ψ is an L_∞-morphism $\mathfrak{g} \to \mathfrak{h}$. Our aim is to understand the behavior of S-actions under twisting. We assume that (*) from Section 6.2 holds.

Assume that $\mathfrak{g}$ and $\mathfrak{h}$ are equipped with an S-action and assume that ψ commutes with this action (as in Proposition 6.3.2). Let $\omega \in \mathfrak{g}_1$ be a solution to the Maurer–Cartan equation. Since twisting does not change the Lie bracket (see (6.4)), S acts on $\mathfrak{g}_\omega$ and $\mathfrak{h}_\omega$ as well. The following is [36, Prop. 7.7.1].

Proposition 6.4.1. *Assume that for $i \geqslant 2$ and all $v \in S$, $\gamma \in \mathrm{S}^{i-1}(\mathfrak{g}[1])$ we have*

$$\psi_i(i_v\omega \cdot \gamma) = 0. \tag{6.8}$$

Then ψ_ω commutes with the S-action on $\mathfrak{g}_\omega$ and $\mathfrak{h}_{\omega'}$.

7. Formality for Lie algebroids

In this section we prove Theorem 1.8. We first prove a more precise result in the ring case. To do so we use the existence of the desired L_∞-quasi-isomorphism in the local case, and extend it to the ring case with the help of coordinate spaces constructed in the previous section. We end the proof by sheafifying the ring case, using appropriate functorial properties.

7.1. The formality in the ring case and its functorial properties

Theorem 7.1. *Let R be a k-algebra. Assume that L is a Lie algebroid over R which is free of rank d. There exists a canonical DG-Lie algebra $\mathfrak{l}^L$ together with L_∞-quasi-isomorphisms*

$$T_{\mathrm{poly}}^L(R) \to \mathfrak{l}^L \leftarrow D_{\mathrm{poly}}^L(R) \tag{7.1}$$

such that the induced map

$$\mu : T_{\mathrm{poly}}^L(R) \to H^*\big(D_{\mathrm{poly}}^L(R)\big)$$

is given by the HKR-formula (4.1).

The DG-Lie algebra $\mathfrak{l}^L$ and the quasi-isomorphisms in (7.1) *are functorial in the following sense: assume that $\phi : (R, L) \to (T, M)$ is an algebraic morphism of Lie algebroids which induces an isomorphism*

$$T \otimes_R L \cong M \tag{7.2}$$

then there is an associated commutative diagram

$$\begin{array}{ccccc} T_{\mathrm{poly}}^L(R) & \longrightarrow & \mathfrak{l}^L & \longleftarrow & D_{\mathrm{poly}}^L(R) \\ \downarrow & & \downarrow{\scriptstyle \mathfrak{l}^\phi} & & \downarrow \\ T_{\mathrm{poly}}^M(T) & \longrightarrow & \mathfrak{l}^M & \longleftarrow & D_{\mathrm{poly}}^M(T) \end{array} \tag{7.3}$$

and $\mathfrak{l}^{\phi\theta} = \mathfrak{l}^\phi \circ \mathfrak{l}^\theta$.

The proof of this theorem will take the greater part of the next two subsections.

7.2. The local formality quasi-isomorphism

Let $\mathbb{K} = k[\![t_1, \dots, t_d]\!]$. Kontsevich (over the reals) and Tamarkin (over the rationals) construct an L_∞-quasi-isomorphism [22,33]

$$\mathcal{U} : T_{\mathrm{poly}}(\mathbb{K}) \to D_{\mathrm{poly}}(\mathbb{K}) \tag{7.4}$$

where $\mathcal{U}_1$ is given by the HKR-formula[7]

$$\mathcal{U}_1(\partial_{i_1} \wedge \cdots \wedge \partial_{i_p}) = (-1)^{p(p-1)/2} \frac{1}{p!} \sum_{\sigma \in S_p} (-1)^\sigma \partial_{i_{\sigma(1)}} \otimes \cdots \otimes \partial_{i_{\sigma(p)}} \tag{7.5}$$

with $\partial_i = \partial/\partial t_i$. This quasi-isomorphism has two supplementary properties which are crucial for its extension to the global case.

(P4) $\mathcal{U}_q(\gamma_1 \cdots \gamma_q) = 0$ for $q \geqslant 2$ and $\gamma_1, \dots, \gamma_q \in T^{\mathrm{poly},1}(\mathbb{K})$.[8]
(P5) $\mathcal{U}_q(\gamma\alpha) = 0$ for $q \geqslant 2$ and $\gamma \in \mathfrak{gl}_d(k) \subset T^{\mathrm{poly},1}(\mathbb{K})$.

For Tamarkin's quasi-isomorphism the fact that properties (P4) and (P5) can be made to hold has been proved in [18].

Finally let us mention the following technical property.

(+) $\mathcal{U}_q$ is a filtered morphism of filtered complete topological vector spaces (in each degree) for the filtrations on $T_{\mathrm{poly}}(\mathbb{K})$ and $D_{\mathrm{poly}}(\mathbb{K})$ by order of differential operator.

This property implies in particular that we can safely base extend $\mathcal{U}$ (e.g., $\tilde{\mathcal{U}}$ in Section 7.3.2 below). The fact that (+) holds for Kontsevich's quasi-isomorphism follows from its description as a linear combination of differential operators indexed by graphs (see Section 9 below). It is not hard to see that Tamarkin's local quasi-isomorphism can be chosen to have a similar description (see, e.g., [29, §3])

7.3. Proof of Theorem 1.8 in the ring case

7.3.1. Resolutions

In this section we construct resolutions of $T^L_{\mathrm{poly}}(R)$ and $D^L_{\mathrm{poly}}(R)$. These are jet analogues of the *Dolgushev–Fedosov resolutions* in [4, Section 2].

Since the action of UL_2 on $C^{\mathrm{aff},L} \hat{\otimes}_{R_1} JL$ commutes with ${}^1\nabla^{\mathrm{aff}}$ we obtain morphisms of DG-Lie algebras:

$$\begin{aligned} T^{L_2}_{\mathrm{poly}}(R_2) &\to T_{\mathrm{poly},C^{\mathrm{aff},L}}\big(C^{\mathrm{aff},L} \hat{\otimes}_{R_1} JL, {}^1\nabla^{\mathrm{aff}}\big), \\ D^{L_2}_{\mathrm{poly}}(R_2) &\to D_{\mathrm{poly},C^{\mathrm{aff},L}}\big(C^{\mathrm{aff},L} \hat{\otimes}_{R_1} JL, {}^1\nabla^{\mathrm{aff}}\big). \end{aligned} \tag{7.6}$$

Proposition 7.3.1. *The morphisms in* (7.6) *are quasi-isomorphisms.*

Proof. This follows from Lemmas 4.3.4 and 5.3.1. □

[7] The sign $(-)^{p(p-1)/2}$ is not present in Kontsevich's setting. In this paper we slightly modify Kontsevich's quasi-isomorphism. See Section 9.

[8] For degree reasons, this is always true if $q > 2$.

7.3.2. The formality map on coordinate spaces

The local L_∞-quasi-isomorphism

$$\mathcal{U}: T_{\text{poly}}(\mathbb{K}) \to D_{\text{poly}}(\mathbb{K})$$

extends linearly to an L_∞-morphism in the category of filtered complete topological vector spaces

$$\tilde{\mathcal{U}}: C^{\text{coord},L} \hat{\otimes} T_{\text{poly}}(\mathbb{K}) \to C^{\text{coord},L} \hat{\otimes} D_{\text{poly}}(\mathbb{K})$$

(using Section 7.2(+)). One easily verifies that the canonical maps

$$\begin{aligned} C^{\text{coord},L} \hat{\otimes} T_{\text{poly}}(\mathbb{K}) &\to T_{\text{poly},C^{\text{coord},L}}\big(C^{\text{coord},L} \hat{\otimes} \mathbb{K}\big), \\ C^{\text{coord},L} \hat{\otimes} D_{\text{poly}}(\mathbb{K}) &\to D_{\text{poly},C^{\text{coord},L}}\big(C^{\text{coord},L} \hat{\otimes} \mathbb{K}\big) \end{aligned} \tag{7.7}$$

are isomorphisms of DG-Lie algebras. Thus we obtain a corresponding L_∞-morphism

$$\tilde{\mathcal{U}}: T_{\text{poly},C^{\text{coord},L}}\big(C^{\text{coord},L} \hat{\otimes} \mathbb{K}\big) \to D_{\text{poly},C^{\text{coord},L}}\big(C^{\text{coord},L} \hat{\otimes} \mathbb{K}\big).$$

In Section 5 (Section 5.2.1) we have constructed an isomorphism of $C^{\text{coord},L}$-DG-algebras

$$\tilde{t}: \big(C^{\text{coord},L} \hat{\otimes}_{R_1} JL, {}^1\nabla^{\text{coord}}\big) \to \big(C^{\text{coord},L} \hat{\otimes} \mathbb{K}, d + \omega\big).$$

Therefore we obtain an isomorphism of Lie algebras

$$\tilde{t} \circ - \circ \tilde{t}^{-1}: D_{\text{poly},C^{\text{coord},L}}\big(C^{\text{coord},L} \hat{\otimes}_{R_1} JL\big) \to D_{\text{poly},C^{\text{coord},L}}\big(C^{\text{coord},L} \hat{\otimes} \mathbb{K}\big). \tag{7.8}$$

The Hochschild differential on the left is sent to the Hochschild differential on the right. The differential $[{}^1\nabla^{\text{coord}}, -]$ on the left is sent to $[d + \omega, -]$ on the right. Then it follows using (6.4) from Section 6.2 that $\tilde{t}$ defines an isomorphism of DG-Lie algebras

$$D_{\text{poly},C^{\text{coord},L}}\big(C^{\text{coord},L} \hat{\otimes}_{R_1} JL, {}^1\nabla^{\text{coord}}\big) \cong D_{\text{poly},C^{\text{coord},L}}\big(C^{\text{coord},L} \hat{\otimes} \mathbb{K}, d\big)_\omega. \tag{7.9}$$

Similarly we have an isomorphism of DG-Lie algebras

$$T_{\text{poly},C^{\text{coord},L}}\big(C^{\text{coord},L} \hat{\otimes}_{R_1} JL, {}^1\nabla^{\text{coord}}\big) \cong T_{\text{poly},C^{\text{coord},L}}\big(C^{\text{coord},L} \hat{\otimes} \mathbb{K}, d\big)_\omega. \tag{7.10}$$

Since ω has Hochschild degree zero (with respect to our shifted grading) and $\tilde{\mathcal{U}}_n$ has degree $1-n$ with respect to the Hochschild grading we deduce that condition (*) in Section 6.2 holds and thus the twisting formalism exhibited in Section 6.2 applies. We obtain an L_∞-morphism

$$\tilde{\mathcal{U}}_\omega: T_{\text{poly},C^{\text{coord},L}}\big(C^{\text{coord},L} \hat{\otimes} \mathbb{K}\big)_\omega \to D_{\text{poly},C^{\text{coord},L}}\big(C^{\text{coord},L} \hat{\otimes} \mathbb{K}\big)_\omega \tag{7.11}$$

since by (P4) (using the notation of (6.6)) one has

$$\omega' = \sum_{j \geqslant 1} \frac{1}{j!} \tilde{\mathcal{U}}_j\big(\omega^j\big) = \tilde{\mathcal{U}}_1(\omega) = \omega.$$

Hence using (7.9) and (7.10) we have an L_∞-morphism

$$\mathcal{V}^{\mathrm{coord}}: T_{\mathrm{poly},C^{\mathrm{coord},L}}\big(C^{\mathrm{coord},L}\hat{\otimes}_{R_1} JL, {}^1\nabla^{\mathrm{coord}}\big) \to D_{\mathrm{poly},C^{\mathrm{coord},L}}\big(C^{\mathrm{coord},L}\hat{\otimes}_{R_1} JL, {}^1\nabla^{\mathrm{coord}}\big). \tag{7.12}$$

7.3.3. The formality map on affine coordinate spaces

First remark that $\tilde{\mathcal{U}}_\omega$ descends under the $\mathfrak{gl}_d(k)$-action. Namely, given the facts that $\tilde{\mathcal{U}}$ clearly commutes with the $\mathfrak{gl}_d(k)$-action (in the sense of Proposition 6.3.2) and that, using (5.8) and (P5), $\tilde{\mathcal{U}}_i(i_{\bar{v}}(\omega)\cdot\gamma) = 0$ for any $v \in \mathfrak{gl}_d(k)$ and $i \geqslant 2$; we can apply the criteria given by Propositions 6.3.2 and 6.4.1 and obtain an L_∞-morphism

$$\begin{aligned}\mathcal{V}^{\mathrm{aff}}: &T_{\mathrm{poly},C^{\mathrm{coord},L}}\big(C^{\mathrm{coord},L}\hat{\otimes}_{R_1} JL, {}^1\nabla^{\mathrm{coord}}\big)^{\mathfrak{gl}_d(k)}\\ &\to D_{\mathrm{poly},C^{\mathrm{coord},L}}\big(C^{\mathrm{coord},L}\hat{\otimes}_{R_1} JL, {}^1\nabla^{\mathrm{coord}}\big)^{\mathfrak{gl}_d(k)}.\end{aligned} \tag{7.13}$$

Here the notation $(-)^{\mathfrak{gl}_d(k)}$ is used in the sense of (6.7) and $\mathfrak{gl}_d(k)$ acts by the derivation of the $\mathrm{Gl}_d(k)$-action on the factor $\Omega_{R^{\mathrm{coord},L}}$ of $C^{\mathrm{coord},L} = \Omega_{R^{\mathrm{coord},L}} \otimes_{\Omega_{R_1}} \bigwedge_{R_1} L_1^*$.

There are morphisms of DG-Lie algebras

$$\begin{aligned}&T_{\mathrm{poly},C^{\mathrm{aff},L}}\big(C^{\mathrm{aff},L}\hat{\otimes}_{R_1} JL, {}^1\nabla^{\mathrm{aff}}\big) \to T_{\mathrm{poly},C^{\mathrm{coord},L}}\big(C^{\mathrm{coord},L}\hat{\otimes}_{R_1} JL, {}^1\nabla^{\mathrm{coord}}\big),\\ &D_{\mathrm{poly},C^{\mathrm{aff},L}}\big(C^{\mathrm{aff},L}\hat{\otimes}_{R_1} JL, {}^1\nabla^{\mathrm{aff}}\big) \to D_{\mathrm{poly},C^{\mathrm{coord},L}}\big(C^{\mathrm{coord},L}\hat{\otimes}_{R_1} JL, {}^1\nabla^{\mathrm{coord}}\big)\end{aligned} \tag{7.14}$$

obtained by extending $C^{\mathrm{aff},L}$-linear poly-vector fields and poly-differential operators to $C^{\mathrm{coord},L}$-linear ones. We claim that these maps yield isomorphisms of DG-Lie algebras

$$\begin{aligned}&T_{\mathrm{poly},C^{\mathrm{aff},L}}\big(C^{\mathrm{aff},L}\hat{\otimes}_{R_1} JL, {}^1\nabla^{\mathrm{aff}}\big) \to T_{\mathrm{poly},C^{\mathrm{coord},L}}\big(C^{\mathrm{coord},L}\hat{\otimes}_{R_1} JL, {}^1\nabla^{\mathrm{coord}}\big)^{\mathfrak{gl}_d(k)},\\ &D_{\mathrm{poly},C^{\mathrm{aff},L}}\big(C^{\mathrm{aff},L}\hat{\otimes}_{R_1} JL, {}^1\nabla^{\mathrm{aff}}\big) \to D_{\mathrm{poly},C^{\mathrm{coord},L}}\big(C^{\mathrm{coord},L}\hat{\otimes}_{R_1} JL, {}^1\nabla^{\mathrm{coord}}\big)^{\mathfrak{gl}_d(k)}.\end{aligned} \tag{7.15}$$

Using Lemma 4.3.4 and using the fact that $T_{\mathrm{poly}}^{L_2}(R_2)$ and $D_{\mathrm{poly}}^{L_2}(R_2)$ are free R_2-modules and that JL is a topologically free R_1-module it is sufficient to prove that

$$\Big(\Omega_{R^{\mathrm{coord},L}} \otimes_{\Omega_{R_1}} \bigwedge\nolimits_{R_1} L_1^*\Big)^{\mathfrak{gl}_d(k)} = \Omega_{R^{\mathrm{aff},L}} \otimes_{\Omega_{R_1}} \bigwedge\nolimits_{R_1} L_1^*. \tag{7.16}$$

Using the notation of Section 5 the isomorphism (7.16) follows from

$$\Omega_{S^{\mathrm{coord}}}^{\mathfrak{gl}_d(k)} = \Omega_{S^{\mathrm{aff}}}.$$

This follows easily from the fact that $\mathrm{Gl}_d(k)$ acts freely on $\operatorname{Spec} S^{\mathrm{coord}}$.

Therefore (7.13) now yields an L_∞-morphism

$$\mathcal{V}^{\mathrm{aff}}: T_{\mathrm{poly},C^{\mathrm{aff},L}}\big(C^{\mathrm{aff},L}\hat{\otimes}_{R_1} JL, {}^1\nabla^{\mathrm{aff}}\big) \to D_{\mathrm{poly},C^{\mathrm{aff},L}}\big(C^{\mathrm{aff},L}\hat{\otimes}_{R_1} JL, {}^1\nabla^{\mathrm{aff}}\big).$$

7.3.4. End of the proof

We have constructed L_∞-morphisms

$$\begin{array}{ccc} T_{\mathrm{poly}}^{L_2}(R_2) & \xrightarrow{\cong} & T_{\mathrm{poly},C^{\mathrm{aff},L}}(C^{\mathrm{aff},L}\,\hat{\otimes}_{R_1}\,JL) \\ & & \downarrow{\scriptstyle \mathcal{V}^{\mathrm{aff}}} \\ D_{\mathrm{poly}}^{L_2}(R_2) & \xrightarrow[\cong]{} & D_{\mathrm{poly},C^{\mathrm{aff},L}}(C^{\mathrm{aff},L}\,\hat{\otimes}_{R_1}\,JL) \end{array} \tag{7.17}$$

such that the horizontal maps are quasi-isomorphisms (by Proposition 7.3.1).

We put $\mathfrak{l}^L = D_{\mathrm{poly},C^{\mathrm{aff},L}}(C^{\mathrm{aff},L}\,\hat{\otimes}_{R_1}\,JL)$. Then the lower horizontal map in (7.17) yields the rightmost quasi-isomorphism in (7.1).

We will prove below that the composition

$$T_{\mathrm{poly}}^{L_2}(R_2) \to H^*\big(\mathfrak{l}^L\big) \xrightarrow{\cong^{-1}} H^*\big(D_{\mathrm{poly}}^{L_2}(R_2)\big)$$

coincides with the HKR-isomorphism. It follows in particular that the diagonal map in (7.17)

$$T_{\mathrm{poly}}^{L_2}(R_2) \to \mathfrak{l}^L$$

is an L_∞-quasi-isomorphism as well. This is the leftmost quasi-isomorphism in (7.1). We leave to the reader the tedious but straightforward verification of the functoriality of $\mathfrak{l}^L$.

To prove that the map on cohomology is given by the HKR-map we regard the complexes occurring in (7.14) as double complexes such that the differential obtained from $C^{\mathrm{aff},L}$ and $C^{\mathrm{coord},L}$ is horizontal. We write the coordinates for the double grading as couples (p,q) where p is the column index.

According to (6.5) $\tilde{\mathcal{U}}_{\omega,1}$ is given by

$$\tilde{\mathcal{U}}_{\omega,1}(\gamma) = \sum_{j\geqslant 0} \frac{1}{j!}\tilde{\mathcal{U}}_{j+1}\big(\omega^j\gamma\big). \tag{7.18}$$

Now $\tilde{\mathcal{U}}_{j+1}$ is homogeneous for the column grading and of degree $1-(j+1)$ for the Hochschild grading (the row grading), thus it has bidegree $(0,-j)$. Since ω lives in $C_1^{\mathrm{coord},L}\,\hat{\otimes}\,T_{\mathrm{poly}}^0(\mathbb{K})$ it has bidegree $(1,0)$, and hence $\tilde{\mathcal{U}}_{j+1}(\omega^j -)$ has bidegree $(j,-j)$.

Let $\tilde{\mathcal{U}}_{\omega,1}^j$ be the component of $\tilde{\mathcal{U}}_{\omega,1}$ indexed by j in (7.18).

Lemma 7.3.2. *We have the following commutative diagram*

$$\begin{array}{ccc} T_{\mathrm{poly}}^{L_2}(R_2) & \longrightarrow & C^{\mathrm{coord},L}\,\hat{\otimes}\,T_{\mathrm{poly}}(\mathbb{K}) \\ {\scriptstyle \mu}\downarrow & & \downarrow{\scriptstyle \tilde{\mathcal{U}}_{\omega,1}^0} \\ D_{\mathrm{poly}}^{L_2}(R_2) & \longrightarrow & C^{\mathrm{coord},L}\,\hat{\otimes}\,D_{\mathrm{poly}}(\mathbb{K}) \end{array} \tag{7.19}$$

where the horizontal arrows are inclusions obtained from the action by derivations of L_2 *on* $C^{\mathrm{coord},L}\mathbin{\hat\otimes}_{R_1} JL \cong C^{\mathrm{coord},L}\mathbin{\hat\otimes}\mathbb{K}$ *(see* (5.1)*) and* μ *is the HKR-map* (4.1)*.*

Proof. This is almost a tautology. Let $l_1,\dots,l_n\in L$ and denote by δ_i the derivation on $C^{\mathrm{coord},L}\mathbin{\hat\otimes}\mathbb{K}$ corresponding to l_i. Then

$$\tilde{\mathcal{U}}^0_{\omega,1}(\delta_1\wedge\cdots\wedge\delta_n)=\tilde{\mathcal{U}}_1(\delta_1\wedge\cdots\wedge\delta_n)=(-1)^{n(n-1)/2}\frac{1}{n!}\sum_{\sigma\in S_n}\epsilon(\sigma)\delta_{\sigma(1)}\otimes\cdots\otimes\delta_{\sigma(n)}.$$

This implies the commutativity of (7.19). □

Since the maps in (7.14) are inclusions $\mathcal{V}_1^{\mathrm{aff}}$ has the same grading properties as $\tilde{\mathcal{U}}_{\omega,1}$. In particular it maps $T_{\mathrm{poly},C^{\mathrm{aff},L}}(C^{\mathrm{aff},L}\mathbin{\hat\otimes}_{R_1} JL)_{p,q}$ to $\bigoplus_j D_{\mathrm{poly},C^{\mathrm{aff},L}}(C^{\mathrm{aff},L}\mathbin{\hat\otimes}_{R_1} JL)_{p+j,q-j}$. Let $\mathcal{V}_1^{\mathrm{aff},j}$ be the component of $\mathcal{V}_1^{\mathrm{aff}}$ indexed by j in this decomposition. Thus we obtain a commutative diagram

$$\begin{array}{ccc} T^{L_2}_{\mathrm{poly}}(R_2) & \longrightarrow & T_{\mathrm{poly},C^{\mathrm{aff},L}}(C^{\mathrm{aff},L}\mathbin{\hat\otimes}_{R_1} JL)\\ \mu\downarrow & & \downarrow \mathcal{V}_1^{\mathrm{aff},0}\\ D^{L_2}_{\mathrm{poly}}(R_2) & \longrightarrow & D_{\mathrm{poly},C^{\mathrm{aff},L}}(C^{\mathrm{aff},L}\mathbin{\hat\otimes}_{R_1} JL).\end{array} \tag{7.20}$$

The following lemma ends the proof of the theorem (see [42, Thm. 7.1]).

Lemma 7.3.3. $\mathcal{V}_1^{\mathrm{aff},0}$ *and* $\mathcal{V}_1^{\mathrm{aff}}$ *induce the same maps on cohomology.*

Proof. We filter $T_{\mathrm{poly},C^{\mathrm{aff},L}}(C^{\mathrm{aff},L}\mathbin{\hat\otimes}_{R_1} JL)$ and $D_{\mathrm{poly},C^{\mathrm{aff},L}}(C^{\mathrm{aff},L}\mathbin{\hat\otimes}_{R_1} JL)$ according to the column index.

The E_1 term of the resulting spectral sequences consists of the cohomology of the columns. Using (4.17) we have to compute the cohomology of $(C^{\mathrm{aff},L}\mathbin{\hat\otimes}_{R_1} JL)\mathbin{\hat\otimes}_{R_2} T^{L_2}_{\mathrm{poly}}(R_2)$ and $(C^{\mathrm{aff},L}\mathbin{\hat\otimes}_{R_1} JL)\mathbin{\hat\otimes}_{R_2} D^{L_2}_{\mathrm{poly}}(R_2)$ for the second factor. We obtain $(C^{\mathrm{aff},L}\mathbin{\hat\otimes}_{R_1} JL)\mathbin{\hat\otimes}_{R_2} T^{L_2}_{\mathrm{poly}}(R_2)$ and $(C^{\mathrm{aff},L}\mathbin{\hat\otimes}_{R_1} JL)\mathbin{\hat\otimes}_{R_2} H^{\cdot}(D^{L_2}_{\mathrm{poly}}(R_2))$ (the latter because $D^{L_2}_{\mathrm{poly}}(R_2)$ is a complex consisting of filtered projective R_2-modules with filtered projective cohomology).

Using Lemma 5.3.1 we obtain that the E_2 terms are given by $T^{L_2}_{\mathrm{poly}}(R_2)$ and $H^{\cdot}(D^{L_2}_{\mathrm{poly}}(R_2))$. It is now clear that $\mathcal{V}_1^{\mathrm{aff},0}$ and $\mathcal{V}_1^{\mathrm{aff}}$ induce indeed the same map on cohomology. □

7.4. Proof of Theorem 1.8 in the sheaf case

As indicated in the Introduction we can prove a result which slightly more general than Theorem 1.8. We work over a ringed site $(\mathcal{C},\mathcal{O})$ and $\mathcal{L}$ is a Lie algebroid locally free of rank d on $(\mathcal{C},\mathcal{O})$. The general formalism of (pre)sheaves on sites is developed in [2, I–III].

The DG-Lie algebras $T^{\mathcal{L}}_{\mathrm{poly}}(\mathcal{O})$, $D^{\mathcal{L}}_{\mathrm{poly}}(\mathcal{O})$ are obtained by sheafifying the presheaves

$$U\mapsto T^{\mathcal{L}(U)}_{\mathrm{poly}}\big(\mathcal{O}(U)\big),$$

$$U \mapsto D_{\text{poly}}^{\mathcal{L}(U)}\big(\mathcal{O}(U)\big)$$

for $U \in \text{Ob}(\mathcal{C})$.

Theorem 7.4.1. *There is an isomorphism between $T_{\text{poly}}^{\mathcal{L}}(\mathcal{O})$ and $D_{\text{poly}}^{\mathcal{L}}(\mathcal{O})$ in* HoLieAlg$(\mathcal{O})$*, the homotopy category of sheaves of DG-Lie algebras, which induces the HKR-isomorphism on cohomology.*

Proof. We replace $\mathcal{C}$ with the full subcategory consisting of $U \in \mathcal{C}$ such that there is an isomorphism $\mathcal{L} \mid U \cong (\mathcal{O} \mid U)^d$ (this does not change the category of sheaves).

If $p : U \to V$ is now a map in $\mathcal{C}$ then since $\mathcal{L}(V) \cong \mathcal{O}(V)^d$, $\mathcal{L}(U) \cong \mathcal{O}(U)^d$ we have that the restriction morphism

$$p^* : \big(\mathcal{O}(V), \mathcal{L}(V)\big) \to \big(\mathcal{O}(U), \mathcal{L}(U)\big)$$

satisfies the condition (7.2), i.e. $\mathcal{O}(U) \otimes_{\mathcal{O}(V)} \mathcal{L}(V) \cong \mathcal{L}(U)$.

Put ${}^p\mathfrak{l}^{\mathcal{L}}(U) = \mathfrak{l}^{\mathcal{L}(U)}$ where $\mathfrak{l}^{\mathcal{L}(U)}$ is as in Theorem 7.1. Then ${}^p\mathfrak{l}^{\mathcal{L}}$ is a presheaf of DG-Lie algebras. Let ${}^pT_{\text{poly}}^{\mathcal{L}}(\mathcal{O})$ and ${}^pD_{\text{poly}}^{\mathcal{L}}(\mathcal{O})$ be respectively the presheaves of DG-Lie algebras of $\mathcal{L}$-poly-vector fields and $\mathcal{L}$-poly-differential operators.

From the commutative diagram (7.3) we now deduce the existence of L_∞-quasi-isomorphisms of presheaves

$${}^pT_{\text{poly}}^{\mathcal{L}}(\mathcal{O}) \to {}^p\mathfrak{l}^{\mathcal{L}} \leftarrow {}^pD_{\text{poly}}^{\mathcal{L}}(\mathcal{O}). \tag{7.21}$$

Let $\mathfrak{l}^{\mathcal{L}}$ be the sheafification of ${}^p\mathfrak{l}^{\mathcal{L}}$. Sheafifying (7.21) finishes the proof. □

8. Atiyah classes and jet bundles

In this section we relate Atiyah classes to jet bundles. That this is possible is well known (see, e.g., [20, §4]) although we could not find the exact result we need (Proposition 8.4.2 below) in the literature.

8.1. Reminder

We define $(\mathcal{C}, \mathcal{O}, \mathcal{L})$ as in Section 7.4. Let $\mathcal{E}$ be an arbitrary $\mathcal{O}$-module. The Atiyah class $A(\mathcal{E}) \in \text{Ext}^1_{\mathcal{O}}(\mathcal{E}, \mathcal{L}^* \otimes \mathcal{E})$ is the obstruction against the existence of an $\mathcal{L}$-connection (not necessarily flat) on $\mathcal{E}$.

Let us briefly recall how $A(\mathcal{E})$ is constructed. By (4.8) we have $J^1\mathcal{L} = \mathcal{O}_1 \oplus \mathcal{L}^* = \mathcal{O}_2 \oplus \mathcal{L}^*$ as $\mathcal{O}_1$ and $\mathcal{O}_2$-algebras.

We consider the short exact sequence of $\mathcal{O}_1$-modules

$$0 \to \mathcal{L}^* \otimes_{\mathcal{O}} \mathcal{E} \to J^1\mathcal{L} \otimes_{\mathcal{O}_2} \mathcal{E} \to \mathcal{E} \to 0. \tag{8.1}$$

The class of this sequence in $\text{Ext}^1_{\mathcal{O}}(\mathcal{E}, \mathcal{L}^* \otimes \mathcal{E})$ is $A(\mathcal{E})$. To see that this is the obstruction against the existence of a connection let $\beta_0 : \mathcal{E} \to J^1\mathcal{L} \otimes_{\mathcal{O}_2} \mathcal{E}$ be the canonical splitting (as sheaves of abelian groups) of (8.1) obtained from the decomposition $J^1\mathcal{L} = \mathcal{O}_2 \oplus \mathcal{L}$. Then any splitting

$\beta : \mathcal{E} \to J^1\mathcal{L} \otimes_{\mathcal{O}_2} \mathcal{E}$ as $\mathcal{O}_1$-modules yields a connection $\nabla : \mathcal{E} \to \mathcal{L}^* \otimes_{\mathcal{O}} \mathcal{E}$ given by $\beta - \beta_0$. It is easy to see that this construction is reversible.

Assume now that $\mathcal{E}$ is locally free of finite rank. Taking powers and symmetrizing we obtain an element $A(\mathcal{E})^n$ in $\mathrm{Ext}^n_{\mathcal{O}}(\mathcal{E}, \bigwedge^n \mathcal{L}^* \otimes \mathcal{E})$. The nth (scalar) Atiyah class $a_n(\mathcal{E}) \in H^n(\mathcal{C}, \bigwedge^n \mathcal{L}^*)$ of $\mathcal{E}$ is the trace of $A(\mathcal{E})^n$.

8.2. *Atiyah classes: algebraic background*

We need some functoriality properties of the Atiyah class. To deduce these cleanly we work in a somewhat more abstract setting, which is loosely inspired by the $\bar{d}$-resolution in the complex analytic case. We also introduce some ad hoc terminology.

Let $\mathrm{Sh}^{\mathrm{bi}}(\mathcal{C})$ be the category of sheaves of abelian groups on $\mathcal{C}$ graded by $\mathbb{Z}^2$. If $\mathcal{F} \in \mathrm{Sh}^{\mathrm{bi}}(\mathcal{C})$ and f is a section of $\mathcal{F}_{i,j}$ then $|f| = i + j$ is the total degree of f. As always apply the Koszul sign convention with respect to total degree.

The category $\mathrm{Sh}^{\mathrm{bi}}(\mathcal{C})$ is equipped with two obvious shift functors each of total degree one

$$\mathcal{F}[1]_{i,j} = \mathcal{F}_{i+1,j},$$
$$\mathcal{F}(1)_{i,j} = \mathcal{F}_{i,j+1}.$$

Definition 8.2.1.

(1) A bigraded *DG-algebra* on $\mathcal{C}$ is a bigraded sheaf of algebras $\mathcal{A}$ on $\mathcal{C}$ equipped with a derivation $\bar{d}_{\mathcal{A}}$ of degree $(1, 0)$ such that $\bar{d}^2_{\mathcal{A}} = 0$.

(2) A *dDG-algebra* A on $\mathcal{C}$ is a bigraded sheaf of DG-algebras on $\mathcal{C}$ equipped with an additional derivation $d_{\mathcal{A}}$ of degree $(0, 1)$ such that $\bar{d}_{\mathcal{A}} d_{\mathcal{A}} + d_{\mathcal{A}} \bar{d}_{\mathcal{A}} = 0$.

(3) Assume that $\mathcal{A}$ is a bigraded DG-algebra. A *DG-$\mathcal{A}$-module* is a bigraded sheaf of $\mathcal{A}$-modules $\mathcal{M}$ equipped with an additive map $\bar{d}_M : \mathcal{M} \to \mathcal{M}$ of degree $(1, 0)$ such that $\bar{d}^2_{\mathcal{M}} = 0$ and such that

$$\bar{d}_{\mathcal{M}}(am) = \bar{d}_{\mathcal{A}}(a)m + (-1)^{|a|} a \bar{d}_{\mathcal{M}}(m)$$

for a, m homogeneous sections of $\mathcal{A}$ and $\mathcal{M}$. We denote the category of DG-modules over $\mathcal{A}$ by $\mathrm{DGMod}(\mathcal{A})$.

(4) Assume that $\mathcal{M}$ is DG-module over a dDG-algebra $\mathcal{A}$. Then a *connection* on $\mathcal{M}$ is an additive map $d_{\mathcal{M}} : \mathcal{M} \to \mathcal{M}$ of degree $(0, 1)$ such that

$$d_{\mathcal{M}}(am) = d_{\mathcal{A}}(a)m + (-1)^{|a|} a d_{\mathcal{M}}(m).$$

(5) The functors $?[1]$ and $?(1)$ change the signs of both $d_{\mathcal{M}}$ and $\bar{d}_{\mathcal{M}}$, when applicable.

(6) Assume that M is a DG-$\mathcal{A}$-module over a dDG-algebra $\mathcal{A}$, equipped with a connection. Then the *curvature* of $\mathcal{M}$ is defined as $R_{\mathcal{M}} = -(d_{\mathcal{M}} \bar{d}_{\mathcal{M}} + \bar{d}_{\mathcal{M}} d_{\mathcal{M}})$. This is a map $R_{\mathcal{M}} : \mathcal{M} \to \mathcal{M}(1)[1]$ in $\mathrm{DGMod}(\mathcal{A})$.

(7) The derived category of $\mathrm{DGMod}(\mathcal{A})$, equipped with the shift functor $?[1]$, is denoted by $\mathsf{D}(\mathcal{A})$.

Example 8.2.2. Let $A \to B$ be a morphism of sheaves of commutative DG-algebras. Then $\Omega_{B/A}$ is a dDG-algebra. The bigrading comes from the internal (coming from B) and external (exterior) degrees. The degree (0, 1) derivation d is the De Rham differential and the degree $(1,0)$ derivation $\bar{d}$ is characterized by the property that it commutes with d and that it coincides with d_B on $B = \Omega^0_{B/A}$.

Assume that $\mathcal{A}$ is a dDG-algebra. We define a bigraded DG-algebra

$$J^1\mathcal{A} = \mathcal{A} \oplus \mathcal{A}\epsilon$$

where ϵ satisfies $\bar{d}_{\mathcal{A}}(\epsilon) = 0 = \epsilon^2$, has degree $(0, -1)$ and

$$a\epsilon = (-1)^{|a|}\epsilon a.$$

We have two algebra morphisms commuting with $\bar{d}_{\mathcal{A}}$:

$$i_1 : \mathcal{A} \to J^1\mathcal{A} : a \mapsto a,$$
$$i_2 : \mathcal{A} \to J^1\mathcal{A} : a \mapsto a + \epsilon d_{\mathcal{A}}(a).$$

We view $J^1\mathcal{A}$ as a DG-$\mathcal{A}$-bimodule via i_1, i_2.

We get an associated exact sequence of $\mathcal{A}$–$\mathcal{A}$-bimodules

$$0 \to \mathcal{A}\epsilon \to J^1\mathcal{A} \to \mathcal{A} \to 0. \tag{8.2}$$

Let $\mathcal{M} \in \mathrm{DGMod}(\mathcal{A})$. Tensoring (8.2) on the right by $\mathcal{M}$ we obtain an exact sequence in $\mathrm{DGMod}(\mathcal{A})$

$$0 \to \mathcal{M}(1) \to J^1\mathcal{A} \otimes_{\mathcal{A}} \mathcal{M} \to \mathcal{M} \to 0 \tag{8.3}$$

with

$$\mathcal{M}(1) \to J^1\mathcal{A} \otimes_{\mathcal{A}} \mathcal{M} : n \mapsto \epsilon \otimes n,$$
$$J^1\mathcal{A} \otimes_A \mathcal{M} \to \mathcal{M} : (a + b\epsilon) \otimes m \mapsto am.$$

Definition 8.2.3. Let $\mathcal{M} \in \mathrm{DGMod}(\mathcal{A})$. The *Atiyah class* $A(\mathcal{M})$ of $\mathcal{M}$ is the element of $\mathrm{Hom}^1_{\mathsf{D}(\mathcal{A})}(\mathcal{M}, \mathcal{M}(1))$ representing the exact sequence (8.3).

Lemma 8.2.4. *If $\mathcal{M}$ has a connection then $A(\mathcal{M}) = R_{\mathcal{M}}$. In other words $A(\mathcal{M})$ is represented by an actual map*

$$R_{\mathcal{M}} : \mathcal{M} \to \mathcal{M}(1)[1]$$

of bigraded $\mathcal{A}$-modules.

Proof. If M has a connection $d_{\mathcal{M}}$ then the map

$$\beta : \mathcal{M} \to J^1\mathcal{A} \otimes_{\mathcal{A}} \mathcal{M} : m \mapsto 1 \otimes m - \epsilon \otimes d_{\mathcal{M}}(m)$$

defines a right splitting of (8.3) as graded $\mathcal{A}$-modules. The corresponding left splitting is

$$\alpha : J^1\mathcal{A} \otimes_{\mathcal{A}} \mathcal{M} \to \mathcal{M}(1) : 1 \otimes m + \epsilon \otimes n \mapsto d_{\mathcal{M}}(m) + n.$$

Since (8.3) is split its corresponding class in $\mathrm{Hom}^1_{\mathsf{D}(\mathcal{A})}(\mathcal{M}, \mathcal{M}(1))$ is given by[9] $-\alpha \bar{d}_{\mathcal{M}} \beta$. One computes that this is equal to $R_{\mathcal{M}}$. $\square$

8.3. Scalar Atiyah classes

Let $\mathcal{A}$ be a dDG-algebra on $\mathcal{C}$ and let $\mathcal{M} \in \mathrm{DGMod}(\mathcal{A})$. We assume in addition that $\mathcal{M}$ is locally free of constant rank e over $\mathcal{C}$. I.e. the topology on $\mathcal{C}$ has a basis $\mathcal{B}$ such that for $U \in \mathcal{B}$ we have that $\mathcal{M}_U \cong \mathcal{A}_U^{\oplus e}$ as bigraded $\mathcal{A}$-modules. We may now view $A(\mathcal{M})^n$ (the nth power of $A(\mathcal{M})$) as an element of $\mathrm{Hom}^n_{\mathsf{D}(\mathcal{A})}(\mathcal{M}, \mathcal{M}(n))$, or since $\mathcal{M}$ is locally free, as an element of

$$\mathbb{H}^n\big(\mathcal{C}, \mathcal{E}nd_{\mathcal{A}}(\mathcal{M})_{*,n}\big)$$

where $\mathbb{H}$ denotes hypercohomology and $\mathcal{E}nd_{\mathcal{A}}(\mathcal{M})_{*,n}$ is equipped with the differential $[d_{\mathcal{M}}, -]$.

It is easy to check locally that the trace map

$$\mathrm{Tr} : \mathcal{E}nd_{\mathcal{A}}(\mathcal{M}) \to \mathcal{A}$$

is in $\mathrm{DGMod}(\mathcal{A})$. Thus we obtain a map on hypercohomology

$$\mathrm{Tr} : \mathbb{H}^n\big(\mathcal{C}, \mathcal{E}nd_{\mathcal{A}}(\mathcal{M})_{*,n}\big) \to \mathbb{H}^n(\mathcal{C}, \mathcal{A}_{*,n}).$$

We call

$$a_n(\mathcal{M}) = \mathrm{Tr}\big(A(\mathcal{M})^n\big) \in \mathbb{H}^n(\mathcal{C}, \mathcal{A}_{*,n})$$

the nth (scalar) Atiyah class of M.

Lemma 8.3.1. *Assume that we have a morphism $\theta : \mathcal{A} \to \mathcal{B}$ of dDG-algebras and assume that $\mathcal{M} \in \mathrm{DGMod}(\mathcal{A})$ is locally free of rank e. Put $\mathcal{N} = \mathcal{B} \otimes_{\mathcal{A}} \mathcal{M}$. Then $\mathcal{N} \in \mathrm{DGMod}(\mathcal{B})$ is locally free of rank e. We have*

$$a_n(\mathcal{N}) = \mathbb{H}^n(\theta)\big(a_n(\mathcal{M})\big)$$

where $\mathbb{H}^n(\theta)$ is the natural map

$$\mathbb{H}^n(\theta) : \mathbb{H}^n(\mathcal{C}, \mathcal{A}_{*,n}) \to \mathbb{H}^n(\mathcal{C}, \mathcal{B}_{*,n}).$$

Proof. The compatibility of (8.2) with $\mathcal{B} \otimes_{\mathcal{A}} -$ yields that $A(\mathcal{N})$ is the image of $A(\mathcal{M})$ under the natural map

$$\mathbb{H}^1\big(\mathcal{C}, \mathcal{E}nd_{\mathcal{A}}(\mathcal{M})_{*,1}\big) \xrightarrow{\mathcal{B} \otimes_{\mathcal{A}} -} \mathbb{H}^1\big(\mathcal{C}, \mathcal{E}nd_{\mathcal{B}}(\mathcal{N})_{*,1}\big).$$

[9] To see this one should think of a degree wise split exact sequence as a shift to the left of a standard triangle constructed from a mapping cone. See [19, I§2].

This yields that $A(\mathcal{N})^n$ is the image of $A(\mathcal{M})^n$ under the induced map

$$\mathbb{H}^n\big(\mathcal{C}, \mathcal{E}nd_{\mathcal{A}}(\mathcal{M})_{*,n}\big) \xrightarrow{\mathcal{B}\otimes_{\mathcal{A}}-} \mathbb{H}^n\big(\mathcal{C}, \mathcal{E}nd_{\mathcal{B}}(\mathcal{N})_{*,n}\big).$$

One verifies locally that there is a commutative diagram of DG-modules

$$\begin{array}{ccc} \mathcal{E}nd_{\mathcal{A}}(\mathcal{M}) & \xrightarrow{\mathcal{B}\otimes_{\mathcal{A}}-} & \mathcal{E}nd_{\mathcal{B}}(\mathcal{N}) \\ {\scriptstyle \mathrm{Tr}}\downarrow & & \downarrow{\scriptstyle \mathrm{Tr}} \\ \mathcal{A} & \longrightarrow & \mathcal{B}. \end{array}$$

This finishes the proof. □

Example 8.3.2. We explain how the Atiyah class constructed in Section 8.1 fits into this framework. We define $\mathcal{A}$ as the De Rham complex $(\bigwedge \mathcal{L}_1^*, d)$ and put it in degrees $(0,*)$ (so that it becomes a dDG-algebra with $\bar{d}=0$). We define $\mathcal{M} = \mathcal{A} \otimes_{\mathcal{O}} \mathcal{E}$.

The Atiyah class $A_{\mathcal{L}}(\mathcal{E}) \overset{\text{def}}{=} A(\mathcal{M})$ now becomes an element of

$$\begin{aligned} \operatorname{Ext}^1_{D(\mathcal{A})}\big(\mathcal{M}, \mathcal{M}(1)\big) &= \operatorname{Ext}^1_{D(\mathcal{A})}\big(\mathcal{A} \otimes_{\mathcal{O}} \mathcal{E}, (\mathcal{A} \otimes_{\mathcal{O}} \mathcal{E})(1)\big) \\ &= \operatorname{Ext}^1_{\mathcal{O}}\big(\mathcal{E}, (\mathcal{A} \otimes_{\mathcal{O}} \mathcal{E})(1)\big) \\ &= \operatorname{Ext}^1_{\mathcal{O}}\big(\mathcal{E}, \mathcal{L}^* \otimes_{\mathcal{O}} \mathcal{E}\big). \end{aligned}$$

It is easy to see that $A_{\mathcal{L}}(\mathcal{E}) \in \operatorname{Ext}^1_{\mathcal{O}}(\mathcal{E}, \mathcal{L}^* \otimes_{\mathcal{O}} \mathcal{E})$ represents the part of degree zero of (8.3). This is

$$0 \to \mathcal{L}^* \otimes_{\mathcal{O}} \mathcal{E} \to J^1\mathcal{L} \otimes_{\mathcal{O}} \mathcal{E} \to \mathcal{E} \to 0.$$

Hence $A_{\mathcal{L}}(\mathcal{E})$ coincides with our previous definition. It is easy to deduce from this that we also get the same $a_{n,\mathcal{L}}(\mathcal{E})$.

8.4. Atiyah classes from jet bundles

We assume we are in the setting from Section 8.1. As outlined in the previous section we will work with bigraded sheaves.

We first consider the $\mathcal{L}_2$-De Rham complex $\bigwedge \mathcal{L}_2^*$ as a dDG-algebra concentrated in degrees $(0,*)$ with $\bar{d}=0$. We then let C be a commutative DG-algebra such that JL is equipped with a flat C-connection ∇. Thus $(C \mathbin{\hat{\otimes}_{\mathcal{O}_1}} J\mathcal{L}, \nabla)$ becomes a DG-algebra (actually a DG-C-algebra). From Lemma 4.3.6 we obtain a morphism of dDG-algebras

$$\theta : \Big(\bigwedge \mathcal{L}_2^*, \bar{d}=0\Big) \to \Omega_{C \hat{\otimes}_{\mathcal{O}_1} J\mathcal{L}/C} \tag{8.4}$$

where the dDG-structure on $\Omega_{C \hat{\otimes}_{\mathcal{O}_1} J\mathcal{L}/C}$ is as in Example 8.2.2.

If we set $\mathcal{M} = ((\bigwedge \mathcal{L}_2^*) \otimes_{\mathcal{O}_2} \mathcal{L}, \bar{d}=0) \in \operatorname{DGMod}(\bigwedge \mathcal{L}_2^*, \bar{d}=0)$ then we obtain from Lemma 8.3.1 and Example 8.3.2

$$\begin{aligned}\mathbb{H}(\theta)\big(a_{n,\mathcal{L}}(\mathcal{L})\big) &= \mathbb{H}(\theta)\big(a_n(\mathcal{M})\big) = a_n(\Omega_{C\hat{\otimes}_{\mathcal{O}_1} J\mathcal{L}/C} \hat{\otimes}_{\bigwedge \mathcal{L}_2^*} \mathcal{M}) \\ &= a_n(\underbrace{\Omega_{C\hat{\otimes}_{\mathcal{O}_1} J\mathcal{L}/C} \hat{\otimes}_{\mathcal{O}_2} \mathcal{L}}_{\stackrel{\text{def}}{=}\mathcal{N}}).\end{aligned} \tag{8.5}$$

Finally recall that by Lemma 4.3.4 $(C \hat{\otimes}_{\mathcal{O}_1} J\mathcal{L}) \hat{\otimes}_{\mathcal{O}_2} \mathcal{L}_2 \cong \mathcal{D}er_C(C \hat{\otimes}_{\mathcal{O}_1} J\mathcal{L})$, therefore

$$\mathcal{N} \cong (\Omega_{C\hat{\otimes}_{\mathcal{O}_1} J\mathcal{L}/C}) \hat{\otimes}_{C\hat{\otimes}_{\mathcal{O}_1} J\mathcal{L}} \underbrace{\mathcal{D}er_C(C \hat{\otimes}_{\mathcal{O}_1} J\mathcal{L})}_{\stackrel{\text{def}}{=}\mathcal{N}_0}.$$

The fact that $\mathbb{H}(\theta)(a_n(\mathcal{L})) = a_n(\mathcal{N})$ provides a mean to compute $a_n(\mathcal{L})$ if we can compute $a_n(\mathcal{N})$. The latter can be accomplished if we can put a connection on $\mathcal{N}$ (see Lemma 8.2.4).

Lemma 8.4.1. *Assume that there is an isomorphism of graded C-algebras*

$$\pi : C \hat{\otimes}_{\mathcal{O}_1} J\mathcal{L} \to C \hat{\otimes}_{\mathcal{O}} \widehat{\mathrm{S}_{\mathcal{O}}(\mathcal{L}^*)} \tag{8.6}$$

which induces the identity map $C \otimes_{\mathcal{O}} \operatorname{gr} J\mathcal{L} \cong C \otimes_{\mathcal{O}} \mathrm{S}_{\mathcal{O}}(\mathcal{L}^*)$. Then $\mathcal{N}$, as introduced in (8.5), *has a connection.*

Proof. We use the isomorphism (8.6) to transport the differential ∇ (defined on $B \stackrel{\text{def}}{=} C \hat{\otimes}_{\mathcal{O}} J\mathcal{L}$) to a differential $\tilde{\nabla}$ on $\tilde{B} \stackrel{\text{def}}{=} C \hat{\otimes}_{\mathcal{O}} \widehat{\mathrm{S}_{\mathcal{O}}(\mathcal{L}^*)}$. Note that this differential does not have a simple expression. As for ∇, we extend $\tilde{\nabla}$ to a unique differential of degree $(1,0)$ on $\Omega_{\tilde{B}/C} \cong C \hat{\otimes} \Omega_{\widehat{\mathrm{S}_{\mathcal{O}}(\mathcal{L}^*)}/\mathcal{O}}$ in such a way that it commutes with the De Rham differential $\tilde{d}_{\mathrm{DR}}$ (which has degree $(0,1)$).

We now put

$$\tilde{\mathcal{N}}_0 = \mathcal{D}er_C(\tilde{B}) \cong \tilde{B} \hat{\otimes}_{\mathcal{O}} \mathcal{L} \quad \text{and} \quad \tilde{\mathcal{N}} = \Omega_{\tilde{B}/C} \hat{\otimes}_{\tilde{B}} \tilde{\mathcal{N}}_0 = \Omega_{\tilde{B}/C} \hat{\otimes}_{\mathcal{O}} \mathcal{L}.$$

The isomorphism (8.6) between B and $\tilde{B}$ yields isomorphisms between $\Omega_{B/C}$ and $\Omega_{\tilde{B}/C}$, between $\mathcal{N}_0$ and $\tilde{\mathcal{N}}_0$ and between $\mathcal{N}$ and $\tilde{\mathcal{N}}$. We can now define a connection on $\tilde{\mathcal{N}} = \Omega_{\tilde{B}/C} \otimes_{\mathcal{O}} \mathcal{L}$ by putting

$$d_{\tilde{\mathcal{N}}}(b \otimes l) = \tilde{d}_{\mathrm{DR}}(b) \otimes l.$$

It is easy to see that this is well defined. Transporting across the isomorphism $\mathcal{N} \cong \tilde{\mathcal{N}}$ yields a connection $d_{\mathcal{N}}$ on $\mathcal{N}$. Let $R_{\mathcal{N}}$ be the curvature of this connection (see Definition 8.2.1(6)). We write

$$\hat{a}_n(\mathcal{N}) = \operatorname{Tr}\big(R_{\mathcal{N}}^n\big) \in (\Omega_{C\hat{\otimes} JL/C})_{n,n}. \tag{8.7}$$

According to Lemma 8.2.4 the cohomology class of $\hat{a}_n(\mathcal{N})$ for $\bar{d}$ is $a_n(\mathcal{N})$. □

The DG-algebras $C^{\mathrm{coord},\mathcal{L}}$ and $C^{\mathrm{aff},\mathcal{L}}$ are equipped with a canonical map $\bigwedge \mathcal{L}^* \to C^{\mathrm{coord},\mathcal{L}}$ and $\bigwedge \mathcal{L}^* \to C^{\mathrm{aff},\mathcal{L}}$ as follows from the definitions in Sections 5.2.1 and 5.3.

In addition condition (8.6) applies with $C = C^{\mathrm{aff},\mathcal{L}}$ and $C = C^{\mathrm{coord},\mathcal{L}}$, see (5.5), (5.14). We have natural morphisms

$$\bigwedge \mathcal{L}_2^* \xrightarrow{\theta} C^{\mathrm{aff},\mathcal{L}} \hat{\otimes}_{\mathcal{O}_1} \Omega_{J\mathcal{L}/\mathcal{O}_1} \xrightarrow{\psi} C^{\mathrm{coord},\mathcal{L}} \hat{\otimes}_{\mathcal{O}_1} \Omega_{J\mathcal{L}/\mathcal{O}_1} \underset{\cong}{\xrightarrow{\mu}} C^{\mathrm{coord},\mathcal{L}} \hat{\otimes} \Omega_{\mathbb{K}/k}$$

where θ is obtained from (8.4), ψ is obtained from the inclusion $C^{\mathrm{aff},\mathcal{L}} \hookrightarrow C^{\mathrm{coord},\mathcal{L}}$ and μ is obtained from the isomorphism $\tilde{t} : C^{\mathrm{coord},\mathcal{L}} \hat{\otimes} J\mathcal{L} \cong C^{\mathrm{coord},\mathcal{L}} \hat{\otimes} \mathbb{K}$ (see (5.5)).

Below we decorate notation referring to $C^{\mathrm{aff},\mathcal{L}}$ and $C^{\mathrm{coord},\mathcal{L}}$ by superscripts "aff" and "coord" respectively. For example we define $\mathcal{N}^{\mathrm{aff}}$ and $\mathcal{N}^{\mathrm{coord}}$ like $\mathcal{N}$ in (8.5) but we replace C by $C^{\mathrm{aff},\mathcal{L}}$ and $C^{\mathrm{coord},\mathcal{L}}$.

Proposition 8.4.2. *Write the Maurer–Cartan form* (*see* (5.7)) *as*

$$\omega = \sum_{i,\alpha} \eta_\alpha \omega_\alpha^i \partial_i$$

with $\partial_i = \partial/\partial t_i$, $\eta_\alpha \in (C^{\mathrm{coord},\mathcal{L}})_1$, $\omega_\alpha^i \in \mathbb{K}$. *Then we have in* $C_n^{\mathrm{coord},\mathcal{L}} \hat{\otimes} \Omega^n_{\mathbb{K}/k}$

$$(\mu\psi)\big(\hat{a}_n\big(\mathcal{N}^{\mathrm{aff}}\big)\big) = \mathrm{Tr}\big(\Xi^n\big)$$

where Ξ *is the matrix with entries*

$$\sum_\alpha \eta_\alpha d_{\mathbb{K}}\big(\partial_j \omega_\alpha^i\big) \in C_1^{\mathrm{coord},\mathcal{L}} \hat{\otimes} \Omega^1_{\mathbb{K}/k}.$$

Furthermore as cohomology classes we have

$$\mathbb{H}(\theta)\big(a_{n,\mathcal{L}}(\mathcal{L})\big) = a_n\big(\mathcal{N}^{\mathrm{aff}}\big). \tag{8.8}$$

Proof. The identity (8.8) is (8.5). We use the canonical connection on $\mathcal{N}^{\mathrm{aff}}$ and $\mathcal{N}^{\mathrm{coord}}$ exhibited in the proof of Lemma 8.4.1 to compute $\hat{a}_n(\mathcal{N}^{\mathrm{aff}})$ and $\hat{a}_n(\mathcal{N}^{\mathrm{coord}})$ (using (8.7)). Since these connections are compatible we get

$$\hat{a}_n\big(\mathcal{N}^{\mathrm{coord}}\big) = \psi\big(\hat{a}_n\big(\mathcal{N}^{\mathrm{aff}}\big)\big)$$

as elements of complexes.

We now compute $a_n(\mathcal{N}^{\mathrm{coord}})$ explicitly. We have identifications (e.g., (5.15) and (5.16))

$$C^{\mathrm{coord},\mathcal{L}} \hat{\otimes}_{\mathcal{O}_1} J\mathcal{L} \cong C^{\mathrm{coord},\mathcal{L}} \hat{\otimes}_{\mathcal{O}_1} \widehat{\mathrm{S}_{\mathcal{O}_1}(\mathcal{L}_1)} \cong C^{\mathrm{coord},L} \hat{\otimes} \mathbb{K}.$$

Using these identifications we have

$$\mathcal{N}^{\mathrm{coord}} = C^{\mathrm{coord},\mathcal{L}} \hat{\otimes}_{\mathcal{O}} \Omega_{\widehat{\mathrm{S}_{\mathcal{O}}(\mathcal{L}^*)}/\mathcal{O}} \hat{\otimes}_{\mathcal{O}} \mathcal{L} = C^{\mathrm{coord},\mathcal{L}} \hat{\otimes} \Omega_{\mathbb{K}} \hat{\otimes} \sum_i k\partial_i.$$

We have

$$\partial_1, \dots, \partial_d \in \mathcal{O}^{\mathrm{coord},L} \hat{\otimes}_{\mathcal{O}} \mathcal{L} \subset C^{\mathrm{coord},\mathcal{L}} \hat{\otimes}_{\mathcal{O}} \Omega_{\widehat{\mathrm{S}_{\mathcal{O}}(\mathcal{L}^*)}/\mathcal{O}} \hat{\otimes}_{\mathcal{O}} \mathcal{L}$$

and since $d_{\mathcal{N}^{\mathrm{coord}}}$ is zero on $\mathcal{O}^{\mathrm{coord},\mathcal{L}} \hat{\otimes}_{\mathcal{O}} \mathcal{L}$ we deduce

$$d_{\mathcal{N}^{\mathrm{coord}}}(\partial_i) = 0.$$

For further computation we use the identification

$$\mathcal{N}^{\mathrm{coord}} = C^{\mathrm{coord},\mathcal{L}} \hat{\otimes} \Omega_{\mathbb{K}} \hat{\otimes} \sum_i k\partial_i$$

where $d_{\mathcal{N}^{\mathrm{coord}}}$ acts as

$$d_{\mathcal{N}^{\mathrm{coord}}}(c \hat{\otimes} \omega \hat{\otimes} \partial_i) = (-1)^{|c|} c \hat{\otimes} d_{\Omega_{\mathbb{K}}}\omega \hat{\otimes} \partial_i.$$

Remember from Section 5.2.1 that the differential $\bar{d}_{B^{\mathrm{coord}}} = {}^1\nabla^{\mathrm{coord}}$ on $C^{\mathrm{coord},\mathcal{L}} \hat{\otimes} \Omega_{\mathbb{K}} \cong \Omega_{B^{\mathrm{coord}}/C^{\mathrm{coord},\mathcal{L}}}$ is given by

$$d_{C^{\mathrm{coord},\mathcal{L}}} \otimes 1 + \sum_{i,\alpha} \eta_\alpha \omega_\alpha^i \partial_i$$

where we think of ∂_i as a Lie derivative. We compute

$$\begin{aligned}\bar{d}_{\mathcal{N}^{\mathrm{coord}}}(\partial_j) &= \left[d_{C^{\mathrm{coord},L}} \otimes 1 + \sum_{i,\alpha} \eta_\alpha \omega_\alpha^i \partial_i, \partial_j\right] \\ &= \sum_{i,\alpha} \left(\eta_\alpha \partial_j \omega_\alpha^i\right)\partial_i\end{aligned}$$

and hence

$$\begin{aligned}R_{\mathcal{N}^{\mathrm{coord}}}(\partial_j) &= -(d_{\mathcal{N}^{\mathrm{coord}}}\bar{d}_{\mathcal{N}^{\mathrm{coord}}} + d_{\mathcal{N}^{\mathrm{coord}}}\bar{d}_{\mathcal{N}^{\mathrm{coord}}})(\partial_j) \\ &= -d_{\mathcal{N}^{\mathrm{coord}}}\left(\sum_{i,\alpha} \eta_\alpha \left(\partial_j \omega_\alpha^i\right)\partial_i\right) \\ &= \sum_{i,\alpha} \eta_\alpha d_{\Omega_{\mathbb{K}}}\left(\partial_j \omega_\alpha^i\right)\partial_i.\end{aligned}$$

Thus $\mu(\hat{a}_n(\mathcal{N}^{\mathrm{coord}})) = \mathrm{Tr}(\Xi^n)$ where Ξ is as in the statement of the proposition. This finishes the proof. □

9. The Kontsevich local formality quasi-isomorphism

9.1. The L_∞-morphism

In this section we assume that k contains the reals and we describe the exact form of the Kontsevich local formality morphism.

As above let $\mathbb{K} = k[\![t_1, \dots, t_d]\!]$ and $T_{\mathrm{poly}}(\mathbb{K})$, $D_{\mathrm{poly}}(\mathbb{K})$ be respectively the Lie algebras of poly-vector fields and poly-differential operators over $\mathbb{K}$. We equip $T_{\mathrm{poly}}(\mathbb{K})$ and $D_{\mathrm{poly}}(\mathbb{K})$ with the shifted Gerstenhaber structures introduced in Section 4.2.2. For $\gamma \in T^n_{\mathrm{poly}}(\mathbb{K})$ we put

$$\gamma^{i_1,\dots,i_{n+1}} = \langle dt_{i_1} \wedge \cdots \wedge dt_{i_{n+1}}, \gamma \rangle$$

where $\langle -, - \rangle$ is the pairing introduced in (4.2).

The Kontsevich local formality isomorphism $\mathcal{U} : T_{\mathrm{poly}}(\mathbb{K}) \to D_{\mathrm{poly}}(\mathbb{K})$ is defined as follows. We put

$$\mathcal{U}_n = \sum_{m \geqslant 0} \sum_{\Gamma \in G_{n,m}} W_\Gamma \mathcal{U}_\Gamma$$

where the W_Γ are some coefficients to be defined below and where $G_{n,m}$ is a set of directed graphs Γ described as follows:

(1) There are n vertices of the "first type" labeled by $1, \dots, n$.
(2) There are m vertices of the "second type" labeled by $1, \dots, m$.
(3) The vertices of the second type have no outgoing arrow.
(4) There are no loops and double arrows.
(5) There are $2n + m - 2$ edges.
(6) All edges carry a distinct label.

For use below we also introduce $G_{n,m,\epsilon}$ which is defined in the same way except that the number of edges of the graphs should be equal to $2n + m - 2 - \epsilon$. The number of edges in a graph is denoted by $|\Gamma|$.

For a vertex v of Γ we denote the incoming and outgoing edges of v by $\mathrm{In}(v)$ and $\mathrm{Out}(v)$ respectively. For each v we choose an arbitrary ordering on $\mathrm{Out}(v)$.

Let Γ_i be the vertices of the ith kind for $i = 1, 2$. Let $\gamma_i \in T_{\mathrm{poly}}(\mathbb{K})$ and put $k_i = |\gamma_i|$. By definition $\mathcal{U}_\Gamma(\gamma_1 \cdots \gamma_n)$ is zero unless $|\mathrm{Out}(i)| = k_i + 1$. In that case

$$\mathcal{U}_\Gamma(\gamma_1 \cdots \gamma_n)(f_1 \cdots f_m) = \prod_{\substack{v \in \Gamma_1 \\ \mathrm{In}(v) = r_1, \dots, r_d \\ \mathrm{Out}(v) = s_1, \dots, s_{k_v+1}}} \partial_{r_1} \cdots \partial_{r_d} \gamma_v^{s_1 \cdots s_{k_v+1}} \prod_{\substack{v \in \Gamma_2 \\ \mathrm{In}(v) = t_1, \dots, t_e}} \partial_{t_1} \cdots \partial_{t_e} f_v$$

where we assume that the ordering on $\mathrm{Out}(v)$ is such that $s_1 < \cdots < s_{k_v+1}$.

The coefficients W_Γ are defined as integrals over configuration spaces. Let $\mathcal{H}$ be the upper half plane and let $\mathbb{R}$ be its horizontal boundary. The group

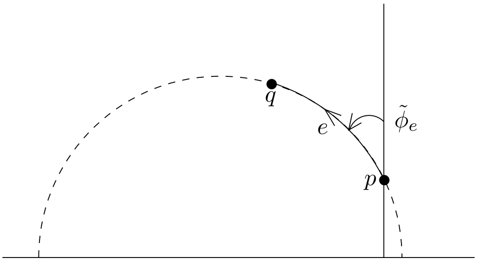
$$G^{(1)} = \{ z \mapsto az + b \mid a, b \in \mathbb{R},\ a > 0 \}$$

acts on $\mathcal{H} \cup \mathbb{R}$. $C^+_{n,m}$ is the quotient $\mathrm{Conf}^+_{n,m} / G^{(1)}$ where $\mathrm{Conf}^+_{n,m}$ is the space of configurations of n distinct points $p_1, \dots, p_n$ in $\mathcal{H}$ and m distinct points $q_1, \dots, q_m$ in $\mathbb{R}$ such that $q_1 < \cdots < q_m$. The manifold $C^+_{n,m}$ will be oriented as follows (see [1]). One puts p_1 in a fixed position and uses the coordinates of the other points to identify $C^+_{n,m}$ with an open subset of the affine space $\mathbb{A} = \mathbb{C}^{n-1} \times \mathbb{R}^m$. One then transfers the standard orientation on $\mathbb{A}$ to $C^+_{n,m}$.

For $\Gamma \in G_{n,m,\epsilon}$ put

$$\kappa_\Gamma = \bigwedge\nolimits_{e\in\{\text{edges of }\Gamma\}} d\phi_e$$

where the (multi-valued) function ϕ_e on $C^+_{n,m}$ is defined as $\tilde{\phi}_e/2\pi$ where $\tilde{\phi}_e$ is computed as in the following image



Thus if e is an edge in Γ from p to q then we embed e as a line in the hyperbolic plane $\mathcal{H}$ and we measure the angle ϕ_e in the counter clockwise direction as indicated in the drawing.

The ordering of the edges in the product $\bigwedge_e d\phi_e$ is first according to the ordering of the starting vertices in the set Γ_1 (the ordering is by label) and then according to the chosen ordering on outgoing edges.

Now we put[10]

$$W_\Gamma = (-1)^{|\Gamma|(|\Gamma|-1)/2} \int_{C^+_{n,m}} \kappa_\Gamma. \tag{9.1}$$

One easily sees that the product $W_\Gamma \mathcal{U}_\Gamma$ is independent of the chosen ordering on outgoing edges.

Assume $n = 1$. In that case $G_{1,m}$ contains only one graph Γ_0 and $\kappa_{\Gamma_0} = (-1)^{m(m-1)/2} 1/m!$. Furthermore

$$\begin{aligned} \mathcal{U}_{\Gamma_0}(\gamma)(f_1, \ldots, f_m) &= \gamma^{i_1\cdots i_m} \partial_{i_1} f_1 \cdots \partial_{i_m} f_m \\ &= \langle df_1 \cdots df_m, \gamma\rangle \end{aligned}$$

where $\langle -, - \rangle$ is as in (4.2). Hence

$$\mathcal{U}_1(\gamma)(f_1, \ldots, f_m) = (-1)^{m(m-1)/2} \frac{1}{m!} \langle df_1 \cdots df_m, \gamma\rangle. \tag{9.2}$$

It is easy to see that $\mathcal{U}_1$ coincides with the HKR-map $T_{\text{poly}}(\mathbb{K}) \to D_{\text{poly}}(\mathbb{K})$ as defined by (7.5).

[10] This definition differs by a sign from Kontsevich's definition. Kontsevich's definition necessitates an unpleasant sign change in the definition of the Lie bracket on $T_{\text{poly}}(\mathbb{K})$ (see [1]). Moreover this sign change destroys the Gerstenhaber property of $T_{\text{poly}}(\mathbb{K})$. A tedious computation shows that with our definition no sign changes for the Lie bracket are necessary.

9.2. Compatibility with cupproduct

Let C be a commutative DG-k-algebra over the category of filtered complete vector spaces. Then (using Section 7.2(+)) we may extend $\mathcal{U}$ to an L_∞-morphism

$$\tilde{\mathcal{U}} : C \hat{\otimes} T_{\mathrm{poly}}(\mathbb{K}) \to C \hat{\otimes} D_{\mathrm{poly}}(\mathbb{K}).$$

Assume now that we have a solution $\omega = \sum_\alpha \eta_\alpha \omega_\alpha$ to the Maurer–Cartan equation in $(C^1 \hat{\otimes} T^0_{\mathrm{poly}}(\mathbb{K}))$ which satisfies (*) or Section 6.2. By property (P4) we know that $\mathcal{U}_1(\omega)$ is a solution to the Maurer–Cartan equation in $C^1 \hat{\otimes} D^0_{\mathrm{poly}}(\mathbb{K})$. It will be convenient to denote $\mathcal{U}_1(\omega)$ also by ω.

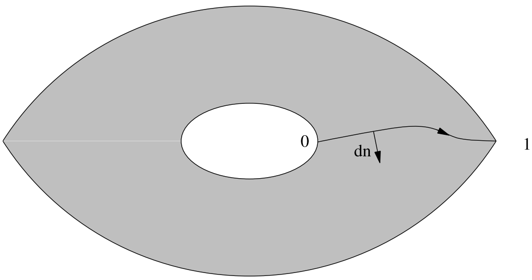
There exists a twisted L_∞-morphism (see Section 7.2)

$$\tilde{\mathcal{U}}_\omega : \left(C \hat{\otimes} T_{\mathrm{poly}}(\mathbb{K})\right)_\omega \to \left(C \hat{\otimes} D_{\mathrm{poly}}(\mathbb{K})\right)_\omega.$$

Kontsevich sketches a proof that $\mathcal{U}_\omega$ commutes with cup product up to homotopy.[11] A more detailed proof in the slightly restricted case that $\omega \in T^1_{\mathrm{poly}}(\mathbb{K})$ (i.e. a Poisson bracket) was given in [24]. In [26] it is even shown that $\tilde{\mathcal{U}}_\omega$ can be extended to an A_∞-morphism (this is again in the case $\omega \in T^1_{\mathrm{poly}}(\mathbb{K})$).

For the benefit of the reader we will state a result below which will be sufficient for the sequel. It can be obtained by copying the proof of [24], taking into account our modified sign conventions.

It is well known that $C^+_{n,m}$ can be canonically compactified as a manifold with corners $\bar{C}^+_{n,m}$ [22]. Let $\bar{C}_{2,0} = \bar{C}^+_{2,0}$ (the "+" is superfluous) be the "Eye" as in the following figure



The upper outer boundary is where p_1 approaches the real line, the lower outer boundary is where p_2 approaches the real line. The inner boundary is where p_1 and p_2 approach each other, away from the real line.

The right corner is the locus where p_1, p_2 both approach the real line with p_1 to the left of p_2. Following Kontsevich [22] we have indicated a path $\xi : [0, 1] \to \bar{C}^+_{2,0}$ from a point on the inner boundary (labeled "0") to the right corner (labeled "1").

Next we consider the map

$$F : C^+_{n,m} \to C_{2,0}$$

[11] Kontsevich's proves this in fact for general solutions of the Maurer–Cartan equation in $C \hat{\otimes} T_{\mathrm{poly}}(\mathbb{K})$.

which is given by projection onto the first two points. One may show that this map can be extended to a map

$$\bar{F}:\bar{C}^{+}_{n,m}\to\bar{C}_{2,0}$$

and we put $Z_{n,m}=\bar{F}^{-1}\xi([0,1])$. We orient Z by the normal dn (as indicated in the above figure).

For $\Gamma\in G_{n,m,1}$ we put

$$\tilde{W}_\Gamma=(-1)^{|\Gamma|(|\Gamma|-1)/2}\int\limits_{Z_{n,m}}\kappa_\Gamma.$$

After a tedious computation, mimicking [24], we obtain the following.

Proposition 9.1. *For any $\alpha,\beta\in T_{\mathrm{poly}}(\mathbb{K})$ we have*

$$\tilde{\mathcal{U}}_{\omega,1}(\alpha)\cup\tilde{\mathcal{U}}_{\omega,1}(\beta)-\tilde{\mathcal{U}}_{\omega,1}(\alpha\cup\beta)+d\bigl(H(\alpha,\beta)\bigr)-H(d\alpha,\beta)-(-1)^{|\alpha|+1}H(\alpha,d\beta)=0$$

where

$$H(\alpha,\beta)=\sum_{n,m\geqslant 0,\,\Gamma\in G_{n,m,1}}(-1)^{m-1}\frac{1}{(n-2)!}\tilde{W}_\Gamma\mathcal{U}_\Gamma\bigl(\alpha\beta\omega^{n-2}\bigr)\tag{9.3}$$

(*the operators $\mathcal{U}_\Gamma$ are extended multilinearly to $C\mathbin{\hat{\otimes}}T_{\mathrm{poly}}(\mathbb{K})$*). *In particular $\tilde{\mathcal{U}}_{\omega,1}$ commutes with cupproduct, up to a natural homotopy.*

10. Proof of Theorem 1.3

In this section we will initially assume that k contains the reals and we let the local formality morphism $\mathcal{U}$ in (7.4) be the one defined by Kontsevich (as in Section 9).

10.1. The local case

Combining the L_∞-morphisms (7.11), (7.9), (7.10), (7.12), (7.15), (7.13) we obtain a commutative diagram

$$\begin{array}{ccccccc}
T^{L_2}_{\mathrm{poly}}(R_2) & \longrightarrow & T_{\mathrm{poly},C^{\mathrm{aff},L}}(C^{\mathrm{aff},L}\hat{\otimes}_{R_1}JL) & \longrightarrow & T_{\mathrm{poly},C^{\mathrm{coord},L}}(C^{\mathrm{coord},L}\hat{\otimes}_{R_1}JL) & \xrightarrow{\cong} & (C^{\mathrm{coord},L}\hat{\otimes}T_{\mathrm{poly}}(\mathbb{K}))_\omega\\
 & & \Big\downarrow{\scriptstyle\mathcal{V}^{\mathrm{aff}}} & & \Big\downarrow{\scriptstyle\mathcal{V}^{\mathrm{coord}}} & & \Big\downarrow{\scriptstyle\tilde{\mathcal{U}}_\omega}\\
D^{L_2}_{\mathrm{poly}}(R_2) & \longrightarrow & D_{\mathrm{poly},C^{\mathrm{aff},L}}(C^{\mathrm{aff},L}\hat{\otimes}_{R_1}JL) & \longrightarrow & D_{\mathrm{poly},C^{\mathrm{coord},L}}(C^{\mathrm{coord},L}\hat{\otimes}_{R_1}JL) & \xrightarrow[\cong]{} & (C^{\mathrm{coord},L}\hat{\otimes}D_{\mathrm{poly}}(\mathbb{K}))_\omega
\end{array}\tag{10.1}$$

where the horizontal maps are strict morphisms (i.e. the only the first Taylor coefficient is nonzero).

Our aim is to sheafify diagram (10.1) and to look at the result in the derived category of $\mathcal{O}$-modules. To determine the result it is sufficient to understand the $(-)_1$ part of (10.1). I.e.

$$\begin{array}{ccccccc}
T_{\mathrm{poly}}^{L_2}(R_2) & \longrightarrow & T_{\mathrm{poly},C^{\mathrm{aff},L}}(C^{\mathrm{aff},L}\hat{\otimes}_{R_1} JL) & \longrightarrow & T_{\mathrm{poly},C^{\mathrm{coord},L}}(C^{\mathrm{coord},L}\hat{\otimes}_{R_1} JL) & \xrightarrow{\cong} & (C^{\mathrm{coord},L}\hat{\otimes}T_{\mathrm{poly}}(\mathbb{K}))_\omega \\
 & & \downarrow \mathcal{V}_1^{\mathrm{aff}} & & \downarrow \mathcal{V}_1^{\mathrm{coord}} & & \downarrow \tilde{\mathcal{U}}_{\omega,1} \\
D_{\mathrm{poly}}^{L_2}(R_2) & \longrightarrow & D_{\mathrm{poly},C^{\mathrm{aff},L}}(C^{\mathrm{aff},L}\hat{\otimes}_{R_1} JL) & \longrightarrow & D_{\mathrm{poly},C^{\mathrm{coord},L}}(C^{\mathrm{coord},L}\hat{\otimes}_{R_1} JL) & \xrightarrow[\cong]{} & (C^{\mathrm{coord},L}\hat{\otimes}D_{\mathrm{poly}}(\mathbb{K}))_\omega.
\end{array} \tag{10.2}$$

Lemma 10.1.1. *The map* $\mathcal{V}_1^{\mathrm{aff}}: T_{\mathrm{poly},C^{\mathrm{aff},L}}(C^{\mathrm{aff},L}\hat{\otimes}_{R_1} JL) \to D_{\mathrm{poly},C^{\mathrm{aff},L}}(C^{\mathrm{aff},L}\hat{\otimes}_{R_1} JL)$ *commutes with the Lie bracket and the cupproduct up to homotopies which are functorial for algebraic Lie algebroid morphisms which satisfy* (7.2).

Proof. For the Lie bracket this is clear since $\mathcal{V}_1^{\mathrm{aff}}$ is obtained from an L_∞-morphism.

For the cupproduct we need to show that the homotopy H defined by (9.3) descends to a map $T_{\mathrm{poly},C^{\mathrm{aff},L}}(C^{\mathrm{aff},L}\hat{\otimes}_{R_1} JL)^2 \to D_{\mathrm{poly},C^{\mathrm{aff},L}}(C^{\mathrm{aff},L}\hat{\otimes}_{R_1} JL)$. This is a computation similar to the proof of Proposition 6.4.1 combined with (5.8). We need the following version of (P5).

- $\tilde{W}_\Gamma \mathcal{U}_\Gamma(\gamma\alpha) = 0$ for $q \geqslant 3$ (q being the number edges of the "first type" in Γ) and $\gamma \in \mathfrak{gl}_d(k) \subset T^{\mathrm{poly},1}(\mathbb{K})$.

This is proved in exactly the same way as (P5). See [22, §7.3.3.1]. □

We will now evaluate the formula for $\tilde{\mathcal{U}}_{\omega,1}(\gamma)$ where we assume $\gamma \in T_{\mathrm{poly}}(\mathbb{K})$.

$$\tilde{\mathcal{U}}_{\omega,1}(\gamma) = \sum_{j\geqslant 0} \frac{1}{j!} \tilde{\mathcal{U}}_{j+1}(\omega^j \gamma).$$

We may write

$$\omega = \sum_\alpha \eta_\alpha \omega_\alpha$$

with $\eta_\alpha \in C_1^{\mathrm{coord}}$ and $\omega_\alpha \in T_{\mathrm{poly}}^0(\mathbb{K})$. Below we suppress the summation sign over α. Thus

$$\tilde{\mathcal{U}}_{j+1}(\omega^j \gamma) = \eta_{\alpha_j} \cdots \eta_{\alpha_1} \mathcal{U}_{j+1}(\omega_{\alpha_1} \cdots \omega_{\alpha_j} \gamma).$$

To understand the (absence of) signs in this formula we note that we consider $\tilde{\mathcal{U}}$ as a degree zero map $S^{j+1}((C^{\mathrm{coord},L}\hat{\otimes} T_{\mathrm{poly}}(\mathbb{K}))[1]) \to (C^{\mathrm{coord},L}\hat{\otimes} D_{\mathrm{poly}}(\mathbb{K}))[1]$. Thus ω has *even* degree when appearing as argument to $\tilde{\mathcal{U}}$. However the ω_α as argument to $\tilde{\mathcal{U}}$ have *odd* degree. By contrast the degree of the η_α is unchanged.

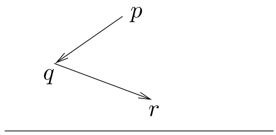
We need to enumerate the graphs contributing to the evaluation of $\mathcal{U}_{j+1}(\omega_{\alpha_1} \cdots \omega_{\alpha_j}\gamma)$. We need to consider the following type of graphs.

(1) There are j vertices of the first type labeled by $\omega_{\alpha_1}, \ldots, \omega_{\alpha_j}$. These have 1 outgoing arrow.
(2) There is 1 vertex of the first type labeled γ which has $p+1$ outgoing arrows.
(3) There are $m = p - j + 1$ vertices of the second type labeled by elements $f_1, \ldots, f_m \in F$.

The edges leaving γ are ordered by their ending vertex where we extend the implied ordering of vertices of the first type to all vertices via $\omega_{\alpha_1} < \cdots < \omega_{\alpha_j} < \gamma < f_1 < \cdots < f_m$.

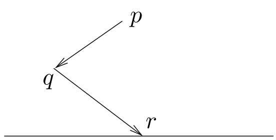
Since there are no loops and double edges we find that γ is connected through an outgoing arrow with all other vertices. It remains to allocate the j arrows emanating from the vertices labeled ω_α.

Recall that by [Ko, §7.3.1.1, §7.3.3.1] that W_Γ is zero if Γ contains one the following subgraphs



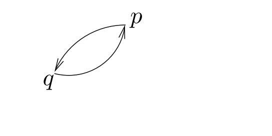
(10.3)

or



(10.4)

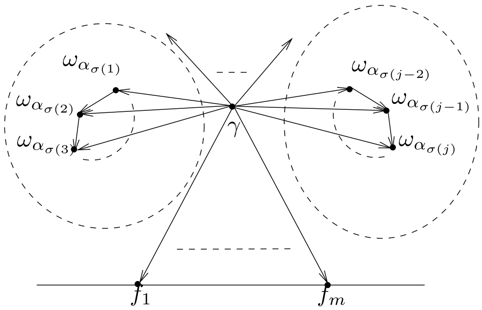
or



(10.5)

where q has no additional incoming or outgoing vertices.

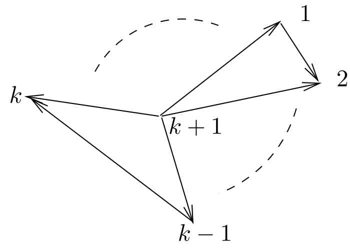
If there is an ω_α which does not have an incoming arrow from another ω_α then we are in one of the situations (10.3), (10.4) or (10.5) and hence $W_\Gamma = 0$. The remaining graphs are of the form



(10.6)

where σ is a permutation of $\{1, \ldots, j\}$.

Let us define Σ_k as the following "opposite wheel"



and put $W_k = W_{\Sigma_k}$. To fix the sign we order the vertices according to their labels and the outgoing edges of the central vertex according to their ending vertex.

Now we compute W_Γ and $\mathcal{U}_\Gamma$ for a graph as in (10.6). First we consider W_Γ. Assume that there are s wheels of size $l_1, \dots, l_s$ respectively.

Write g_i for the edge emanating in ω_{α_i} for $i = 1, \dots, j$ and e_i for the edge connecting γ to ω_{α_i} for $i = 1, \dots, j$. Finally write h_i for the edge connecting γ to f_i for $i = 1, \dots, p + 1 - j = m$. Then

$$W_\Gamma = (-1)^{(m+2j)(m+2j-1)/2} \int d\phi_{g_1} \cdots d\phi_{g_j}\, d\phi_{e_1} \cdots d\phi_{e_j}\, d\phi_{h_1} \cdots d\phi_{h_{p+1-j}}.$$

To evaluate the integral we may put γ in $i \in \mathcal{H}$. This reduces the symmetry group $G^{(1)}$ to the identity. We may clearly choose the ϕ_{h_i} freely apart from the fact that $\phi_{h_1} < \cdots < \phi_{h_{p+1-j}}$. Thus we get

$$\begin{aligned}
W_\Gamma &= (-1)^{(m+2j)(m+2j-1)/2} \frac{1}{m!} \int d\phi_{g_1} \cdots d\phi_{g_j}\, d\phi_{e_1} \cdots d\phi_{e_j} \\
&= (-1)^{(m+2j)(m+2j-1)/2} \frac{1}{m!} \int d\phi_{g_{\sigma(1)}} \cdots d\phi_{g_{\sigma(j)}}\, d\phi_{e_{\sigma(1)}} \cdots d\phi_{e_{\sigma(j)}} \\
&= (-1)^{\sum_{p<q} l_p l_q} (-1)^{(m+2j)(m+2j-1)/2} \frac{1}{m!} \int d\phi_{g_{\sigma(1)}} \cdots d\phi_{g_{\sigma(l_1)}}\, d\phi_{e_{\sigma(1)}} \cdots d\phi_{e_{\sigma(l_1)}} \cdots \\
&= (-1)^{\sum_{p<q} l_p l_q} (-1)^{(m+2j)(m+2j-1)/2} (-1)^{\sum_i 2l_i(2l_i-1)/2} \frac{1}{m!} W_{l_1} \cdots W_{l_s} \\
&= (-1)^{\sum_{p<q} l_p l_q} (-1)^{(m+2j)(m+2j-1)/2} (-1)^j \frac{1}{m!} W_{l_1} \cdots W_{l_s}
\end{aligned}$$

where in the last line we have used the identities $(2l(2l-1))/2 \equiv l \bmod 2$ and $\sum_i l_i = j$.

Now we compute $\mathcal{U}_\Gamma(\omega_{\alpha_1} \cdots \omega_{\alpha_j} \gamma)(f_1, \dots, f_m)$. We are short of symbols so we use the same symbol for an edge and for its corresponding index. Of course we use the same ordering of the edges as above. We find

$$\begin{aligned}
&\mathcal{U}_\Gamma(\omega_{\alpha_1} \cdots \omega_{\alpha_j} \gamma)(f_1, \dots, f_m) \\
&\quad = \big(\partial_{e_{\sigma(1)}} \partial_{g_{\sigma(l_1)}} \omega_{\alpha_{\sigma(1)}}^{g_{\sigma(1)}}\big)\big(\partial_{e_{\sigma(2)}} \partial_{g_{\sigma(1)}} \omega_{\alpha_{\sigma(2)}}^{g_{\sigma(2)}}\big)\big(\partial_{e_{\sigma(3)}} \partial_{g_{\sigma(2)}} \omega_{\alpha_{\sigma(3)}}^{g_{\sigma(3)}}\big) \cdots (\partial_{h_1} f_1 \cdots \partial_{h_m} f_m) \gamma^{e_1 \cdots e_j h_1 \cdots h_m}.
\end{aligned}$$

We need a more concise way of writing this. Let Ξ_α be the matrix of 1-forms $d(\partial_i \omega_\alpha^j) = \partial_k \partial_i (\omega_\alpha^j)\, dt^k$. Then $\mathcal{U}_\Gamma(\omega_{\alpha_1} \cdots \omega_{\alpha_j} \gamma)(f_1, \dots, f_m)$ is equal to

$$\begin{aligned}
&(-1)^\sigma \big\langle \operatorname{Tr}(\Xi_{\alpha_{\sigma(1)}} \cdots \Xi_{\alpha_{\sigma(l_1)}}) \cdots \operatorname{Tr}(\Xi_{\alpha_{\sigma(l_1+\cdots+l_{s-1}+1)}} \cdots \Xi_{\alpha_{\sigma(l_1+\cdots+l_{s-1}+l_s)}})\, df_1 \cdots df_m, \gamma \big\rangle \\
&\quad = (-1)^\sigma \big\langle df_1 \cdots df_m, \operatorname{Tr}(\Xi_{\alpha_{\sigma(l_1+\cdots+l_{s-1}+l_s)}} \cdots \Xi_{\alpha_{\sigma(l_1+\cdots+l_{s-1}+1)}}) \cdots \operatorname{Tr}(\Xi_{\alpha_{\sigma(l_1)}} \cdots \Xi_{\alpha_{\sigma(1)}}) \wedge \gamma \big\rangle \\
&\quad = (-1)^\sigma (-1)^{\frac{m(m-1)}{2}} m! \\
&\qquad \times \operatorname{HKR}\big(\operatorname{Tr}(\Xi_{\alpha_{\sigma(l_1+\cdots+l_{s-1}+l_s)}} \cdots \Xi_{\alpha_{\sigma(l_1+\cdots+l_{s-1}+1)}}) \cdots \operatorname{Tr}(\Xi_{\alpha_{\sigma(l_1)}} \cdots \Xi_{\alpha_{\sigma(1)}}) \wedge \gamma\big)(f_1, \dots, f_m)
\end{aligned}$$

where the first equality follows from (4.3) and the second equality is (9.2). So

$$\begin{aligned}
&\mathcal{U}_\Gamma(\omega_{\alpha_1} \cdots \omega_{\alpha_j} \gamma) \\
&\quad = (-1)^\sigma (-1)^{\frac{m(m-1)}{2}} m! \\
&\qquad \times \operatorname{HKR}\big(\operatorname{Tr}(\Xi_{\alpha_{\sigma(l_1+\cdots+l_{s-1}+l_s)}} \cdots \Xi_{\alpha_{\sigma(l_1+\cdots+l_{s-1}+1)}}) \cdots \operatorname{Tr}(\Xi_{\alpha_{\sigma(l_1)}} \cdots \Xi_{\alpha_{\sigma(1)}}) \wedge \gamma\big).
\end{aligned}$$

Put

$$\tilde{\mathcal{U}}_\Gamma\big(\omega^j \gamma\big)(f_1, \dots, f_n) = \eta_{\alpha_j} \cdots \eta_{\alpha_1} \mathcal{U}_\Gamma(\omega_{\alpha_1} \cdots \omega_{\alpha_j} \gamma)(f_1, \dots, f_m).$$

An easy computation yields

$$\tilde{\mathcal{U}}_\Gamma\big(\omega^j \gamma\big) = (-1)^{j(j-1)/2} (-1)^{m(m-1)/2} m! \operatorname{HKR}\big(\operatorname{Tr}\big(\Xi^{l_s}\big) \cdots \operatorname{Tr}\big(\Xi^{l_1}\big) \wedge \gamma\big)$$

where we have extended $- \wedge -$ and HKR$(-)$ to operations over $C^{\mathrm{coord},L}$ and where Ξ is the matrix $\eta_\alpha d(\partial_i \omega_\alpha^j)$ of elements of $C_1^{\mathrm{coord}} \hat{\otimes} \Omega^1_{\mathbb{K}}$. The entries of Ξ have even total degree so the traces $\operatorname{Tr}(\Xi^l)$ commute.

Now note the following simple identities

$$\begin{aligned}
\frac{j(j-1)}{2} = \frac{(\sum_i l_i)(\sum_i l_i - 1)}{2} = \sum_{i<j} l_i l_j + \sum_i \frac{l_i(l_i-1)}{2}, \\
\frac{(m+2j)(m+2j-1)}{2} = \frac{m(m-1)}{2} + j \bmod 2.
\end{aligned}$$

Collecting all signs we deduce

$$W_\Gamma \tilde{\mathcal{U}}_\Gamma\big(\omega^j \gamma\big) = (-1)^{\sum_i l_i(l_i-1)/2} W_{l_1} \cdots W_{l_s} \operatorname{HKR}\big(\operatorname{Tr}\big(\Xi^{l_1}\big) \cdots \operatorname{Tr}\big(\Xi^{l_s}\big) \wedge \gamma\big).$$

Putting temporarily

$$X_l = (-1)^{l(l-1)/2} W_l \operatorname{Tr}\big(\Xi^l\big)$$

we find

$$W_\Gamma \tilde{\mathcal{U}}_\Gamma\big(\omega^j \gamma\big) = \operatorname{HKR}(X_{l_1} \cdots X_{l_s} \wedge \gamma).$$

Now we have to enumerate the number of possible graphs Γ. Ordering the size of the wheels in increasing order we get a partition $\tau = (1 \cdots 1 \cdots r \cdots r)$ where i occurs τ_i times. The number distinct graphs corresponding to such a partition is

$$\frac{j!}{\tau_1! \cdots \tau_r! 1^{\tau_1} \cdots r^{\tau_r}}.$$

Thus we find that

$$\sum_{j\geqslant 0} \frac{1}{j!} \tilde{\mathcal{U}}_{j+1}(\omega^j \gamma) = \sum_{\tau_1,\tau_2,\ldots} \frac{1}{\tau_1!\tau_2! \cdots 1^{\tau_1} 2^{\tau_2} \cdots} \mathrm{HKR}\big(X_1^{\tau_1} X_2^{\tau_2} \cdots \wedge \gamma\big).$$

Formally we have

$$e^{X_r/r} = \sum_{\tau_r} \frac{1}{\tau_r! r^{\tau_r}} X_r^{\tau_r}$$

so that we find

$$\sum_{j\geqslant 0} \frac{1}{j!} \tilde{\mathcal{U}}_{j+1}(\omega^j \gamma) = \mathrm{HKR}\big(e^{X_1 + X_2/2 + \cdots} \wedge \gamma\big).$$

So if we put

$$\Theta = \sum_l (-1)^{l(l-1)/2} \frac{1}{l} W_l \Xi^l$$

then

$$\begin{aligned}\sum_{j\geqslant 0} \frac{1}{j!} \tilde{\mathcal{U}}_{j+1}(\omega^j \gamma) &= \mathrm{HKR}\big(e^{\mathrm{Tr}(\Theta)} \wedge \gamma\big) \\ &= \mathrm{HKR}\big(\det\big(e^{\Theta}\big) \wedge \gamma\big).\end{aligned}$$

This formula was proved under the assumption that $\gamma \in T_{\mathrm{poly}}(\mathbb{K})$. However by linear extension it follows that it remains true if $\gamma \in C^{\mathrm{coord},L} \hat{\otimes} T_{\mathrm{poly}}(\mathbb{K})$. Thus our final formula is

$$\tilde{\mathcal{U}}_{\omega,1} = \mathrm{HKR}\big(\det\big(e^{\Theta}\big) \wedge -\big). \tag{10.7}$$

We now analyze the series Θ is more detail. We need to know the value of W_l. As explained to us by Torossian this can be obtained from the work of Cattaneo and Felder on the quantization of coisotropic submanifolds [11,12]. See [38, Thm. 18]. As an alternative one can use a tedious but elementary computation using Stokes theorem [35, (1.1)]. The result is the following.

Lemma 10.1.2. *We have*

$$W_l = -(-1)^{(l+1)l/2} l ß_l \tag{10.8}$$

where $ß_n$ *is the nth modified Bernoulli number* $ß_n$ *which is defined by*

$$\sum_l ß_l x^l = \frac{1}{2}\log\frac{e^{x/2}-e^{-x/2}}{x}.$$

Let us finally also mention that a result similar to (10.8) was announced by Shoikhet in [31, §2.3.1]. It can presumably be obtained from the methods in [30].

Substituting we find

$$\begin{aligned}\Theta &= -\sum_l (-1)^l ß_l \Xi^l \\ &= -\frac{1}{2}\log\frac{e^{\Xi/2}-e^{-\Xi/2}}{\Xi}\end{aligned}$$

and hence

$$e^{\Theta} = \sqrt{\frac{\Xi}{e^{\Xi/2}-e^{-\Xi/2}}}. \tag{10.9}$$

It will be convenient for a module N with a connection to introduce the modified Todd class as follows

$$\widetilde{\mathrm{td}}(N) = \det\bigl(\tilde{q}\bigl(A(N)\bigr)\bigr)$$

where

$$\tilde{q}(x) = \frac{x}{e^{x/2}-e^{-x/2}}.$$

It follows from Proposition 8.4.2 that the following diagram is commutative on the level of complexes

$$\begin{array}{ccc} T_{\mathrm{poly},C^{\mathrm{aff},L}}(C^{\mathrm{aff},L}\hat{\otimes}_{R_1} JL) & \longrightarrow & C^{\mathrm{coord},L}\hat{\otimes} T_{\mathrm{poly}}(\mathbb{K}) \\ \Big\downarrow {\scriptstyle \widetilde{\mathrm{td}}(\mathcal{N}^{\mathrm{aff}})^{1/2}\wedge -} & & \Big\downarrow {\scriptstyle \det e^{\Theta}\wedge -} \\ T_{\mathrm{poly},C^{\mathrm{aff},L}}(C^{\mathrm{aff},L}\hat{\otimes}_{R_1} JL) & \longrightarrow & C^{\mathrm{coord},L}\hat{\otimes} T_{\mathrm{poly}}(\mathbb{K}). \end{array}$$

Since the horizontal maps are monomorphisms we get by comparing with (10.2)

$$\mathcal{V}_1^{\mathrm{aff}} = \mathrm{HKR}\circ\bigl(\widetilde{\mathrm{td}}\bigl(\mathcal{N}^{\mathrm{aff}}\bigr)^{1/2}\wedge -\bigr). \tag{10.10}$$

10.2. The global case

Now we globalize things. Let $(\mathcal{C},\mathcal{O},\mathcal{L})$ be as in Section 7.4. Then it follows from Proposition 8.4.2 (specifically (8.8)) that the following diagram is commutative in $\mathrm{D}(\mathcal{C})$ (the derived category of sheaves of vector spaces over $\mathcal{C}$)

$$\begin{array}{ccc} T_{\mathrm{poly}}^{\mathcal{L}}(\mathcal{O}) & \xrightarrow{\cong} & T_{\mathrm{poly},C^{\mathrm{aff},\mathcal{L}}}(C^{\mathrm{aff},L}\,\hat{\otimes}_{\mathcal{O}_1} J\mathcal{L}) \\ {\scriptstyle \widetilde{\mathrm{td}}(\mathcal{L})^{1/2}\wedge-}\downarrow & & \downarrow{\scriptstyle \widetilde{\mathrm{td}}(\mathcal{N}^{\mathrm{aff}})^{1/2}\wedge-} \\ T_{\mathrm{poly}}^{\mathcal{L}}(\mathcal{O}) & \xrightarrow[\cong]{} & T_{\mathrm{poly},C^{\mathrm{aff},\mathcal{L}}}(C^{\mathrm{aff},L}\,\hat{\otimes}_{\mathcal{O}_1} J\mathcal{L}) \end{array}$$

so that we get a commutative diagram over $\mathrm{D}(\mathcal{C})$

$$\begin{array}{ccc} T_{\mathrm{poly}}^{\mathcal{L}}(\mathcal{O}) & \xrightarrow{\cong} & T_{\mathrm{poly},C^{\mathrm{aff},\mathcal{L}}}(C^{\mathrm{aff},L}\,\hat{\otimes}_{\mathcal{O}_1} J\mathcal{L}) \\ {\scriptstyle \mathrm{HKR}(\widetilde{\mathrm{td}}(\mathcal{L})^{1/2}\wedge-)}\downarrow & & \downarrow{\scriptstyle \mathcal{V}_1^{\mathrm{aff}}} \\ D_{\mathrm{poly}}^{\mathcal{L}}(\mathcal{O}) & \xrightarrow[\cong]{} & D_{\mathrm{poly},C^{\mathrm{aff},\mathcal{L}}}(C^{\mathrm{aff},L}\,\hat{\otimes}_{\mathcal{O}_1} J\mathcal{L}) \end{array}$$

where $\widetilde{\mathrm{td}}(\mathcal{L})^{1/2}$ is as defined in the Introduction. Since the horizontal isomorphisms as well as $\mathcal{V}_1^{\mathrm{aff}}$ are Gerstenhaber algebra morphisms (Lemma 10.1.1) the same holds for $\mathrm{HKR}(\widetilde{\mathrm{td}}(\mathcal{L})^{1/2}\wedge -)$ as well. This finishes the proof of Theorem 1.3 in the case k contains the reals and the Todd class is replaced by the modified Todd class.

10.3. Proof for the ordinary Todd class

We have

$$\begin{aligned} \widetilde{\mathrm{td}}(\mathcal{L}) &= \mathrm{td}(\mathcal{L})\det\bigl(e^{-A(\mathcal{L})/2}\bigr) \\ &= \mathrm{td}(\mathcal{L})e^{-\mathrm{Tr}(A(\mathcal{L}))/2} \\ &= \mathrm{td}(\mathcal{L})e^{-a_1(\mathcal{L})/2}. \end{aligned}$$

In other words it is sufficient to prove that $e^{-a_1(\mathcal{L})/4}\wedge -$ defines an automorphism of $T_{\mathrm{poly}}^{\mathcal{L}}(\mathcal{O})$ as a Gerstenhaber algebra in $\mathsf{D}(\mathcal{C})$.

We may as well prove that $e^{-\mathrm{Tr}(\Xi)/4}\wedge -$ is compatible with the Lie bracket and the cupproduct on $C^{\mathrm{coord}}\,\hat{\otimes}\,T_{\mathrm{poly}}(\mathbb{K})$ or equivalently that $\mathrm{Tr}(\Xi)\wedge -$ is a derivation for these operations. We have $\mathrm{Tr}(\Xi)=\sum_{i,\alpha}\eta_\alpha d(\partial_i\omega_\alpha^i)$. Put $b_\alpha=\sum_i\partial_i\omega_\alpha^i$. Since everything is $C^{\mathrm{coord},\mathcal{L}}$-linear it is sufficient to prove that $db_\alpha\wedge -$ is a derivation for the Lie algebra and cupproduct on $T_{\mathrm{poly}}(\mathbb{K})$.

Since this fact is clear for the cupproduct we only look at the Lie bracket. For $D,E\in T_{\mathrm{poly}}^0(\mathbb{K})$ we have

$$\begin{aligned}
db_\alpha \wedge [D,E] &= [D,E](b_\alpha)\\
&= D\big(E(b_\alpha)\big) - E\big(D(b_\alpha)\big)\\
&= D(db_\alpha \wedge E) - E(db_\alpha \wedge D)\\
&= [D, db_\alpha \wedge E] + [db_\alpha \wedge D, E]
\end{aligned}$$

finishing the proof.

10.4. Arbitrary base fields

Now we let k be arbitrary (of characteristic zero) and we choose an embedding $k \subset \mathbb{C}$. It follows from the formulas (10.7) and (10.9) that $\tilde{\mathcal{U}}_{\omega,1}$, while initially defined over $\mathbb{C}$ ($\mathbb{R}$ in fact), actually descends to k. We will denote the descended morphism by u.

Now note the following.

Proposition 10.4.1.

(1) *There exist $w_\Gamma \in k$ for $\Gamma \in G_{m,n}$, which are zero when W_Γ is zero, such that for*

$$l(\alpha,\beta) = \sum_{n,m\geqslant 0,\ \Gamma\in G_{n,m}} w_\Gamma \mathcal{U}_\Gamma\big(\alpha\beta\omega^{n-2}\big)$$

we have

$$\big[u(\alpha), u(\beta)\big] - u\big([\alpha,\beta]\big) + d\big(l(\alpha,\beta)\big) - l(d\alpha,\beta) - (-1)^{|\alpha|} l(\alpha, d\beta) = 0. \tag{10.11}$$

(2) *There exist $\tilde{w}_\Gamma \in k$ for $\Gamma \in G_{m,n,1}$, which are zero when $\tilde{W}_\Gamma$ is zero, such that for*

$$h(\alpha,\beta) = \sum_{n,m\geqslant 0,\ \Gamma\in G_{n,m,1}} \tilde{w}_\Gamma \mathcal{U}_\Gamma\big(\alpha\beta\omega^{n-2}\big)$$

we have

$$u(\alpha)\cup u(\beta) - u(\alpha\cup\beta) + d\big(h(\alpha,\beta)\big) - h(d\alpha,\beta) - (-1)^{|\alpha|} h(\alpha, d\beta) = 0. \tag{10.12}$$

Proof. Eqs. (10.11), (10.12) are linear in w_Γ, $\tilde{w}_\Gamma$. By the fact that $\tilde{\mathcal{U}}_\omega$ is an L_∞-morphism and Proposition 9.1 there is a solution over $\mathbb{C}$. Hence there is a solution over k (for example obtained by applying an arbitrary projection $\mathbb{C} \to k$). $\square$

We can now proceed as in Sections 10.1, 10.2 (using an analogue of Lemma 10.1.1) to construct a commutative diagram

$$\begin{array}{ccccccc}
T^{\mathcal{L}_2}_{\mathrm{poly}}(\mathcal{O}_2) & \longrightarrow & T_{\mathrm{poly},C^{\mathrm{aff},\mathcal{L}}}(C^{\mathrm{aff},\mathcal{L}} \hat{\otimes}_{\mathcal{O}_1} J\mathcal{L}) & \longrightarrow & T_{\mathrm{poly},C^{\mathrm{coord},\mathcal{L}}}(C^{\mathrm{coord},\mathcal{L}} \hat{\otimes}_{\mathcal{O}_1} J\mathcal{L}) & \xrightarrow{\cong} & (C^{\mathrm{coord},\mathcal{L}} \hat{\otimes}\, T_{\mathrm{poly}}(\mathbb{K}))_\omega\\
& & \Big\downarrow{\scriptstyle v^{\mathrm{aff}}} & & \Big\downarrow{\scriptstyle v^{\mathrm{coord}}} & & \Big\downarrow{\scriptstyle u}\\
D^{\mathcal{L}_2}_{\mathrm{poly}}(\mathcal{O}_2) & \longrightarrow & D_{\mathrm{poly},C^{\mathrm{aff},\mathcal{L}}}(C^{\mathrm{aff},\mathcal{L}} \hat{\otimes}_{\mathcal{O}_1} J\mathcal{L}) & \longrightarrow & D_{\mathrm{poly},C^{\mathrm{coord},\mathcal{L}}}(C^{\mathrm{coord},\mathcal{L}} \hat{\otimes}_{\mathcal{O}_1} J\mathcal{L}) & \xrightarrow[\cong]{} & (C^{\mathrm{coord},\mathcal{L}} \hat{\otimes}\, D_{\mathrm{poly}}(\mathbb{K}))_\omega
\end{array}$$

where v^{aff} commutes both with the Lie bracket and the cupproduct up to a global homotopy. Using the fact that the formula (10.7) continues to hold

$$u = \mathrm{HKR} \circ \big(\det(\Theta) \wedge -\big)$$

we can now continue as in Section 10.2 to finish the proof of Theorem 1.3.

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