Invariant Eigendistributions on a Semisimple Lie Algebra and Homology Classes on the Conormal Variety. II. Representations of Weyl Groups

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We construct representations of $W$ in the homology of certain subvarieties of the cotangent bundle of the flag manifold of $g$, in particular in the homology of the conormal variety and of Springer varieties. Using the integral formula we prove that in top degree the former is isomorphic with the natural representation of $W$ on coherent families of invariant eigendistributions and decomposes into a sum of the latter according to the decomposition of the nilpotent variety into $G_0$-orbits. As consequences we obtain Springer's theorem on irreducible representations of Weyl groups, a formula for irreducible characters of $G_0$ as integrals over characteristic varieties of $D$ modules, an identification of the harmonic polynomials occurring in the asymptotic expansion at zero of invariant eigendistributions as homology classes on the flag manifold, a formula for the Fourier transforms of nilpotent $G_0$ orbits, and a proof of a conjecture of Joseph on the characteristic polynomials of intersections of nilpotent $G_0$-orbits with $n_0$.

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1. INTRODUCTION

This second part contains applications of the Integral Formula, as promised in the introduction to the first part. The main results I consider to be Theorems (5.1) and (7.1), even though neither is hard to prove. Apart from being of some interest in themselves, they imply a number of results which are generally considered hard, if I am not mistaken, and certainly occupy considerable space in the literature. Among the results that follow from these theorems one might mention a formula for the global character of a \((g_0, K_0)\)-module with regular integral infinitesimal character as an integral over the characteristic cycle of the corresponding \((\mathcal{D}, K)\)-module (for \(g_0\) complex); a formula for the harmonic polynomials studied in [15, 16, 20, 29] through an interpretation of these polynomials as cohomology classes on the flag manifold; a proof of a conjecture of Joseph [17], and various other things.

The core of the paper is logically self-contained, except for reference to the Integral Formula and some basic facts about flag manifolds and conormal varieties, but factually indebted to many sources, as I shall point out where appropriate. Some peripheral results, however, rely on theorems not proved here (and beyond what might reasonably be called "basic facts"). In Section 6, for example, I use a theorem of Kashiwara and Tanisaki [18, 27] and in Section 11 I quote a result of Hotta [12]. On the whole, I have made some effort to keep the paper as elementary and self-contained as possible without becoming repetitious.

As mentioned, some of the results presented here among the applications of the Integral Formula are known; I included them when I felt that the present approach sheds some additional light thereon. An example is the Kazhdan–Lusztig [19] completeness theorem for the Weyl group representation on the top homology of the conormal variety, which here appears in a very simple and explicit form as a consequence of Theorem 5.1. Another example is a formula for measures on nilpotent orbits proved by Barbasch and Vogan [3, 4] for special orbits and by Hotta and Kashiwara [13] in general. Here this formula falls out as a byproduct of Theorem 7.1. (I am indebted to Michèle Vergne for correcting a mistake in my proof of that theorem.) The table of contents may serve as a further guide to the topics treated.

Some related papers, which have appeared or come to my attention since this paper was written, are cited in Refs. [30–34]. In particular, the recent work of Joseph [32] and Vergne [34] provides an interesting alternative approach to some of the questions discussed in Sections 7 and 10.
2. CONSTRUCTION OF WEYL GROUP REPRESENTATIONS

We keep the notation introduced in part I, except that (until further notice) \( g \) can be any complex, semisimple Lie algebra (not necessarily \( \cong g_0 \times g_0 \)), \( b_1 \) any Borel subalgebra of \( g \) containing the Cartan subalgebra \( h \). For any (not necessarily regular) \( \lambda \in h^* \) we set

\[ \Omega_\lambda = \{ \xi \in g^* : p(\xi) = p(\lambda) \text{ for all } G\text{-invariant polynomials } p \text{ on } g^* \}. \]

The map

\[ p_\lambda = p_{h_1, \lambda} : \mathcal{B}^* \to \Omega_\lambda, u \cdot (b_1, v) \to u \cdot (\lambda + v) \]

with \( u \in U \) (compact form of \( G \), \( v \in b_1^+ \) is then well defined and surjective for all \( \lambda \in h^* \).

Assume now \( \lambda \) regular. Then \( p_\lambda \) is bijective and for any \( w \in W \) we may define a transformation \( a_\lambda(w) = a_{h_1, \lambda}(w) \) of \( \mathcal{B}^* \) by

\[ a_\lambda(w) = p_{w, \lambda}^{-1} \circ p_\lambda : \mathcal{B}^* \to \mathcal{B}^*. \]

It is evident that

\[ a_\lambda(\omega w) = a_\lambda(w) a_\lambda(\omega). \quad (1) \]

If one could set \( \lambda = 0 \) in this equation, one would obtain an action of \( W \) on \( \mathcal{B}^* \), which would leave the map \( p_0 : \mathcal{B}^* \to \Omega_0 \) invariant, so that \( W \) would permute the fibers of this map. This is of course only trivially possible, as the fibers are generically single points: the map \( p_0 \) is the Springer map, which is a desingularization \( \pi : \mathcal{B}^* \to \mathcal{N} \) of the nilpotent cone \( \Omega_0 = \mathcal{N} \) in \( g^* \) [26]. Borrowing an idea of Kazhdan and Lusztig [19] we look for what one might call a (proper) homotopy action of \( W \) on \( \mathcal{B}^* \), meaning a homomorphism of \( W \) into the group of proper homotopy equivalences of \( \mathcal{B}^* \), rather than a genuine action. But we shall not use the Kazhdan–Lusztig construction (which in fact they could not prove to give a homotopy action); instead we use the \( a_\lambda(w) \). (We shall prove in an appendix that our construction agrees with Kazhdan–Lusztig's, proving incidentally that their construction gives a homotopy action after all. Of course, what we are ultimately interested in is a representation of \( W \) in homology, and that they did obtain—at least for the top-dimensional homology. Several other constructions of (variants of) this representation of \( W \) are known, first of all Springer's original construction [24, 25], another construction of Lusztig [21], and others.)

We write out explicitly the definition of \( a_\lambda(w) \),

\[ a_\lambda(w)(u \cdot (b_1, v)) = u' \cdot (b_1, v'), \quad \text{where } u' \cdot (w \lambda + v') = u(\lambda + v). \]
Thus
\[ u' \cdot v' - u \cdot v = -u' \cdot w + u \cdot \lambda. \]
hence
\[ |u' \cdot v' - u \cdot v| \leq const |\lambda|. \]

Thus for \( \lambda \) close to 0 (but regular) the transformations \( a_\lambda(w) \) leave the Springer map \( \pi: B^* \to \mathcal{N} \) approximately invariant in the sense that
\[ |\pi(a_\lambda(w)(b, v)) - \pi(b, v)| \leq const |\lambda|. \quad (2) \]

For any subset \( V \) of \( \mathcal{N} \) let
\[ B^*(V) = \{(b, v) \in B^* : v \in V\}, \]
the inverse image of \( V \) in \( B^* \). We wish to construct a proper homotopy action of \( W \) on \( B^*(V) \). This requires a regularity condition on \( V \). Namely, for fixed \( \varepsilon > 0 \), let \( U \) be the \( \varepsilon \)-neighbourhood of \( V \) in \( \mathcal{N} \),
\[ U = \{v \in \mathcal{N} : |v - v'| < \varepsilon \text{ for some } v' \in V\}, \]
and
\[ B^*(U) = \{(b, v) \in B^* : |v - v'| < \varepsilon \text{ for some } v' \in V\} \quad (3) \]
its inverse image in \( B^* \). We require that

for sufficiently small \( \varepsilon > 0 \), the inclusion \( i: B^*(V) \to B^*(U) \) should admit a proper homotopy inverse \( p: B^*(U) \to B^*(V) \), i.e.,
\[ p \circ i \sim 1 \text{ on } B^*(V), \text{ and } i \circ p \sim 1 \text{ on } B^*(U) \quad (4) \]
with "\( \sim \)" meaning "properly homotopic."

We observe that his condition is satisfied in either the following two cases:

(a) \( V \) is a finite subpolyhedron of a triangulation of \( \mathcal{N} \),
(b) \( V \) is a constructible subset of \( \mathcal{N} \), stable under scalar multiplications.

The first condition may be explained as follows. Let \( \bar{g}^* = P(g^* \oplus C) \) be the projective completion of \( g^* \). \( \tilde{\mathcal{N}} \) the closure of \( \mathcal{N} \) in \( \bar{g}^* \), \( \bar{B}^* \) the closure of \( B^* \subset B \times \mathcal{N} \) in \( B \times \tilde{\mathcal{N}} \). As a projective algebraic variety, \( \tilde{\mathcal{N}} \) admits a triangulation \([11]\) hence (5) makes sense.

That (5) implies (4) is seen as follows. Assume (5). Then \( B^*(V) \) is also a finite subpolyhedron of a triangulation of \( B^* \). It is an elementary fact that a finite subpolyhedron has a retractable neighbourhood
[1, Sects. 6.1–6.2]. In particular $\mathcal{B}^*(V)$ has a retractible neighbourhood $W$ in $\mathcal{B}^*$. Since $\pi$ is a proper map, there is a neighbourhood $U$ of $V$ in $\mathcal{N}$ so that $\pi^{-1}(U) \subset W$, which gives (4).

That the condition (6) also implies (4) is seen as follows. One may suppose that $V$ excludes 0, as a neighbourhood of 0 may be treated separately. Because of the assumption that $V$ is stable under scalar multiplications and excludes 0, one may then replace $V$ by $V \cap \{ |v| = 1 \}$ in order to prove (4). Since $V$ is also constructible, one may apply (5).

From now on we shall assume that all sets $V$ under consideration satisfy either (5) or (6), so that (4) applies. (This may well be unnecessarily restrictive, but is sufficient for the applications we have in mind.) It is clear from (2) and (3) that

$$a_{\lambda}(w) \mathcal{B}^*(V) \subset \mathcal{B}^*(U)$$

for $\lambda \in h^*$ sufficiently close to 0. Thus the transformation

$$a_{\lambda, V}(\omega) = p \cdot a_{\lambda}(w) \cdot i$$

(7)

of $\mathcal{B}^*(V)$ is defined for all regular $\lambda$ in a small ball about 0 in $h^*$. Since these $\lambda$ from a connected set, the proper homotopy class $a_{\lambda}(w)$ of $a_{\lambda, V}(w)$ is independent of $\lambda$, and Eq. (1) implies that

$$a_{\lambda}(wy) = a_{\lambda}(w) a_{\lambda}(y)$$

so that the $a_{\lambda}(w)$ give a proper homotopy action of $W$ on $\mathcal{B}^*(V)$.

As a consequence we have a representation of $W$ in the homology with integral coefficients, denoted $H_*(\mathcal{B}^*(V), \mathbb{Z})$. As in [19] “homology” may here be understood either as “Borel–Moore homology” or as relative homology $H_*(\mathcal{B}^*(V), \partial \mathcal{B}^*(V); \mathbb{Z})$ of the finite polyhedron $\mathcal{B}^*(V)$ with respect to its finite subpolyhedron $\partial \mathcal{B}^*(V)$ (see the explanations in connection with (5)). The coefficient ring $\mathbb{Z}$ will be omitted from the notation when understood or unimportant.

For the further analysis of the representations of $W$ in these $H_*(\mathcal{B}^*(V))$ we follow Kazhdan–Lusztig [19]. Suppose $X \subset Y$ are closed subvarieties of $\mathcal{N}$, and put $U = Y - X$. There is a long exact sequence

$$\cdots \to H_{i+1}(\mathcal{B}^*(U)) \to H_i(\mathcal{B}^*(X)) \to H_i(\mathcal{B}^*(Y))$$$$H_i(\mathcal{B}^*(U)) \to H_{i-1}(\mathcal{B}^*(X)) \to \cdots.$$ (8)

In top degree, when $i = 2m$, $m = \dim_c \mathcal{B}^*(Y)$, $H_{2m+1}(\mathcal{B}^*(U)) = 0$, trivially, and the boundary map $H_{2m}(\mathcal{B}^*(U)) \to H_{2m-1}(\mathcal{B}^*(X))$ is $= 0$ because the topological boundary of a complex variety has real codimension at least 2.
So (8) leads to the short exact sequence

\[ 0 \rightarrow H_{2m}(\mathcal{C}(X)) \rightarrow H_{2m}(\mathcal{C}(Y)) \rightarrow H_{2m}(\mathcal{C}(U)) \rightarrow 0. \]  

(9)

The maps (7) and (9) are \( W \)-maps.

3. Specialization

In the above construction, choose \( V = \{v\} \), a single point. Then

\[ \mathcal{B}^*(v) = \{(h, v) | v \in h \downarrow\}, \]

which may be identified with

\[ \mathcal{B}^v = \{h \in \mathcal{B} | v \in h \downarrow\}. \]

This is the fixed point set of (the one parameter group generated by) \( v \), when \( g^* \) is identified with \( g \). We thus have a representation of \( W \) in \( H_\bullet(\mathcal{B}^v) \).

The component group \( A(v) \) of the stabilizer of \( v \) in \( G \) acts on \( H_\bullet(\mathcal{B}^v) \).

This action commutes with the action of \( W \), because the elements of \( A(v) \) have representatives in \( U \) (the compact form of \( G \)) and the operators \( a_\alpha(w) = p_{w,1}^{-1}p_\alpha \) commute with the action of \( U \) on \( \mathcal{B}^* \), the \( p_\alpha \) being \( U \)-equivariant.

The Weyl group \( W \) acts on the flag-manifold \( \mathcal{B} \cong U/T \) by

\[ a(w) \cdot u \cdot b_1 = u \cdot w^{-1}b_1 \quad (u \in U). \]

In the context of Springer's construction, the analogue of the following lemma is the Specialization Theorem of Hotta–Springer [14].

**Lemma 3.1.** The inclusion \( \mathcal{B}^v \rightarrow \mathcal{B} \) induces a \( W \)-map \( H_\bullet(\mathcal{B}^v) \rightarrow H_\bullet(\mathcal{B}) \).

This map factors through the projection \( H_\bullet(\mathcal{B}^v) \rightarrow H_\bullet(\mathcal{B})^{A(v)} \) onto the \( A(v) \)-invariants.

**Proof.** It suffices to show that the homotopy equivalence of \( \mathcal{B}^v \) constructed above has a representative \( \mathcal{B}^v \rightarrow \mathcal{B}^v \), denoted \( a^v(w) \), so that the inclusion \( i: \mathcal{B}^v \rightarrow \mathcal{B} \) satisfies

\[ i \circ a^v(w) \sim a(w) \circ i. \]

Choose a neighbourhood \( V \) of \( \mathcal{B}^v \) for which the inclusion \( k: \mathcal{B}^v \rightarrow V \) has a homotopy inverse \( q: V \rightarrow \mathcal{B}^v \):

\[ q \circ k \sim 1 \quad \text{on} \ \mathcal{B}^v, \quad k \circ q \sim 1 \quad \text{on} \ V. \]
Let \( U = \{(b, v') : b \in V, |v'| \leq R \} \) for some fixed \( R > |v| \). Define maps

\[
j : B^r \to U, j(b) = (k(b), v) = (b, v), \quad p : V \to B^r, p(b, v') = q(b).
\]

We show that

\[
p \circ j \sim 1 \quad \text{on } B^r, \quad j \circ p \sim 1 \quad \text{on } U. \quad (1)
\]

The first relation is clear since

\[
p \circ j(h) = q \circ k(h) \quad \text{and} \quad q \circ k \sim 1 \quad \text{on } B^r.
\]

To see the second relation, use \( k \circ q \sim 1 \) to choose a homotopy \( q_s : V \to V, 0 \leq s \leq 1, \) from \( q_0 = 1_v \) to \( q_1 = k \circ q \). Perform successively the following homotopies of maps \( U \to U \):

1. \( (h, v') \to (h, sv'), s \) going from 1 to 0,
2. \( (h, v') \to (q_s(h), 0), s \) going from 0 to 1,
3. \( (h, v') \to (k \circ q_s(h), sv), s \) going from 0 to 1.

This gives a homotopy from the identity \( 1_U : (b, v') \to (h, v') \) to \( p \circ j : (b, v') \to (k \circ q_s(h), v) \) as required.

For \( \lambda \) close to 0, \( a_{\lambda}(w) \circ j(B^r) \) stays in the neighbourhood \( U \) of \( j(B^r) \). The relation (1) shows that the \( p, j \) can take the place of the \( p, i \) in the definition (2.7) of \( a'(w) \). For such \( \lambda \), the map \( p \circ a_{\lambda}(w) \circ j : B^r \to B^r \) therefore represents the homotopy class \( a'(w) \):

\[
a'(w) \sim p \circ a_{\lambda}(w) \circ j : B^r \to B^r. \quad (2)
\]

For \( 0 \leq s \leq 1 \), define maps

\[
j_s : B^r \to U, j_s(b) = (b, sv), \quad p_s : U \to V, p_s(b, v') = q_s(b)
\]

with \( q_s \) as above. Consider the homotopy of maps

\[
p_s \circ a_{\lambda}(w) \circ j_s : B^r \to V, \quad 0 \leq s \leq 1. \quad (3)
\]

For \( s = 0 \) we obtain

\[
p_0 \circ a_{\lambda}(w) \circ j_0(b) = q_0 \circ a_{\lambda}(w)(b, 0) = a(w) b
\]

because \( a_{\lambda}(w) \) and \( a(w) \) coincide on \( B \) considered as the zero section in \( B^* \); this is clear from the definition of \( a_{\lambda}(w) \). So for \( s = 0 \), (3) reduces to \( a(w) \circ i \). On the other hand, for \( s = 1 \) we obtain

\[
p_1 \circ a_{\lambda}(w) \circ j_1(b) = p \circ a_{\lambda}(w)(b, v) = p \circ a_{\lambda}(w) \circ j(b) = a'(w) b
\]
by (2). So for \( s = 1 \), (3) reduces to \( a^t(w) \circ i \). Hence (3) provides the desired homotopy \( i \circ a^t(w) \sim a(w) \circ i \).

To see the second assertion of the lemma one only has to note that the action of the stabilizer of \( v \) in \( U \), which induces the action of \( A(v) \) in \( H_*(\mathcal{B}^v) \), becomes trivial in \( H_*(\mathcal{B}) \) as \( U \) is connected. This proves the lemma.

**Corollary 3.2.** In top dimension the \( W \)-map

\[
H_{2e(v)}(\mathcal{B}^v)^{A(v)} \to H_{2e(v)}(\mathcal{B}), \quad e(v) = \dim_C \mathcal{B}^v,
\]

is an isomorphism onto its image.

**Proof.** This is because the representation of \( W \) on \( H_{2e(v)}(\mathcal{B}^v)^{A(v)} \) is irreducible, as we shall see in (4.1).

### 4. Springer Theory

We shall now apply the above construction with \( G_0 \) replaced by \( G \cong G_0 \times G_0 \). \( \mathcal{B} \) will again denote the flag manifold of \( g \), \( \mathcal{B}_0 \) that of \( G_0 \), so that \( \mathcal{B} = \mathcal{B}_0 \times \mathcal{B}_0 \). \( \mathcal{N} \cong \mathcal{N}_0 \times \mathcal{N}_0 \) is the nilpotent cone in \( g^* \), \( \mathcal{N}_0 \) that in \( g_0^* \).

We again set \( z_1 = b_0 \times b_0 \) and \( z_n = w^{-1}z_1 \). For \( V \) we now take

\[
\mathcal{N} \cap k^+ = \{ (v, -v) \mid v \in \mathcal{N}_0 \} \cong \mathcal{N}_0.
\]

Then \( \mathcal{B}^*(V) \) becomes

\[
\mathcal{L} = \{ (b, b'; v, -v) \mid v \in b^+ \cap b'^+ \}
\]

the conormal variety of the \( K \)-orbits on \( \mathcal{B} \). Our construction therefore gives a representation of \( W \cong W_0 \times W_0 \) on \( H_*(\mathcal{L}, Z) \).

We may identify \( \mathcal{N}_0 \) with the subset \( \mathcal{N} \cap k^+ \) of \( \mathcal{N} \). For a subset \( V \) of \( \mathcal{N}_0 \) we write \( \mathcal{L}(V) \) for \( \mathcal{B}^*(V) \). When \( V \) is a closed subvariety of \( \mathcal{N} \), then \( H_{2n}(\mathcal{L}(V)) \) is naturally a \( W \)-submodule of \( H_{2n}(\mathcal{L}) \), by (2.9). When \( V \) is only constructible then \( H_{2n}(\mathcal{L}(V)) \) is naturally a \( W \)-subquotient,

\[
H_{2n}(Z(V)) \approx H_{2n}(\mathcal{L}(\bar{V}))/H_{2n}(\mathcal{L}(\partial V)),
\]

where \( \bar{V} \) is the closure of \( V \) and \( \partial V \) is the topological boundary of \( V \).

The decomposition of \( \mathcal{N}_0 \) into \( G_0 \)-orbits \( \mathcal{C} \) (equivalently: the decomposition of \( \mathcal{N} \cap k^+ \) into \( K \)-orbits) leads to a filtration of \( H_{2n}(\mathcal{L}) \) according to the closure relations among the orbits.

\[
\mathcal{C}' \subset \mathcal{C} \implies H_{2n}(\mathcal{L}(\bar{C})) \subset H_{2n}(\mathcal{L}(\bar{C}')).
\]
The subquotients of this filtration are

\[ H_{2n}(\mathcal{L}(C)) = H_{2n}(\mathcal{L}(\bar{C})) \bigg/ \sum_{\bar{C}' < C} H_{2n}(\mathcal{L}(\bar{C}')), \]

where \( \bar{C}' < C \) means \( \bar{C}' \supseteq \bar{C} \). The associated graded group is

\[ \text{gr } H_{2n}(\mathcal{L}) \approx \bigoplus H_{2n}(\mathcal{L}(C)) \tag{1} \]

sum over all \( G_0 \)-orbits on \( \mathcal{L} \). Equation (1) is a \( W \)-decomposition; but one should keep in mind that the \( H_{2n}(\mathcal{L}(C)) \) are naturally \( W \)-subquotients of \( H_{2n}(\mathcal{L}) \), not \( W \)-submodules. This in spite of the fact that \( H_{2n}(\mathcal{L}(C)) \) may naturally be realized as a subgroup of \( H_{2n}(\mathcal{L}) \): it has as a basis the fundamental cycles of the components of \( \mathcal{L}(C) \) and is therefore isomorphic to the subgroup of \( H_{2n}(\mathcal{L}) \) spanned by the fundamental cycles of the closures in \( \mathcal{L} \) of these components. But the latter subgroup of \( H_{2n}(\mathcal{L}) \) is generally not \( W \)-stable. It should also be noted that according to Steinberg [26] each \( \mathcal{L}(C) \) has uniform dimension \( \dim \mathcal{L} = n \), so its components are certain \( \mathcal{L}_n \)'s. More precisely, the components of \( \mathcal{L}(C) \) are dense parts of those \( \mathcal{L}_n \) for which \( C \) intersects the fibre \( \mathcal{L}_n \cap k^+ \) of \( \mathcal{L} \to \mathcal{B} \) over \( \mathcal{L}_n \) densely. We denote them \( \mathcal{L}_n(C) := \mathcal{L}(C) \cap \mathcal{L}_n \).

From the fibration \( \mathcal{L}(C) \to C \) one obtains that

\[ H_{2n}(\mathcal{L}(C)) \approx H_{2e(v)}(B^v)^{A_0(v)}, \tag{2} \]

where \( v \in C, \ e(v) = \dim_C B^v - 2e_0(v), \ e_0(v) = \dim_C B_0 \), and \( A_0(v) \) is the component group of the stabilizer of \( v \) in \( G_0 \), a quotient of the fundamental group of \( C \). \( H_{2e(v)}(B^v)^{A_0(v)} \) denotes the \( A_0(v) \)-invariants in \( H_{2e(v)}(B^v) \).

The isomorphism (2) is explicitly seen as follows. As just mentioned, \( H_{2n}(\mathcal{L}(C)) \) has a basis consisting of the fundamental cycles of those \( \mathcal{L}_n \) of \( \mathcal{L} \) which make up \( \mathcal{L}(C) \). On such a \( \mathcal{L}_n \) the fibration \( \mathcal{L}(C) \to C \) restricts to a fibration \( \mathcal{L}_n(C) \to C \) whose fibre over \( v \) is exactly an \( A_0(v) \)-orbit of components of \( B^v = B^v_0 \times B^v_0 \) (according to Steinberg [26]) and these \( A_0(v) \)-orbits of components of \( B^v \) form a basis of \( H_{2e(v)}(B^v)^{A_0(v)} \).

The action of the component group \( A(v) = A_0(v) \times A_0(v) \) of the \( G \)-stabilizer of \( v \) on \( H_{2e(v)}(B^v) = H_{2e_0(v)}(B_0^v) \otimes H_{2e_0(v)}(B_0^v) \) commutes with the action of \( W = W_0 \times W_0 \), as we know, and the invariants of the diagonal \( A_0(v) \) in \( A_0(v) \times A_0(v) \) decompose as

\[ \sum_\phi \chi_{v,\phi} \otimes \chi_{v,\phi}, \]

where \( \phi \) runs over the irreducible characters of \( A_0(v) \) which occur in \( H_{2e_0(v)}(B_0^v) \) and \( \chi_{v,\phi} \) is the character of \( W_0 \) on the subspace of \( H_{2e_0(v)}(B_0^v) \).
which transforms according to $\phi$. (A priori one might have to extend scalars to $\mathbb{C}$ for this decomposition, but it follows from the known structure of the $A_0(v)$ that it suffices to work over $\mathbb{Q}$.) If one knew that

the representation of $W = W_0 \times W_0$ on $H_{2n}(\mathcal{F}, \mathbb{C})$ is the biregular representation on $\mathbb{C}[W/W_0] = \mathbb{C}[W_0]$. \hfill (3)

which is the main result of the Kazhdan–Lusztig paper [19], this argument (which they attribute to Springer) would prove

**Springer's Theorem 4.1.** The $\gamma_{\cdot, \epsilon}$ are exactly the irreducible characters of $W_0$.

In the present context the missing link (3) will be supplied by the Integral Formula, as we shall now show.

### 5. Coherent Families of Eigendistributions

To bring in the Integral Formula we need to pass from the conormal variety $\mathcal{Z}$ of the $K$-action to the conormal variety $\mathcal{F}$ of the $G_0$-action by means of the involution $i: (x, y) \mapsto (x, \tilde{y})$ of $g = g_0 \times g_0$ which interchanges $g_0 = \{(x, x)\}$ and $k = \{(x, x)\}$. Use this automorphism $i$ to transfer the representation of $W$ on $H_\bullet(\mathcal{F})$ constructed above from $\mathcal{Z}$ to $\mathcal{F}$. It is still induced by the transformations $a_{i, \epsilon}(w) = p_{\omega, \lambda}(w)^{-1} \cdot p_\lambda$ with $p_\lambda: \mathfrak{g}^* \to \mathcal{O}_\lambda, u \cdot (b_1, v) \mapsto u \cdot (\lambda + v)$, except that for $\mathcal{F}$ we need to take $b_1 = s_1 = b_0 \times b_0$ as base-point in the definition of $p_\lambda$ (because $\phi_{\lambda, \epsilon}(w) = a_{\epsilon, \lambda}(w) \sim a_{\epsilon, \lambda}(w)$).

Recall the Integral Formula: for $\Gamma \in H_{2n}(\mathcal{F}, \mathbb{C})$, and regular $\lambda \in \mathfrak{h}^*$,

$$ \int_{\rho, \Gamma} e^{i\epsilon \cdot \mu} = \frac{1}{\pi(x)} \sum_{\gamma \in W} m_{\gamma} e^{i\gamma \cdot \epsilon}. \hfill (1) $$

We also know that for $\Gamma = \mathcal{F}_w$, $m_{\gamma}$ is up to a sign the local Euler number of $\mathcal{F}_w$ at $s_w$,

$$ m_{\gamma} = (-1)^{m_0 + (\tilde{\eta}(w) - \tilde{\eta})} \text{Eu}_\gamma(S_w) = (-1)^{m_0 + (\tilde{\eta}(w) - \tilde{\eta})} \text{Eu}_\gamma(Z_w) \quad (\text{for } \Gamma = \mathcal{F}_w). $$

Equation (1) is interpreted as an invariant eigendistribution on $g_0$, which we denoted $\theta_{\Gamma}(\lambda)$:

$$ \theta_{\Gamma}(\lambda) = \frac{1}{(2\pi i)^n} \int_{\rho, \Gamma} e^{i\lambda \cdot \mu}. $$
The right side of (1) may be written as
\[ \theta_f(\lambda) = \frac{1}{\pi} \sum_{\gamma \in W/W_0} m_{\gamma} \varphi_{\gamma}(\lambda), \]
where
\[ \varphi_{\gamma}(\lambda) = \frac{1}{\pi} \sum_{w \in W_0} e^{(\pi i w)^{-1} \lambda}. \tag{2} \]

Any family \( \theta(\lambda) \) of \( G_0 \)-invariant eigendistributions depending continuously on \( \lambda \in h^* \) which for regular \( \lambda \) is given by a formula
\[ \theta(\lambda) = \frac{1}{\pi} \sum_{\gamma \in W} m_{\gamma} e^{\pi i \gamma^{-1} \lambda}. \tag{3} \]
with \( m_{\gamma z} = m_{\gamma} \) for \( z \in W_0 \), or equivalently
\[ \theta(\lambda) = \frac{1}{\pi} \sum_{\gamma \in W/W_0} m_{\gamma} \varphi_{\gamma}(\lambda). \tag{4} \]
will be referred to as a coherent family of invariant eigendistributions on \( g_0^* \). It is evident that a coherent family \( \theta \) is determined by its value \( \theta(\lambda) \) at a single regular \( \lambda \). Furthermore, any \( \theta(\lambda) \) given by (3) or (4) for a single regular \( \lambda \) extends uniquely to a coherent family: to see this it suffices to know that the \( \varphi_{\gamma}(\lambda) \) extend, and that will become clear shortly. We shall denote the space of coherent families of invariant eigendistributions on \( g_0 \) by \( \text{CH}(g_0, \mathbb{C}) \) or \( \text{CH}(g_0, \mathbb{Z}) \) depending on whether we use complex or integral coefficients \( m_{\gamma} \) in (3) or (4). The \( \mathbb{C} \) or \( \mathbb{Z} \) will be omitted when understood or unimportant.

The Weyl group \( W = W_0 \times W_0 \) operates on the \( \theta \) in the obvious way,
\[ (w \cdot \theta)(\lambda) = \theta(w^{-1} \lambda), \]
and the resulting representation of \( W \) on \( \text{CH}(g_0, \mathbb{C}) \) is evidently the biregular representation on \( \mathbb{Z}[W/W_0] \approx \mathbb{Z}[W_0] \), with the \( \varphi_{w, w} \), \( w \in W/W_0 \), corresponding to the basis elements \( wW_0 \) of \( \mathbb{Z}[W/W_0] \). On the other hand, we have the representation of \( W \) on \( H_{2n}(\mathcal{S}, \mathbb{Z}) \) constructed above.

**Theorem 5.1.** The map
\[ H_{2n}(\mathcal{S}, \mathbb{Z}) \to \text{CH}(g_0, \mathbb{Z}), \quad \Gamma \to \theta_f, \]
given by the Integral Formula is a \( W \)-isomorphism.
More precisely, the homology classes \( w \cdot \mathcal{I}_1 \), \( w \in W/W_0 \), form a basis for \( H_{2n}(\mathcal{I}, \mathbb{Z}) \). They correspond to the basis \((-1)^{\phi_w} \varphi_w \), \( w \in W/W_0 \), of \( \text{CH}(g_0, \mathbb{Z}) \) under the bijection \( \Gamma \to \theta_{\varphi} \).

Proof. That the map \( H_{2n}(\mathcal{I}) \to \text{CH}(g_0) \) is a bijection we know from the Integral Formula: its matrix \((-1)^{\phi_w + \ell(w) - 1} \text{Eu}_w(\mathcal{I}_w)\) with respect to the bases \( \mathcal{I}_w \) of \( H_{2n}(\mathcal{I}) \) and \( \varphi_w \) of \( \text{CH}(g_0) \), is integral and unipotent-triangular with respect to the Bruhat order on \( W/W_0 \).

It suffices to prove the second assertion which says explicitly that for regular \( \lambda \in h^* \),

\[
\frac{1}{(2\pi i)^{n}} \int_{p_{\lambda} \omega s_{1}} e^{\nu_{\lambda} - \sigma_{\lambda}} = \frac{(-1)^{\phi_\lambda}}{\pi(x)} \sum_{y \in W_0} e^{(\nu y)^{-1} \lambda(y)}. \tag{5}
\]

For \( w = 1 \) this is a special case of the Integral Formula: the \( K \)-orbit \( \mathcal{I}_1 = K \cdot z_1 \) is smooth, so \( \text{Eu}_w(z_1) = 1 \) or 0, according as \( y \in W_0 \) or not. To see that (5) holds for all \( w \in W \) we only need to identify the \( p_{\lambda}w \mathcal{I}_1 \). The cycle \( p_{\lambda} \mathcal{I}_1 \) is the image under the map

\[ p_{\lambda} : \mathcal{B} \to \Omega_\lambda , \quad u \cdot (b_1, v) \to u \cdot (\lambda + v) \]

of the conormal bundle of the closed \( G_0 \)-orbit \( G_0 \cdot s_1 \), where \( s_1 = b_0 \times b_0 \). That conormal bundle is

\[ G_0 \cdot \{(s_1, v) \mid v \in s_1 \cap ig_0^+ \}. \]

Since \( G_0 = K_0 B_0 \) and \( b_0 \subset s_1 \) this conormal is also

\[ = K_0 \cdot \{(b_1, v) \mid v \in ib_0^\perp \}, \]

where \( b_0^\perp \subset g_0^* \) is the orthogonal of \( b_0 \) in \( g_0^* \) (not in \( g^* \)). Its image under \( p_{\lambda} \) becomes \( K_0 \cdot \{ \lambda + ib_0^\perp \} \).

From the definition of the action of \( W \) on \( H_{2n}(\mathcal{I}) \) one finds that the cycle \( p_{\lambda}w \cdot \mathcal{I}_1 \) on \( \Omega_\lambda \) which figures in (5) is properly homotopic to \( K_0 \cdot \{ w^{-1}\lambda + ib_0^\perp \} \),

\[ p_{\lambda}w \cdot \mathcal{I}_1 \sim K_0 \cdot \{ w^{-1}\lambda + ib_0^\perp \} \quad \text{on} \ \Omega_\lambda . \]

taken with the appropriate orientation. So (5) says

\[
\frac{1}{(2\pi i)^{n}} \int_{K_0 \cdot \{ w^{-1}\lambda + ib_0^\perp \}} e^{\nu_{\lambda} - \sigma_{\lambda}} = \frac{(-1)^{\phi_\lambda}}{\pi(x)} \sum_{y \in W_0} e^{(\nu y)^{-1} \lambda(y)}. \tag{6}
\]

We know that this formula holds for \( w = 1 \) (and regular \( \lambda \)). Replacing \( \lambda \) by \( w \lambda \) one sees that it holds for all \( w \in W \). Furthermore, writing the integral
as a double integral, first over $b^+_0$, then over $K_0$, one sees that (6) exists for all $\lambda \in h^*$ (regular or not). This means that

$$\varphi_n(\lambda) \text{ is entire analytic in } \lambda \in h^*$$

as promised above. This finishes the proof of the theorem.

We record explicitly the special case when $\Gamma = \mathcal{F}_n$:

**Corollary 5.2.**

$$\mathcal{F}_n = \sum_{y \in W/W_0} (-1)^{l(w) - l(y)} \mathcal{E}_{\lambda} (S_w) \cdot \mathcal{F}_{1}.$$ or equivalently

$$\mathcal{F}_n = \sum_{y \in W/W_0} (-1)^{l(w) - l(y)} \mathcal{E}_{\lambda} (Z_w) \cdot \mathcal{F}_{1}.$$  

6. **Characters and Characteristic Cycles**

As a first and immediate application of Theorem 5.1 we derive a formula for the global character of a $(g, K)$-module as a contour integral over the characteristic cycle of the corresponding $(D, K)$-module.

We start with some general remarks about global characters. A $(g, K)$-module $M$ with infinitesimal character $\lambda \in h$ is the Harish-Chandra module of an admissible representation of $G_0$, whose global character is an invariant eigendistribution $\Theta = \text{ch}(M)$ on $G_0$. It follows from results of Harish-Chandra [9] that one has an identity of distributions in a neighbourhood of 0 in $g_0$,

$$\Theta(\exp x) = f(x)^{-1} \theta(x),$$

where $\theta$ is an invariant eigendistribution on $g_0$ with infinitesimal character $\lambda$ and

$$f(x) = \det^{1/2} \left( e^{ax/2} - e^{-ax/2} \right) = A(x) / n(x) \quad \text{on } h_0.$$  

Here $n = n_{\text{std}} + \delta$ as before, and

$$A = \prod_{x \in \Delta_+} (e^{x/2} - e^{-x/2})$$

is the Weyl denominator. The products are over the roots $x$ of $h_0$ in $h_0^\perp$. If
\( \lambda \in h^* \) is regular, as we shall now assume, the Integral Formula allows us to write \( \theta = \theta_\lambda(\lambda) \) for a unique \( \Gamma \in H_{2n}(\mathcal{L}) \). We shall call this homology class \( \Gamma \), or any \( 2n \)-cycle representing it, the character contour of the \((g, K)\)-module \( M \), and denote it \( \mathcal{C}(M) \). It should be noted that \( \mathcal{C}(M) \) depends on the choice of the element \( \lambda \) in the \( W \)-orbit in \( h^* \) determined by the infinitesimal character of \( M \) and on the Borel subalgebra \( s_1 \) used to define the map \( p_s \).

Recall the Beilinson-Bernstein [5] correspondence between \((g, K)\)-modules and \((\mathcal{D}, K)\)-modules. We consider only the case of regular integral infinitesimal character; one may as well assume (as we now do) that the infinitesimal character is represented by \(-\rho\), where \( \rho = (\rho_0, \rho_0) \) is half the sum of the roots of \( h \) in \( z_i = b_0 \times h_0 \).

The Beilinson-Bernstein correspondence based on the data \((z_1, -\rho)\) associates to each \((g, K)\)-module \( M \) with infinitesimal character \(-\rho\) the \((\mathcal{D}, K)\)-module \( \mathcal{M} = \mathcal{D} \otimes M \). (\( \mathcal{D} \) is the sheaf of differential operators on \( \mathcal{B} \) as in [5]; the tensor product is over \( U(g) \) as explained there.) This correspondence is an equivalence of categories.

For each \( K \)-orbit \( \mathcal{Z}_s = K \cdot y^{-1}z_1 \) there is an induced \( \mathcal{D} \)-module which we shall denote \( \mathcal{I}_s \). Write \( I_s \) for the corresponding \((g, K)\)-module. It is known that \( I_s \) is an induced (principal series) \((g, K)\)-module. The corresponding representation of \( G_0 \) has global character

\[
\frac{1}{\mathfrak{A}} \sum_{w \in W} e^{-(yw)^{-1}\rho}.
\]

(This is a special case of the relation between two of the three classifications of \((g, K)\)-modules, of Hecht et al. [10].)

Under the correspondence (1) between invariant eigendistributions on \( G_0 \) and on \( g_0 \) the global character corresponds to the distribution

\[
\frac{1}{\pi} \sum_{w \in W} e^{-(yw)^{-1}\rho}.
\]  

(2)

In analogy with the data \((z_1, -\rho)\) for the Beilinson-Bernstein correspondence it seems natural to base the character contours on the data \((s_1, -\sigma)\) which correspond to \((z_1, -\rho)\) under the automorphism \( \iota \) interchanging \( k \) and \( g_0 : s_1 = b_0 \times b_0 \) and \( \sigma = (\rho_0, -\rho_0) \). We therefore write (2) as

\[
\frac{1}{\pi} \sum_{w \in W} e^{-(w, y)\sigma^{-1}} = \varphi_{w_1}(\sigma),
\]

(3)

where \( w_1 \in W \) is the Weyl group element with \( w, \rho = \sigma \).
It follows from Theorem 5.1 that the character contour of the \((g, K)\)-module \(I_\varepsilon\) (based on the data \((s, \sigma)\)) is

\[
\mathcal{C}(I_\varepsilon) = w_\varepsilon \cdot \mathcal{F}_1.
\]  

On the other hand, to a \((D, K)\)-module \(M\) one associate a characteristic cycle; this is an algebraic cycle of complex dimension \(n\) on \(\mathfrak{B}^*\) which is known to lie on the conormal variety \(\mathcal{I}\) of the \(K\)-action on \(\mathfrak{B}\) for these modules \(M\). It determines therefore a homology class in \(H_{2n}(\mathcal{I})\), denoted \(\text{Ch}(M)\). By a result of Tanisaki [27] (or by the corresponding result of Kashiwara–Tanisaki [18] for \((D, B)\)-modules, which amounts to essentially the same thing when \(g_0\) itself is complex, as here) the characteristic cycle of the induced \(D\)-module \(\mathcal{F}_1\) is

\[
\text{Ch}(\mathcal{F}_1) = y \cdot \mathcal{F}_1.
\]  

Since the \(\mathcal{F}_\xi\) and \(I_\varepsilon\) form bases for the respective \(K\)-groups one finds by comparing (4) and (5):

**Theorem 6.1.** The character contour \(\mathcal{C}(M)\) of a \((g, K)\)-module \(M\) and the characteristic variety \(\text{Ch}(M)\) of the corresponding \(D\)-module \(M\) are related by

\[
\text{Ch}(M) = w_\varepsilon \cdot \mathcal{C}(M).
\]

In this equation \(\iota\) denotes the map from the conormal variety of \(K\)-action of \(\mathfrak{B}\) to the conormal variety of the \(G_0\)-action induced by the involution \(\iota\) of \(g\) which interchanges \(k\) and \(g_0\) and induces the \(W\)-isomorphism of the homology groups of the conormal varieties. It is further understood that the data \((z, \rho)\) entering into the Beilinson–Bernstein corresponding and the data \((s, \sigma)\) entering into the definition of character contours are related by this automorphism \(\iota\). Explicitly, (6) says that the global character \(\text{ch}(M)\) of \(M\) is given by the formula

\[
\text{ch}(M) = f(x)^{-1} \int_{w_\varepsilon^{-1} \text{Ch}(M)} e^{w_{\varepsilon^{-1}} \cdot \tau_{w_{\varepsilon^{-1}}} \cdot x}.
\]

7. ASYMPTOTICS AT ZERO

Recall the \(W\)-filtration of \(\mathcal{I}\) by the inverse images \(\mathcal{I}(\mathcal{C})\) of nilpotent \(K\)-orbits \(\mathcal{C}\) on \(N \cap k^+\), which gave rise to the decomposition (4.1) of \(\text{gr} H_*(\mathcal{I})\). Passing from \(\mathcal{I}\) to \(\mathcal{F}\), the analogous filtration of \(\mathcal{F}\) by the inverse images \(\mathcal{F}(\mathcal{C})\) of \(G_0\)-orbits on \(N \cap i g_0^+\) gives a filtration of \(H_{2n}(\mathcal{F})\)
by the subgroups $H_{2n}(\mathcal{F}(\mathcal{O}))$ according to the closure relation among the $\mathcal{O}$'s leading to the decomposition

$$\text{gr } H_{2n}(\mathcal{F}) \approx \sum \text{gr } H_{2n}(\mathcal{F}(\mathcal{O})).$$

(1)

As in (4.2),

$$H_{2n}(\mathcal{F}(\mathcal{O})) \approx H_{2n}(\mathcal{G}(\mathcal{B}^r))^{A_{1(r)}}$$

(2)

and the inclusion $\mathcal{B}^r \subseteq \mathcal{B}$ induces a $W$-injection

$$H_{2n}(\mathcal{G}(\mathcal{B}^r))^{A_{1(r)}} \subseteq H_{2n}(\mathcal{B}).$$

(3)

(Note that there is $A_{0}(v)$ in (2), $A_{0}(v) \times A_{0}(v)$ in (3).) We choose the same base point $h_1 = s_1$ to define the $W$-action on $H_{2n}(\mathcal{G}(\mathcal{B}^r))^{A_{1(r)}}$ and on $H_{2n}(\mathcal{B})$.

We recall Borel's description of the cohomology ring of $\mathcal{B}$ [6]. For $\lambda \in h^*$ denote $\tau_\lambda$ the $U$-invariant two-form on $\mathcal{B}$ which at the base point $s_1$ is given by

$$\tau_\lambda(x \cdot s_1, y \cdot s_1) = \lambda([x, y]) \text{ for } x, y \in u.$$  

(4)

Set

$$\omega_\lambda = -\frac{1}{2\pi i} \tau_\lambda.$$  

The map $\lambda \mapsto \omega_\lambda$ extends to a map $f \mapsto \omega_f$ from the ring $C[h]$ of polynomial functions on $h$ to the algebra of differential forms on $\mathcal{B}$. It annihilates the $W$-invariants with zero constant term in $C[h]$, denoted $I^+$, and induces an isomorphism

$$C[h]/I^+ \cong H^*(\mathcal{B}), \quad [f] \mapsto [\omega_f],$$

where $[f]$ is the class of $f$, $[\omega_f]$ the class of $\omega_f$ (in de Rahm cohomology).

The transpose of (5) is an isomorphism

$$H_\mathcal{B}(\mathcal{B}, C) \cong \mathcal{H}(h^*), \quad \gamma \mapsto c_\gamma,$$

(6)

where $\mathcal{H}(h^*)$ consists of the $W$-harmonic polynomials on $h^*$, i.e., the polynomials annihilated by the $W$-invariant constant coefficient operators without constant term. Explicitly, the isomorphism (6) is defined by

$$\int c_\gamma = \langle c_\gamma, f \rangle.$$  

(7)
The right side is the natural pairing $\langle x^i, \lambda^j \rangle [i!] = \langle x, \lambda \rangle^j \delta_{ij}$. Observe that $c_\gamma$ may also be defined directly by the simple equation

$$c_\gamma(\lambda) = \int x e^{x \lambda}.$$  

(8)

This one sees by taking $f = e^{x \lambda}$ in (7) using $\langle c, e^{x \lambda} \rangle = c(\lambda)$. Composing (3) and (6) gives a map

$$H_{2e(v)}(\mathcal{B}^*)^{A(v)} \to \mathscr{H}_{e(v)}(h^*),$$  

(9)

where $\mathscr{H}_{e(v)}(h^*)$ denotes the homogeneous polynomials of degree $e(v)$ in $\mathcal{H}(h^*)$. Using further (2) we obtain a map

$$H_{2n}(\mathcal{P}(\overline{\mathcal{O}})) \to \mathscr{H}_{e(v)}(h^*), \quad \Gamma \to c_\Gamma.$$  

(10)

This last map (10) is of course not an isomorphism: it factors through the projection of $H_{2n}(\mathcal{P}(\overline{\mathcal{O}}))$ onto $H_{2n}(\mathcal{P}(\mathcal{O}))$ as well as through the projection of $H_{2e(v)}(\mathcal{B}^*)^{A_0(v)}$ onto $H_{2e(v)}(\mathcal{B}^*)^{A(v)}$.

THEOREM 7.1. For any nilpotent $G_0$-orbit $\mathcal{O}$ in $\mathfrak{g}_0^*$ and any $2n$-cycle $\Gamma \in H_{2n}(\mathcal{P}(\overline{\mathcal{O}}))$ over $\overline{\mathcal{O}}$,

$$\frac{1}{(2\pi i)^n} \int_{p,r} e^{v \cdot x \cdot \sigma_x} = c_{\Gamma}(\lambda) \frac{1}{(2\pi i)^d} \int e^{v \cdot x \cdot \sigma_x} + o(|\lambda|^\epsilon).$$  

(11)

Explanation and Remarks. $d = d(\mathcal{O}) = \dim_{\mathcal{O}} \mathcal{O}$, $e = e(\mathcal{O}) = \dim_{\mathcal{O}} \mathcal{B}^*(\mathcal{O})$; $x_\epsilon$ is $x$ considered as a function on $\mathcal{O}$; $\sigma_\epsilon$ is the canonical two-form on $\mathcal{O}$,

$$\sigma_\epsilon(x \cdot v, y \cdot v) = \langle v, [x, y] \rangle.$$  

Equation (11) is understood as an identity of distributions on $\mathfrak{g}_0$, as usual. It will be abbreviated to

$$\theta_{\Gamma}(\lambda) = c_{\Gamma}(\lambda) \theta_{\epsilon} + o(|\lambda|^\epsilon).$$  

(12)

with

$$\theta_{\Gamma}(\lambda) = \frac{1}{(2\pi i)^n} \int_{p,r} e^{v \cdot x \cdot \sigma_x},$$  

$$\theta_{\epsilon} = \frac{1}{(2\pi i)^d} \int e^{x \cdot \epsilon \cdot \sigma_\epsilon}.$$
Equation (11) can also be written as an asymptotic relation as \( x \to 0 \) in \( g_0 \) in the sense of Barbasch and Vogan [2],

\[
\theta_r(\lambda) \sim c_r(\lambda) \theta_c \quad \text{as} \quad x \to 0 \quad \text{in} \quad g_0.
\]  

(13)

This means that \( c_r(\lambda) \theta_c \) is the leading term in the asymptotic expansion of \( \theta_r(\lambda) \) at \( x=0 \) described in [2, Theorem 1.3].

**Proof of the Theorem.** Fix the \( G_0 \)-orbit \( \mathcal{C} \) on \( \mathcal{N} \cap ig_0^* \) and \( \Gamma \in H_{2n}(\mathcal{H}(\mathcal{C})) \). For \( f \in C^\infty_c(g) \), set

\[
\varphi(\xi) = \int_{g} e^{i(\xi)f}(x) \, dx.
\]

The theorem says that for all such \( f \)

\[
\frac{(-1)^n}{(2\pi i)^n n!} \int_{\mu, \Gamma} \phi^{\xi}_\mu = c_r(\lambda) \frac{(-1)^d}{(2\pi i)^d d!} \int_{\mu} \phi^d_\mu + o(|\lambda|^n). \quad (14)
\]

To determine the asymptotic behaviour of the integral on the left we need some general remarks on symplectic structure. Let \( \mathcal{L} \) be an orbit of the complex group \( G \) on the nilpotent cone \( \mathcal{N} \) in \( g^* \). As before, put

\[
\mathcal{G}^*(\mathcal{L}) = \{(h, v) | v \in \mathcal{G} \}.
\]

Then there are injections

\[
\mathcal{B}^*(\mathcal{L}) \subseteq \mathcal{B} \times \mathcal{L} \quad (15)
\]

\[
\mathcal{G}^*(\mathcal{L}) \subseteq \Omega_\lambda \quad (16)
\]

the first map being the natural inclusion, the second one the restriction of the bijection

\[
p_\lambda : \mathcal{B}^* \to \Omega_\lambda, \quad u \cdot (s_1, v) \to u \cdot (\lambda + v)
\]

\((u \in U, v \in s_1^\perp)\). In view of (15) and (16) we can restrict to \( \mathcal{B}^*(\mathcal{L}) \) the two-form \( \tau_\lambda + \sigma_\lambda \) on \( \mathcal{B} \times \mathcal{L} \) and the two-form \( \sigma_\lambda \) on \( \Omega_\lambda \).

**Lemma 7.2.** The two-forms \( \tau_\lambda + \sigma_\lambda \) and \( \sigma_\lambda \) agree on \( \mathcal{B}^*(\mathcal{L}) \).

**Proof of the Lemma.** Because of \( U \)-invariance it suffices to show that the forms agree at a point \((s_1, v)\) with \( v \subset s_1^\perp \cap \mathcal{L} \). Let

\[
u(t) \cdot (s_1, v(t) \cdot v)
\]

(17)
be a smooth curve on $B^*(Z)$ with $u(t) \in U$, $v(t) \in G$, $v(t) \cdot v \in s^1$, $u(0) = 1$, $v(0) = 1$. Its tangent vector at $t = 0$ is

$$(u' \cdot s_1, (u' + v') \cdot v)$$ (18)

$u'$, $v' \in g$ being the tangent vectors of $u(t)$ and $v(t)$ at $t = 0$. The two-form $\tau_z + \sigma_{\bar{e}}$ assigns to two such tangent vectors at $(s_1, v)$ the value

$$\langle \lambda, [u', u''] \rangle + \langle v, [u', u''] \rangle + \langle v, [u', v''] \rangle$$

$$+ \langle v, [v', u''] \rangle + \langle v, [v', v''] \rangle.$$ (19)

The curve on $\Omega_2$ corresponding to (17) under the injection (16) is

$$u(t) \cdot \lambda + u(t) \cdot v(t) \cdot v$$

and its tangent vector at $\lambda + v$, corresponding to the vector (18) at $(s_1, v)$, is

$$u' \cdot (\lambda + v) + v' \cdot v.$$ The two-form $\sigma_{\bar{e}}$ assigns to two such tangent vectors at $\lambda + v$ the value

$$\langle \lambda, [u', u''] \rangle + \langle v, [u', u''] \rangle + \langle v, [u', v''] \rangle$$

$$+ \langle v, [v', u''] \rangle + \langle \lambda + v, [v', v''] \rangle.$$ (20)

The last term of (20) is 0, as $\lambda + s^1 \subset \Omega_2$ is an isotropic (in fact Lagrangian) submanifold of $\Omega_2$, which is clear. The same is true of the last term of (19), but this is less clear: it is known that, for any $b$, $b^\perp \cap Z$ is a co-isotropic subvariety of $Z$ [17, Lemmas 7.5 and 9.6(i)] of dimension $= \frac{1}{2} \dim Z$ (as follows from [23; 26, Section 4]), hence in fact Lagrangian. Thus (19) agrees with (20), proving the lemma.

Remark. The fact that $b^\perp \cap Z$ is Lagrangian on $Z$ is mentioned in [8, Proposition 4.3]. The proof indicated there uses general facts from symplectic geometry.

We return to the proof of the theorem. Apply the above lemma to the $G$-orbit $Q$ containing the $G_0$-orbit $C$. The fibration

$$B^r \subseteq B^*(Z) \to Z$$ (21)

restricts to

$$B^r \subseteq \mathcal{P}(C) \to C.$$ (22)
For $\Gamma$ it suffices to take a component of $\mathcal{S}(\mathcal{C})$. As explained for $\mathcal{S}(\mathcal{C})$ in connection with (4.2), a component of $\mathcal{S}(\mathcal{C})$ is a dense part of an $\mathcal{S}_w$, denoted $\mathcal{S}_w(\mathcal{C})$. The fibration (22) restricts to a fibration

$$
\mathcal{B}^v_w \subseteq \mathcal{S}_w(\mathcal{C}) \rightarrow \mathcal{C}.
$$

(23)

According to [26] the fibre $\mathcal{B}^v_w = \mathcal{S}_w(\mathcal{C}) \cap \mathcal{B}^v$ over $v \in \mathcal{C}$ in (23) is a single orbit of components of $\mathcal{B}^v$ under the component group $A_0(v)$ of the stabilizer of $v$ in $G_0$. These components all have the same dimension $e = e(\mathcal{C}) = \dim_c \mathcal{B}^v$, by a result of Spaltenstein [23]. The fundamental class of such a component in $H_{2e}(\mathcal{B})$ is independent of $v$ in $\mathcal{C}$, as $\mathcal{C}$ is connected. Note, incidentally, that

$$
d + e = n
$$

because of the fibration (14). Now calculate

$$
\frac{(-1)^n}{(2\pi i)^n n!} \int_{\mathcal{S}_w(\mathcal{C})} \varphi \sigma^n_{\lambda},
$$

$$
= \frac{(-1)^n}{(2\pi i)^n n!} \int_{\mathcal{S}_w(\mathcal{C})} (\varphi \circ p_\lambda)(\tau_{\lambda} + \sigma_\lambda)^n
$$

$$
= \frac{(-1)^n}{(2\pi i)^n d! e!} \int_{e} \left\{ \int_{\mathcal{S}_w(\mathcal{C})} \varphi(p_\lambda(b, v)) \tau_{\lambda}^n(db) \right\} \sigma_{\lambda}^d(dv) + o(|\lambda|^r),
$$

(24)

where we used the fibration (22) and Lemma 7.2 to write the integral over $\mathcal{S}_w(\mathcal{C})$ as an integral over the fibre $\mathcal{B}^v_w$ followed by an integral over the base $\mathcal{C}$.

In the integral (24) write $(b, v) = u \cdot (s, v_1)$ with $u \in U$ and $v_1 \in s_+^\perp$. Then the integrand becomes

$$
\varphi(p_\lambda(b, v)) = \varphi(u \cdot (\lambda + v_1))
$$

$$
= \varphi(u \cdot v_1) + o(|\lambda|)
$$

$$
= \varphi(v) + o(|\lambda|).
$$

Thus

$$
\text{Eq. (24)} = \frac{(-1)^n}{(2\pi i)^n d! e!} \int_{e} \varphi(v) \left\{ \int_{\mathcal{S}_w(\mathcal{C})} \tau_{\lambda}^n(db) \right\} \sigma_{\lambda}^d(dv) + o(|\lambda|^r)
$$

$$
= \frac{(-1)^d}{(2\pi i)^d d!} \int_{e} \varphi(v) \langle c^\perp, c^\perp \rangle \sigma_{\lambda}^d(dv) + o(|\lambda|^r)
$$

$$
= \frac{(-1)^d}{(2\pi i)^d d!} c(\lambda) \int_{e} \varphi(v) \sigma_{\lambda}^d(dv) + o(|\lambda|^r),
$$

(25)
where $c = c_w \in \mathcal{H}_*(h^*)$ is the harmonic polynomial on $h^*$ representing the fundamental class of $\mathcal{B}_w$ in $H_*(\mathcal{B})$ under Borel's isomorphism (6). As in (7), $\langle c, f \rangle$ is the natural pairing of polynomials on $h^*$ and formal power series on $h$, so that $\langle c, e^t \rangle = c(\lambda)$. This proves the theorem.

8. HARMONIC POLYNOMIALS

We study in some more detail the harmonic polynomials $c_\Gamma \in \mathcal{H}_*(h^*)$ associated to a $2n$-cycle $\Gamma \in H_{2n}(\mathcal{L}(\mathcal{O}))$ over $\mathcal{O}$ by the relation

$$\theta_\Gamma(\lambda) = c_\Gamma(\lambda) \theta_e + o(|\lambda|^e).$$

These polynomials, or variants thereof, have a history. When $\theta_\Gamma$ represents a coherent family of virtual characters of Harish-Chandra modules, then $c_\Gamma$ is the character polynomial studied by King [20] using results of Joseph [16] except that in King's definition the polynomials are not canonically normalized: there they are defined up to a constant factor which depends on an arbitrary regular element in the Cartan $h_0$. King shows that these polynomials are essentially the same as the polynomials introduced by Jantzen [15], Joseph [16], and Vogan [29]. The approach taken here is logically independent of (but strongly inspired by) these developments; it opens up some (from the previous point of view) surprising perspectives, among them the relation of the $c_\Gamma$ to the homology of the flag variety, and a formula for the $c_\Gamma$ in terms of Euler numbers.

Write

$$\Gamma = \sum_{y \in W/W_0} m_y y \cdot \mathcal{F}_y.$$  \hspace{1cm} (2)

Then

$$\theta_\Gamma(\lambda) = \frac{1}{\pi} \sum_{y \in W} m_y e^{-1} y \cdot \mathcal{F}_y.$$  \hspace{1cm} (3)

Expanding the exponential gives

$$\theta_\Gamma(\lambda) = \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{y \in W} m_y y^{-1} \lambda^k.$$  \hspace{1cm} (4)

Comparing (1) and (4) one finds that

$$\sum_{y \in W} m_y y^{-1} \lambda^k = 0 \quad \text{for} \quad k < e.$$  \hspace{1cm} (5)
and

$$\sum_{y \in W} m_y y^{-1} \pi - c(\lambda) p_c,$$

(6)

where

$$p_c = e^{\theta_c}.$$

(7)

From Eq. (7) it appears that $p_c$ is a $G_0$-invariant distribution on $g_0$ (the $W_0$-invariant polynomial $\pi$ on $h_0$ extends to a $G_0$-invariant polynomial on $g_0$), but from (6) one sees that on $h_0$

$$p_c$$

is a $W_0$-invariant polynomial on $h$,

homogeneous of degree $e = e(\mathcal{C})$.

(8)

(In this interpretation we used implicitly Harish-Chandra’s regularity theorem, which guarantees that $\theta_c$ is a locally integrable function on $g_0$.)

The relation (6) is an amazingly powerful tool for analyzing the polynomials $c_{\Gamma}$. First of all we obtain from (6) a formula for the $c_{\Gamma}$ for arbitrary $\Gamma \in H_{2n}(\mathcal{S}(\mathcal{C}))$:

If $\Gamma = \sum_{y \in W/W_0} m_y y \cdot \mathcal{S}_1$, then $c_{\Gamma} = \frac{1}{p_c(x)} \sum_{y \in W} m_y y \cdot x^e$

(9)

for any $x \in h$ with $p_c(x) \neq 0$. In particular, for $\Gamma = \mathcal{S}_\omega$ (any $\omega \in W/W_0$) we take for $\mathcal{C} = \mathcal{C}(\omega)$ the unique nilpotent $G_0$-orbit in $g^*_0$ which intersects $s^+ \cap ig^*_0$ densely (which amounts to $\mathcal{C}$ intersecting $h^+_0 \cap w^{-1} h^+_0$ densely if we think of $\mathcal{C}$ as a $G_0$-orbit on $g^*_0$ and represent $w \in W/W_0$ as $(1, w)$); then $\mathcal{S}_\omega \in H_{2n}(\mathcal{S}(\mathcal{C}(\omega)))$ and according to Corollary 5.2

$$\mathcal{S}_\omega = \sum_{y \in W/W_0} (-1)^{\ell(\mathcal{S}_\omega) - \ell(\mathcal{C})} \mathcal{E} u(S^{-1}_w) y \cdot \mathcal{S}_1.$$

(10)

$\mathcal{S}_\omega$ corresponds to the fundamental cycle $[\mathcal{B}_w^\mathcal{C}] \in H_{2n}(\mathcal{B})$ of the component $\mathcal{B}_w^\mathcal{C}$ of $\mathcal{B}^\mathcal{C}$ under the map

$$H_{2n}(\mathcal{S}(\mathcal{C})) \to H_{2n}(\mathcal{S}(\mathcal{C})) \approx H_{2n}(\mathcal{B}^\mathcal{C}) \to H_{2n}(\mathcal{B}).$$

From the definition of $c_{\Gamma}$ the corresponding $c_{\Gamma} = c_n$ is explicitly given by the formula

$$c_n(\lambda) = \int_{\mathbb{R}^n} e^{i\omega \cdot \lambda}.$$

(11)
On the other hand, the formula (9) says that

\[ c_n = \frac{1}{\varphi(x)} \sum_{\lambda \in W} (-1)^{\ell(\lambda)} \varphi(\lambda^\vee) \mathbb{E}_\lambda(S_n) \cdot y \cdot x^c \]  

(12)

for any \( x \in h \) with \( \varphi(x) \neq 0 \). Comparing (11) and (12) leads to the curious integral formula

\[ \int_{\mathfrak{h}} e^{\omega_{\mathfrak{h}}} = \frac{1}{\varphi(x)} \sum_{\lambda \in W} (-1)^{\ell(\lambda)} \mathbb{E}_\lambda(S_n) \langle \lambda, y \cdot x \rangle^c \]  

(13)

for any \( x \in h \) with \( \varphi(x) \neq 0 \).

The polynomials \( c_\lambda \) have some positivity and integrality properties which should be mentioned. When \( \lambda \in h^* \) is the \( s_\lambda \)-highest weight of a finite-dimensional representation \( V_\lambda \) of \( G \), the integral on the left of (13) can be thought of as follows.

Embed the flag-manifold \( \mathcal{B} \) of \( G \) into the projective space \( PV_\lambda \) of \( V_\lambda \) by

\[ i_\lambda : \mathcal{B} \to PV_\lambda, \quad h \mapsto [h \cdot \text{highest weight vector}] \]  

(14)

Then

\[ \omega_\lambda = i_\lambda^* \omega_{PV_\lambda}, \]

where \( \omega_{PV_\lambda} \) is the Kähler two-form of the Fubini-Study metric on \( PV_\lambda \). Thus for a complex subvariety \( V \) of \( \mathcal{B} \),

\[ \int_{\mathcal{B}} e^{\omega_\mathcal{B}} = \text{vol}_\lambda(V) = \frac{1}{(\dim V)!} \text{deg}_\lambda(V), \]  

(15)

where \( \text{vol}_\lambda(V) \) is the volume of \( V \) with respect to the metric on \( \mathcal{B} \) coming from the embedding (14) and \( \text{deg}_\lambda(V) \) is the degree of the projective variety \( i_\lambda(V) \) in \( PV_\lambda \). The formula (11) says that for \( \lambda \) regular, \( s_\lambda \)-positive, integral

\[ c_n(\lambda) = \text{vol}_\lambda(\mathcal{B}_n) = e^c \text{deg}_\lambda(\mathcal{B}_n). \]  

(16)

Thus

\[ e^c c_n(\lambda) \]  

is positive integral for regular, \( s_\lambda \)-positive, integral \( \lambda \). (17)

We want a more explicit formula for the \( W_0 \)-invariant polynomial \( \varphi_{\lambda} \). For that purpose we make a particular choice for \( \Gamma \in H_{2n}(\mathcal{I}(\mathcal{V})) \) as follows.

We know that the \( W \)-quotient \( H_{2n}(\mathcal{I}(\mathcal{V})) \) of \( H_{2n}(\mathcal{I}(\mathcal{V})) \) is \( \approx H_{2n}(\mathcal{V})^{A_0} \) and contains the \( A_0 \)-invariants \( H_{2n}(\mathcal{V})^{A_0} \) as an irreducible subrepresen-
tation (isomorphic with the space $\mathcal{H}_c(h^*)$ of $W$-harmonic polynomials on $h^*$). Let 

$$\chi_c = \text{character of } W_0 \text{ on } H_{2n}(\mathcal{B}_0^*)^{d(v)}.$$ 

(Note that we passed from the irreducible representation $H_{2n}(\mathcal{B}_0^*)^{d(v)}$ of $W = W_0 \times W_0$ to the corresponding irreducible representation $H_{2n}(\mathcal{B}_0^*)^{d(v)}$ of $W_0$.) It follows from these remarks that we can choose $\Gamma = \Gamma_c$ in $H_{2n}(\mathcal{P}(\mathcal{E}))$ so that

$$m_\chi = \frac{e_0}{|W|} \chi_c(y), \quad (18)$$

where

$$e_0 = \deg \chi_c = \dim \mathcal{B}_0,$$

and we identify $W/W_0 = W_0$ via $(y_1, y_2) \rightarrow y_1 y_2^{-1}$ to think of $\chi_c$ as a function on $W/W_0$ (or as a $W_0$-bi-invariant function on $W$). Explicitly

$$\chi_c(y) = \chi_c(y_1 y_2^{-1})$$

when we write $y \in W = W_0 \times W_0$ as $y = (y_1, y_2)$. With this choice of $m_\chi$, the element

$$\sum_{y \in W} m_\chi y^{-1} = \frac{e_0}{|W|} \sum_{y \in W} \chi_c(y) y^{-1} \quad (19)$$

of the group ring of $W$ operates as the projection on the $W_0$-invariants of type $\chi_c \otimes \chi_c$, which have dimension 1 in the irreducible $W$-module of this type.

Write out Eq. (6) with $m_y$ given by (18),

$$c_\varepsilon(\lambda) p_\varepsilon(x) = \frac{e_0}{|W|} \sum_{y \in W} \chi_\varepsilon(y) \langle \lambda, y \cdot x \rangle. \quad (20)$$

If one identifies $h$ and $h^*$ by a $W$-invariant, C-bilinear, symmetric, inner product, then the right side of (20) is symmetric in $\lambda$ and $x$, since $\chi_\varepsilon(y) = \chi_\varepsilon(y^{-1})$ as a function of $W$. As a consequence,

$$c_\varepsilon = \text{const } p_\varepsilon \quad (21)$$

for some constant depending only on $C$. In particular, $p_\varepsilon$ belongs to $\mathcal{H}_\varepsilon(h^*)$, and consequently

$$p_\varepsilon = \frac{e_0}{c_\varepsilon(\lambda)|W|} \sum_{y \in W} \chi_\varepsilon(y) y^{-1} \cdot \lambda^x \quad (22)$$
is the (up to a constant factor) unique $W_0$-invariant, harmonic polynomial on $h_0$ which transforms under $W$ by $\chi_c \otimes \chi_c$. In (22) we can take any $\lambda$ for which $c_\lambda(\lambda) \neq 0$. For the same values (18) of $m_w$ the relations (5) and (6) imply

The $W$-representation of type $\chi_c \otimes \chi_c$ occurs with multiplicity one in degree $e = e(\mathcal{C})$ in $\mathbb{C}[h^*]$ and not at all in lower degree.

This follows from (5), (6), and (22), since the $\lambda^k$ span the homogeneous polynomials of degree $k$ on $h^*$ and each irreducible subspace of type $\mathbb{C}[h^*]$ contains exactly one $W_0$-invariant (up to scalar multiples). On the other hand, $\mathbb{C}[h^*] \cong \mathbb{C}[h^*_W] \otimes \mathbb{C}[h^*_W]$ as $W = W_0 \times W_0$ module, so we can pass from $W$ to $W_0$ to obtain the following result of Borho-Macpherson [7]:

**Lemma 8.1.** The $W_0$-representation of type $\chi_c$ occurs with multiplicity one in degree $e_0 = e_0(\mathcal{C})$ in $\mathbb{C}[h^*_W]$ and not at all in lower degree.

To make use of the formula (22) for $p_c$ one needs to know when $c_\lambda(\lambda) \neq 0$. This happens exactly when the linear functional

$$c_\lambda[h^*] \to \mathbb{C}, \quad c \to c(\lambda)$$

has a nonzero value on the $W_0$-invariant therein. That functional is the restriction of the natural pairing of polynomials on $h^*$ with the formal power series $e^\lambda$ on $h$. Hence

$$c_\lambda(\lambda) \neq 0 \iff \lambda^\alpha$$

has a nonzero component along the $W_0$-invariant in $\mathbb{C}_c[h^*_W]$.

A simple sufficient condition is

$$c_\lambda(\lambda) \neq 0 \text{ whenever } \lambda \text{ is regular in } h^*.$$  
To see this, recall that for regular $\lambda$ the map

$$\mathcal{H}(h^*) \to \mathbb{C}[W], \quad c \to \sum_w c(w \cdot \lambda) w$$

is a $W$-isomorphism. Under this isomorphism the linear functional $c \to c(\lambda)$ corresponds to evaluation at 1 in the regular representation of $W$, hence has the nonzero component $\deg \chi$ along each irreducible character $\chi$ of $W_0$ (considered as a $W_0$-bi-invariant function on $W$). We summarize what has transpired in a theorem.

**Theorem 8.2.** Let $\Gamma \in H_{2n}(\mathcal{P}(\mathcal{C}))$ be a $2n$ - cycle $\Gamma$ over $\mathcal{C}$ on the conormal variety $\mathcal{F}$ of the $G_0$-orbits on $\mathcal{B}$, $c_\Gamma \in \mathcal{H}_c(h^*)$ the associated harmonic polynomial on $h^*$,

$$\theta_\Gamma(\lambda) = c_\Gamma(\lambda) \theta_c + o(|\lambda|^\gamma).$$
If $\Gamma = \sum_{e \in W \cdot W_0} m_{e} \cdot g$, then $c_{\Gamma} = \left(1/p_{e}(x)\right) \sum_{e \in W} m_{e} \cdot x^e$. Here $p_{e}$ is the (up to a constant factor) unique $W_0$-invariant, harmonic polynomial on $h$ which transforms under $W$ by $\chi_{e} \otimes \chi_{e}$; $\chi_{e}$ is the (irreducible) character of $W_0$ on $H_{2e}(\mathbb{A}_w)^{(e)}$, $x \in h$ is arbitrary subject to $p_{e}(x) \neq 0$, which is the case whenever $x$ is regular. The $W_0$-representation of type $\chi_{e}$ occurs with multiplicity one in degree $e_0 = e_0(c)$ in $C[h_0^*]$ and not at all in lower degree. In particular, if $\Gamma = \mathcal{F}_w$, the fundamental cycle to the conormal of the $G_0$-orbit $S_w$, then $c_{\Gamma} = c_{w}$ is given by

$$c_{w} = \frac{1}{p_{e}(x)} \sum_{e \in W} (-1)^{h(w)} e^{(x)} \psi(x)(S_w) y \cdot x^e.$$ 

9. Univalence

If one uses explicitly the identification $h = h_0 \times h_0$ and $h_0^* = h_0$ one can obtain a formula for $c_{\Gamma}(\lambda)$ which does not involve the arbitrary regular $x \in h$ and an analogous formula for $p_{\Gamma}$. To see this we can place ourselves momentarily in the following general situation envisaged by Lusztig and Spaltenstein [22].

Until further notice, let $h_0$ denote any finite dimensional complex vector space, $W_0$ any finite group of linear transformations of $h_0$. $C[h_0]$ denotes the complex polynomial functions on $h_0$, $C[W_0]$ the group algebra of $W_0$. We shall assume that $h_0^* \approx h_0$ as $W_0$-module so that we can identify the two whenever convenient. (Nevertheless we generally distinguish $h_0^*$ from $h_0$ in notation. The assumption $h_0^* \approx h_0$ could be avoided at the expense of some complications irrelevant here.) Call an irreducible representation $\sigma$ of $W_0$ univalent if it occurs with multiplicity one in the homogeneous polynomials of degree $e_0$ on $h_0$ and does not occur in lower degree (some $e_0$ defined by this condition). Fix such a univalent representation $\sigma$ of $W_0$ and write $\chi_{\sigma}$ for its character. Denote by $C_0[W_0]$ the subspace of $C[W_0]$ transforming by $\sigma \otimes \sigma$ under the biregular representation of $W_0 \times W_0$. For any $a = \sum a(w)w$ in $C_\sigma[W_0]$ define functions $\varphi_a$ and $f_a$ on $h_0^* \times h_0$ by

$$\varphi_a(\lambda, x) = \sum_{y \in W_0} a(y) e^{\langle \lambda, y \cdot x \rangle}$$  

$$f_a(\lambda, x) = \frac{1}{e_0} \sum_{y \in W_0} a(y) \langle \lambda, y \cdot x \rangle^{e_0}. $$

$f_a(\lambda, x)$ is evidently a polynomial on $h_0^* \times h_0$, homogeneous of degree $e_0$ in either variable separately. Write $C^{e_0,e_0}[h_0^* \times h_0]$ for the space of all such polynomials, and indicate by a subscript "\sigma" the part thereof transforming by $\sigma \otimes \sigma$ under $W_0 \times W_0$. 
Lemma 9.1. The map
\[ C_\sigma[W_0] \rightarrow C_\sigma^{\sigma_0,\sigma_0}[h_0^* \times h_0]. \ a \rightarrow f_a, \]
is a \( W_0 \times W_0 \)-isomorphism. One has
\[ \phi_a(\lambda, x) = f_a(\lambda, x) + o(|\lambda|^{\sigma_0}, |x|^{\sigma_0}) \]
with \( o(|\lambda|^{\sigma_0}, |x|^{\sigma_0}) \) indicating a power series in \( \lambda \) and \( x \) whose terms are of degree \( >e_0 \) both in \( \lambda \) and in \( x \).

Proof. This is obvious from the univalence property of \( \sigma \).

Write \( h = h_0 \times h_0 \) and \( W = W_0 \times W_0 \) with \( W_0 \) embedded as the diagonal in \( W \). An element \( a \in C_\sigma[W_0] \) may be considered as an element of \( C[W] \) via
\[ a(w_1, w_2) = a(w_1w_2^{-1}). \]

As element of \( C[W] \), \( a \) is right \( W_0 \)-invariant and transforms by \( \sigma \otimes \sigma \) by \( W \) on the left. We denote the subspace of these \( a \) in \( C[W] \) by \( C_\sigma[W/W_0] \).

For such \( a \) define functions \( \Phi \) and \( F \) on \( h^* \times h \) by
\[ \Phi_a(\lambda, x) = \sum_{y \in W} a(y) \langle \lambda, y \cdot x \rangle \]
\[ F_a(\lambda, x) = \frac{1}{e_0} \sum_{y \in W} a(y) \langle \lambda, y \cdot x \rangle^{e_0}, \]
where we put \( e = 2e_0 \). \( \Phi_a(\lambda, x) \) and \( F_a(\lambda, x) \) transform according to \( \sigma \otimes \sigma \) in \( \lambda \) and are \( W_0 \)-invariant in \( x \). \( F(\lambda, x) \) is also polynomial, homogeneous of degree \( e \) in \( \lambda \) and \( x \) separately.

Lemma 9.2.
\[ \Phi_a(\lambda, x) = F_a(\lambda, x) + o(|\lambda|^{e_0}, |x|^{e_0}). \]

Furthermore,
\[ F_a(\lambda, x) = f_a(\lambda_1, \lambda_2) f_a(x_1, x_2), \]
where \( \lambda = (\lambda_1, \lambda_2), x = (x_1, x_2) \), and \( h_0^* \) is identified with \( h_0; f_a \) is defined as in (2),
\[ f_a(\lambda_1, \lambda_2) = \frac{1}{e_0} \sum_{y \in W_0} a(y) \langle \lambda_1, y \cdot \lambda_2 \rangle^{e_0} \]
and $f_\sigma$ is defined by

$$f_\sigma(x_1, x_2) = \frac{1}{e_0!} \sum_{y \in W_0} \chi_\sigma(y) \langle x_1, y \cdot x_2 \rangle^{e_0}. \quad (5)$$

**Proof.** A typical term in the expansion of $\Phi_\sigma(\hat{\lambda}, x)$ looks like

$$\frac{1}{k_1!k_2!} \sum_{y \in W} a(y) \langle \hat{\lambda}_1, y_1 \cdot x_1 \rangle^{k_1} \langle \hat{\lambda}_2, y_2 \cdot x_2 \rangle^{k_2},$$

where $y = (y_1, y_2)$. By univalence, this sum = 0 if either $k_1$ or $k_2 < e_0$. For the same reason

$$F_\sigma(\hat{\lambda}, x) = \frac{1}{e_0!e_0!} \sum_{y \in W} a(y) \langle \hat{\lambda}_1, y_1 \cdot x_1 \rangle^{e_0} \langle \hat{\lambda}_2, y_2 \cdot x_2 \rangle^{e_0}. \quad (6)$$

The assumption that $a \in \mathbb{C}[W]$ transforms by $\sigma \otimes \sigma$ on the left and is invariant by $W_0$ on the right allows one to write for $w \in W$

$$a(y) = \frac{\deg \sigma}{|W|} \sum_{w \in W} \chi_\sigma(y^{-1}w) a(w),$$

where $\chi_\sigma(y) = \chi_\sigma(y_1, y_2^{-1})$ if $y = (y_1, y_2)$ in $W_0 \times W_0$. In terms of the pairing $\langle a, b \rangle = \sum a(w) b(w)$ on $\mathbb{C}[W]$ this may be written as

$$a(y) = \frac{\deg \sigma}{|W|} \langle y \cdot \chi_\sigma, a \rangle.$$

Also, in terms of the pairing $\langle \hat{\lambda}^i, x^j \rangle/i!j! = \langle \hat{\lambda}, x \rangle^i \delta_{ij}$ on polynomials on $h_0$ and $h_0^*$, one checks that

$$\langle \hat{\lambda}_1, y_1 \cdot x_1 \rangle^{e_0} \langle \hat{\lambda}_2, y_2 \cdot x_2 \rangle^{e_0} = \frac{1}{e_0!e_0!} \langle (\hat{\lambda}_1 \otimes \hat{\lambda}_2)^{e_0}, y \cdot (x_1 \otimes x_2)^{e_0} \rangle.$$

Thus (6) can be written as

$$F_\sigma(\hat{\lambda}, x) = \frac{\deg \sigma}{e_0!e_0!} \sum_{y \in W} \langle y \cdot \chi_\sigma, a \rangle \langle (\hat{\lambda}_1 \otimes \hat{\lambda}_2)^{e_0}, y \cdot (x_1 \otimes x_2)^{e_0} \rangle.$$

In this equation $(\hat{\lambda}_1 \otimes \hat{\lambda}_2)^{e_0}$ and $(x_1 \otimes x_2)^{e_0}$ may be replaced by their components $(\hat{\lambda}_1 \otimes \hat{\lambda}_2)^{e_0}_{\alpha}$ and $(x_1 \otimes x_2)^{e_0}_{\beta}$ which transform by $\sigma \otimes \sigma$ under $W$, by Schur's relations for irreducible matrix coefficients, which give further that

$$F_\sigma(\hat{\lambda}, x) = \frac{1}{e_0!e_0!} \langle (\hat{\lambda}_1 \otimes \hat{\lambda}_2)^{e_0}, a \rangle \langle \chi_\sigma, (x_1 \otimes x_2)^{e_0} \rangle.$$

(7)
where the pointed brackets denote a $W$-invariant, nondegenerate, bilinear form on $C_\sigma[W] \times C_\sigma^{e_0,e_0}[h]$. Lemma 9.1 says that under the identification $h = h_0 \times h_0$ such a form is given by

$$\langle (x_1 \otimes x_2)^e_{c_0}, a \rangle = \sum_{y \in W_0} a(y)\langle x_1, y \cdot x_2 \rangle^{e_0}.$$ 

Thus (7) becomes

$$F_\sigma(\lambda, x) = f_\sigma(\lambda_1, \lambda_2) f_\sigma(x_1, x_2)$$

with $f_\sigma$ and $f_\sigma$ as specified. This is just the assertion of the lemma.

Returning now to the situation of Section 8, we have the following:

**ADDENDUM TO THEOREM 8.2.** The polynomials $c_\tau$ and $p_\epsilon$ are given by

$$c_\tau(\lambda) = k_\epsilon \sum_{y \in W_0} m_\epsilon \langle \lambda_1, y \cdot \lambda_2 \rangle^{e_0} \quad (8)$$

$$p_\epsilon(x) = \frac{1}{\epsilon! k_\epsilon} \sum_{y \in W_0} \chi_\epsilon(y)\langle x_1, y \cdot x_2 \rangle^{e_0} \quad (9)$$

for some constant $k_\epsilon$ depending only on $\epsilon$.

(Note that in these equations $m_\epsilon = m_{(\lambda_1, \lambda_2)}$ in accordance with the identification $W/W_0 = W_0$.)

**Proof:** This follows by comparing the relation (8.6),

$$\sum_{y \in W} m_\epsilon \langle \lambda, y \cdot x \rangle^\epsilon = c_\tau(\lambda) p_\epsilon(x),$$

with the relation

$$\frac{1}{\epsilon!} \sum_{y \in W} m_\epsilon \langle \lambda, y \cdot x \rangle^\epsilon = f_\sigma(\lambda_1, \lambda_2) f_\sigma(x_1, x_2)$$

from Lemma 9.2 when we take $\chi_\sigma = \chi_\epsilon$ and $a =$ the component of $m$ which transforms by $\chi_\epsilon \otimes \chi_\epsilon$ on the left. In the formula

$$f_\sigma(\lambda_1, \lambda_2) = \sum_{y \in W_0} a(y)\langle \lambda_1, y \cdot \lambda_2 \rangle^{e_0}$$

we may then again replace $a(y)$ by $m_\epsilon$, because the component of $m$ of type $\chi_\epsilon$ is the only one which contributes in degree $e_0$ in this sum, as follows from

$$f \in H_{2n}(\mathcal{F}(\mathcal{O}')) \approx H_{2n}(\mathcal{F}(\mathcal{O}))+ \sum_{\epsilon' < \epsilon} H_{2n}(\mathcal{F}(\mathcal{O}')).$$
10. Nilpotent Orbital Integrals

For each $G_0$-orbit $\mathcal{O}$ in $i\mathfrak{g}_0^*$ let $\mu_{\mathcal{O}}$ be the distribution on $g_0^*$ defined by

$$
\langle \mu_{\mathcal{O}}, f \rangle = \frac{1}{(2\pi i)^d} \int_{\mathcal{O}} f e^{\sigma_{\mathcal{O}}}
$$

with $d = \dim_C \mathcal{O}$. $\mu_{\mathcal{O}}$ is a tempered distribution on $i\mathfrak{g}_0^*$. (Of course, we use $i\mathfrak{g}_0^*$ rather than $g_0^*$ or $g_0$ only to conform with the definition of the distributions $\theta_R$, which will simplify the notation.) We denote by $\hat{\mu}_{\mathcal{O}}$ the Fourier transform of $\mu_{\mathcal{O}}$, a tempered distribution on $g_0$, given by a locally integrable function. In this sense

$$
\hat{\mu}_{\mathcal{O}}(x) = \frac{1}{(2\pi i)^d} \int_{\mathcal{O}} e^{x^\mathcal{O}} e^{\sigma_{\mathcal{O}}}.
$$

For regular $\lambda \in \mathfrak{h}^*$ we set $\mathcal{O}_\lambda = G_0 \cdot \lambda$ and $\hat{\mu}_{\mathcal{O}_\lambda} = \mu_{\mathcal{O}_\lambda}$. Since $G_0 \cdot \lambda = K \cdot (\lambda + i\mathfrak{b}_0^*) = p_{\lambda} x'_\lambda$ for such $\lambda$ we find that its Fourier transform is

$$
\hat{\mu}_{\mathcal{O}_\lambda}(x) = \frac{1}{(2\pi i)^d} \int_{\mathcal{O}_\lambda} e^{\sigma_{\mathcal{O}_\lambda}} = \frac{1}{\pi(x)} \sum_{y \in W_0} e^{y \cdot \lambda}(x)
$$

as tempered distributions on $g_0$.

Now take for $\mathcal{O}$ a nilpotent $G_0$-orbit in $i\mathfrak{g}_0^*$. From (8.7) we know that its Fourier transform $\hat{\mu}_{\mathcal{O}} = \theta_{\mathcal{O}}$ is

$$
\hat{\mu}_{\mathcal{O}} = \frac{1}{e!} \frac{p_{\mathcal{O}}}{\pi},
$$

where $p_{\mathcal{O}}$ is the distribution on $g_0$ which on $h_0$ is given by the up to scalars unique $W_0$-invariant polynomial of degree $e = e(\mathcal{O})$ on $h$ transforming according to the irreducible character $\chi_\mathcal{O} \otimes \chi_\lambda$ under $W = W_0 \times W_0$. Using (8.20) we have for any regular $\lambda \in \mathfrak{h}^*$ a formula

$$
\hat{\mu}_{\mathcal{O}}(x) = \text{const} \sum_{y \in W} \chi_\mathcal{O}(y) y^{-1} \cdot \lambda^\mathcal{O}
$$

for a constant depending on $\mathcal{O}$ and $\lambda$, namely

$$
\text{const} = \frac{e_0}{e! c_\mathcal{O}(\lambda) |W|}
$$

in the notation of Section 8. Alternatively, using (9.9),

$$
\hat{\mu}_{\mathcal{O}}(x) = \text{const} \sum_{y \in W_0} \chi_\mathcal{O}(y) \langle x_1, y \cdot x_2 \rangle^\mathcal{O},
$$

for a constant depending on $\mathcal{O}$. Thus $\hat{\mu}_{\mathcal{O}}$ is a tempered distribution on $g_0$.
where this time the constant depends only on $O$, namely
\[ \text{const} = \frac{1}{k'e!}. \]

Equation (4) is a rather explicit formula for the invariant eigendistributions $\hat{\mu}_\epsilon$ on $g_0$, which have infinitesimal character $0$. Compare this with Harish-Chandra's formula for eigendistributions with regular infinitesimal character $\lambda$,
\[ \theta = \frac{1}{\pi} \sum_{x \in W} m_x e^{x^{-1}i}. \quad (6) \]
(Of course, (6) is part of the genesis of (4)).

Formula (3) quickly leads to a formula for $\mu_\epsilon$ itself as follows. Let $\partial_{\epsilon, \lambda}$ be the constant coefficient operator on $h^*$ corresponding to the polynomial $p_\epsilon$ on $h$. It satisfies
\[ \partial_{\epsilon, \lambda} e^{\langle \lambda, x \rangle} \big|_{\lambda = 0} = p_\epsilon(x), \]
where the subscript $\lambda$ on $\partial_{\epsilon, \lambda}$ means "differentiation with respect to $\lambda$.

Differentiating Eq. (2) with respect to $\lambda$ and evaluating at $\lambda = 0$ one obtains
\[ \lim_{\lambda \to 0} \partial_{\epsilon, \lambda} \hat{\mu}_\lambda(x) = \lim_{\lambda \to 0} \partial_{\epsilon, \lambda} \left\{ \frac{1}{\pi} \sum_{x \in W_0} e^{\langle \lambda, x \rangle} \right\} = \frac{|W_0|}{\pi} p_\epsilon(x) = e! |W_0| \hat{\mu}_\lambda(x). \]

Inverting the Fourier transforms we arrive at

**Theorem 10.1.** Let $C$ be a nilpotent $G_0$-orbit in $i g_0^\*$, then
\[ \mu_\epsilon = e! |W_0| \lim_{\lambda \to 0} \partial_{\epsilon, \lambda} \hat{\mu}_\lambda \]

and
\[ \hat{\mu}_\lambda(x) = \text{const}(\lambda, C) \frac{1}{\pi} \sum_{x \in W_0} \chi_\lambda(y) \langle \lambda, y \cdot x \rangle^e \]
\[ = \text{const}(\epsilon) \frac{1}{\pi} \sum_{x \in W_0} \chi_\lambda(y) \langle x_1, y \cdot x_2 \rangle^e. \]

The formula (7) has some history. For $C = \{0\}$, $\chi_\lambda = \text{sign}$ representation of $W_0$, $p_\epsilon = \text{const} \pi$, and (7) becomes Harish-Chandra's Limit Formula
\[ \mu_\epsilon = \text{const} \lim_{\lambda \to 0} \partial_{\epsilon, \lambda} \hat{\mu}_\lambda(x). \]
For special orbits formula (7) was proved by Barbasch–Vogan [3, 4] by case by case computations and conjectured to hold in general. This conjecture was then proved by Hotta–Kashiwara [13] (using the theory of holonomic systems).

11. A Conjecture of Joseph

The asymptotic relation
\[ \theta_r(\lambda) = c_r(\lambda) \theta_c + o(|\lambda|^c) \]  
may be used to prove a conjecture of Joseph [17, Conjecture 9.8]. The conjecture may be explained as follows. Let
\[ \mathcal{F} = \{(b, v) | b \in \mathcal{B}_0, v \in b^\perp \cap b_0^\perp \}. \]
This is the conormal variety of the action of the fixed Borel subgroup \( B_0 \) of \( G_0 \) on the flag manifold \( \mathcal{B}_0 \) of \( G_0 \). (The complexification of \( G_0 \) does not enter into the picture here.) \( \mathcal{F} \) is also the inverse image \( \mathcal{B}_0^* (b_0^\perp) \) of \( b_0^\perp \) under the Springer map \( \mathcal{B}_0^* \rightarrow \mathcal{N}_0 \). Its dimension is \( n_0 = \dim C \mathcal{B}_0^* \) and our general construction gives a representation of \( W_0 \) on \( H_{2n}(\mathcal{F}) \). In top degree, \( H_{2n}(\mathcal{F}) \) has as basis the fundamental cycles of the components of \( \mathcal{F} \). These components are the closures of the conormal bundles \( \mathcal{F}_w \) of the \( B_0 \)-orbits \( B_0 \cdot w^{-1}b_0 \),
\[ \mathcal{F}_w = \{ B_0 \cdot (w^{-1}b_0, v) | v \in b_0^\perp \cap w^{-1}b_0^\perp \}. \]
For each \( w \in W_0 \) there is a unique \( G_0 \)-orbit \( \mathcal{C} = \mathcal{C}(w) \) on \( \mathcal{N}_0 \) which intersects \( b_0^\perp \cap w^{-1}b_0^\perp \) densely; and for each \( \mathcal{C} \) the components of \( \mathcal{C} \cap b_0^\perp \) are the closures in \( \mathcal{C} \cap b_0^\perp \) of the dense parts of the \( B_0 \cdot (b_0^\perp \cap w^{-1}b_0^\perp) \) cut out by \( \mathcal{C} \) when \( \mathcal{C}(w) = \mathcal{C} \). For a given \( \mathcal{C} \) these components \( \mathcal{C}(\omega) \) of \( \mathcal{C} \cap b_0^\perp \) all have the same dimension \( \frac{1}{2} \dim \mathcal{C} \). (For these results see [26, 17].) To each such subvariety \( \mathcal{C}(w) \) of \( \mathcal{N}_0 \) Joseph associates a polynomial \( p_{\mathcal{C}(w)} \) on \( h_0^* \), homogeneous of degree \( n_0 - \frac{1}{2} \dim \mathcal{C} \). This degree = \( e_0 = \dim B_0 \) for \( v \in \mathcal{C} \). Joseph's conjecture says that if one writes
\[ \mathcal{Z}_w = \sum_{v \in W_0} A(w, (y, 1))(y, 1) \cdot \mathcal{Z}_1 \]  
in the homology \( H_{2n}(\mathcal{Z}) \) of the conormal variety \( Z \), then
\[ p_{\mathcal{C}(w)} = \text{const} \sum_{v \in W_0} A(w, (y, 1)) y \cdot \rho_0^{eq}, \]
and the exponent $\epsilon_0 = \dim \mathcal{B}_0$, in this relation is the least for which the right side is nonzero. $\rho_0$ denotes half the sum of the roots of $h_0$ in $h_0^\perp$, which must here be thought of as an element of $h_0$ by means of a $W_0$-invariant bilinear form (Joseph identifies $h_0$ and $h_0^\ast$); in (5) we take $(y, 1)$ as element of $W = W_0 \times W_0$ and use the $W$-action on the homology of $\mathcal{L}$ defined earlier (which by Section 12 is also the action constructed by Kazhdan–Lusztig [19], to which Joseph refers. Joseph writes this action of $W_0 \times W_0$ as a $W_0 - W_0$ bimodule.)

To prove the conjecture we consider the projection $\mathcal{B}^\ast = \mathcal{B}_0^\ast \times \mathcal{B}_0^\ast \to \mathcal{B}_0$ through the second factor. It restricts to a fibration

$$\mathcal{T} \subseteq \mathcal{L} \to \mathcal{B}_0$$

with $\mathcal{T}$ as in (2) as fibre over $b_0$. The components $\mathcal{L}_w$ of $\mathcal{L}$ meet the fibre $\mathcal{T}$ in the components $\mathcal{T}_w$ of $\mathcal{T}$. For any $v \in b_0^\perp$ there is a diagram

$$\mathcal{B}_0^\ast \subseteq \mathcal{T} \to b_0^\perp$$

$$\cap \cap \cap$$

$$\mathcal{B}_v^\ast \subseteq \mathcal{T} \to \mathcal{N}$$

For a given $G_0$-orbit $\mathcal{O}$ on $\mathcal{N}_0$ and $v \in \mathcal{O} \cap b_0^\perp$, $\mathcal{L}_w$ meets the fibre $\mathcal{B}_v^\ast$ of $\mathcal{L} \to \mathcal{N}$ in a single $A_0(v)$-orbit of components; $\mathcal{T}_w$ meets the fibre $\mathcal{B}_w^\ast$ of $\mathcal{T} \to b_0^\perp$ in a union of components which lie in a single $A_0(v)$-orbit and determine element of $H_{2e_0}(\mathcal{B}_w^\ast)^{4d(v)}$. (For these facts see [23].) According to a result of Hotta [12], the map which sends $p_{\mathcal{L}^w}$ to this element of $H_{2e_0}(\mathcal{B}_w^\ast)^{4d(v)}$ extends to a $W_0$-isomorphism of the space spanned by the $p_{\mathcal{L}^w}$ onto $H_{2e_0}(\mathcal{B}_w^\ast)^{4d(v)}$.

On the other hand, the inclusion $\mathcal{B}_w^\ast \subseteq \mathcal{B}_0$ gives a $W_0$-injection $H_{2e_0}(\mathcal{B}_0^\ast)^{4d(v)} \to H_{2e_0}(\mathcal{B}_0) \approx \mathcal{H}(h_0)$, as we know from Sections 3 and 7. Hence the element of $H_{2e_0}(\mathcal{B}_v^\ast)^{4d(v)}$ mentioned above gives rise to another (harmonic) polynomial on $h_0^\ast$, homogeneous of degree $e_0$. Because of Lemma 8.1, this polynomial must be $p_{\mathcal{L}^w}$ up to a (nonzero) factor depending only on $\mathcal{O}$.

To make use of the relation (1) note that the fibration (6) leads to a $W_0 \times \{1\}$-isomorphism $H_{2e_0}(\mathcal{L}) \to H_{2e_0}(\mathcal{T})$ which sends the fundamental cycle of $\mathcal{L}_w$ to the fundamental cycle of $\mathcal{T}_w$. From Theorem 5.1 we know that the relation

$$\mathcal{L}_w = \sum_{y \in W_0} A(w, (y, 1))(y, 1) \cdot \mathcal{L}_1$$

implies that

$$\theta_{\mathcal{L}_w} = \frac{(-1)^n}{\pi} \sum_{y \in W_0} A(w, (y, 1)) e^{(y^{-1}, 1) \cdot d}.$$
Of course, we also know that
\[ A(w, y) - (-1)^{l(w)} \cdot \text{Eu}_y(Z_w), \tag{9} \]
but this will not even be needed. Rather, we use the formula (9.8) to write
\[ c_n(\lambda) = k_c \sum_{y \in W_0} A(w, (y, 1)) \langle \lambda_1, y \cdot \lambda_2 \rangle^{\infty}. \tag{10} \]

On the other hand, the polynomial \( c_n \) represents the fundamental cycle of the component \( B_w^* \) of \( B^* \) in \( H_{2\ell}(B) \cong H^*(h) \). Since \( B^* = B_0^* \times B_0^* \) that component must factor \( B_w^* = C_w \times C_w' \), as a product of a pair of components of \( B_0^* \) depending on \( w \). These two components correspond to two harmonic polynomials \( p_w, p_w' \) on \( h_0^* \) and the factorization \( B_w^* = C_w \times C_w' \) means that
\[ c_n(\lambda) = p_n(\lambda_1) p_n'(\lambda_2) \tag{11} \]
if \( \lambda = (\lambda_1, \lambda_2) \) in \( h^* = h_0^* \times h_0^* \). From the discussion around diagram (7) it is clear that
\[ p_w = \text{const} \ p_{\tau(w)}. \]

To prove Joseph's conjecture we may therefore replace \( p_{\tau(w)} \) by \( p_w \) in (5). From (10) and (11) one finds
\[ p_n(\lambda_1) = \frac{k_c}{p_n'(\lambda_2)} \sum_{y \in W_0} A(w, (y, 1)) \langle \lambda_1, y \cdot \lambda_2 \rangle^{\infty} \]
provided \( p_n'(\lambda_2) \neq 0 \), which know to be the case for regular \( \lambda_2 \). Choosing \( \lambda_2 = \rho_0 \) we obtain the formula conjectured by Joseph:
\[ p_{\tau(w)} = \text{const} \sum_{y \in W_0} A(w, (y, 1)) y \cdot \rho_0^{\infty}. \]

For the record we point out once more that we have actually proved the more precise formula
\[ p_{\tau(w)} = \text{const} \sum_{y \in W; W_0} (-1)^{l(w)} \cdot l^+(y) \cdot \text{Eu}_y(Z_w) \cdot y \cdot \rho_0^{\infty}. \tag{12} \]
for a constant depending only on \( \ell \).
12. APPENDIX.
COMPARISON WITH THE KAZHDAN–LUSTIG CONSTRUCTION

For every simple reflection $s$ of $g_0, h_0$ ("simple" with respect to the fixed Borel $b_0$ of $g_0$) Kazhdan and Lusztig [19] define a proper homotopy equivalence $\alpha$, of the conormal variety $\mathcal{Y}$ of the $K$-action on $\mathcal{B} = \mathcal{B}_0 \times \mathcal{B}_0$:

$$\mathcal{Y} = \{ (b, b', v, v') | b, b' \in \mathcal{B}_0, v' = -v \in b^{-1} \cap b'^{-1} \}.$$ 

Their procedure is equivalent to the following. Choose a neighbourhood $\mathcal{U}$ of $\mathcal{Y}$ in $\mathcal{B}_0 \times \mathcal{B}_0 \times \mathcal{N}_0 \times \mathcal{N}_0$ so that the inclusion $i: \mathcal{Y} \to \mathcal{U}$ has a proper homotopy inverse $p: \mathcal{U} \to \mathcal{Y}$. Choose a complex-valued continuous function $\mu$ on $h_0^+$ so that $(k \exp(\mu(v)) s \cdot b_0, k' \cdot b_0, k \cdot v, k' \cdot v') \in \mathcal{U}$ for $(k \cdot b_0, k' \cdot b_0, k \cdot v, k' \cdot v') \in \mathcal{Y}$ and $|v| > \mu(v)$ Here $k, k' \in K_0$, the fixed maximal compact subgroup of $G$ and $\exp(v)$ is defined for $v \in h^+$ by thinking of $v$ as an element of the nilradical $n$ of $h$. (One may take $\mu$ real, positive, as do Kazhdan and Lusztig.) Define

$$a: \mathcal{Y} \to \mathcal{U}, (k \cdot b_0, k' \cdot b_0, k \cdot v, k' \cdot v')$$
$$\to (k \exp(\mu(v)) s \cdot b_0, k' \cdot b_0, k \cdot v, k' \cdot v'). \quad (1)$$

Then $\alpha$ is defined to be the proper homotopy class of

$$p \cdot a \circ i: \mathcal{Y} \to \mathcal{U}. \quad (2)$$

In our construction the proper homotopy equivalence $a(s, 1): \mathcal{Y} \to \mathcal{Y}$ which gives the homotopy action of $(s, 1) \in W = W_0 \times W_0$ on $\mathcal{Y}$ is defined by the same procedure, except that $\mathcal{U}$ is taken as a suitable neighbourhood of $\mathcal{Y}$ in $\mathcal{B}^* = \{ (b, b', v, v') | v \in h^-, v' \in h'^{-1} \}$ and the map (1) is replaced by the map

$$a_{(\lambda, \lambda}^{(s, 1)}(s, 1): \mathcal{Y} \to \mathcal{U}, (k \cdot b_0, k' \cdot b_0, k \cdot v, k' \cdot v')$$
$$\to (h \cdot b_0, h' \cdot b_0, h \cdot \eta, h' \cdot \eta'), \quad (3)$$

where $(\lambda, \lambda') \in h^* = h_0^* \times h_0^*$ is chosen sufficiently close to $(0, 0)$, but regular, and $k, k', h, h' \in K_0$, $v, v', \eta, \eta' \in h_0^+$ are related by

$$(k \cdot (\lambda + v), k' \cdot (\lambda' + v')) = (h \cdot (s\lambda + \eta), h' \cdot (\lambda' + \eta')).$$

(Of course here $h' = k'$ and $\eta' = v'$ and $k \cdot v = -k' \cdot v'$ on $\mathcal{Y}$.) To prove that (1) and (3) give rise to the same homotopy equivalence of $\mathcal{Y}$ by the construction (2) it suffices to show that if we take for $\mathcal{U}$ in (3) the $\mathcal{U}$ in (1), then (1) and (2) are properly homotopic as maps $\mathcal{Y} \to \mathcal{U}$. 

We may assume that \( \mathcal{U} \) consists of all points \((b, b', v, v')\) of \( \mathcal{B}_0 \times \mathcal{B}_0 \times \mathcal{N}_0 \times \mathcal{N}_0 \) satisfying

\[
\text{dist}(v, b^\perp) < \varepsilon, \quad \text{dist}(v', b'^\perp) < \varepsilon, \quad \text{dist}(v, v') < \varepsilon, \quad (4)
\]

where "\text{dist}" refers to a \( U \)-invariant Euclidean distance in \( g^* \). The homotopy between these two maps will take the form

\[
\mathcal{X} \to \mathcal{U}, (k \cdot b_0, k' \cdot b_0, k \cdot v, k' \cdot v') \rightarrow (k \cdot b_0, k' \cdot b_0, k \cdot v_1, k' \cdot v_1); \quad (5)
\]

\( b \) and \( v \) will depend only on \((t, v) \in [0, 1] \times b^\perp_0; \ t \in [0, 1] \) is the homotopy parameter.

In order that (5) be a homotopy from (1) to (3) we need that

\[
(b_1, v_1) = \begin{cases} (\exp(\mu(v) s \cdot b_0, v) & \text{for } t = 0, \\ (h \cdot b_0, h \cdot \eta) & \text{for } t = 1, \end{cases} \quad (6)
\]

where \( h \in K_0 \) and \( v, \eta \in b^\perp_0 \) are related by

\[
(\lambda + v) = h \cdot (s \cdot \lambda + \eta). \quad (7)
\]

It will be convenient to write (5) in the equivalent form

\[
\mathcal{X} \to \mathcal{U}, \quad k \cdot (b_0, b, v, -v) \rightarrow k \cdot (b_0, b, v_1, -v). \quad (5')
\]

We need to insure that the image of (5') remains in the neighbourhood \( \mathcal{U} \) of \( \mathcal{X} \) in \( \mathcal{B}_0 \times \mathcal{B}_0 \times \mathcal{N}_0 \times \mathcal{N}_0 \). For this it suffices that

\[
\text{dist}(v_1, b^\perp_1) < \varepsilon \quad \text{and} \quad \text{dist}(v_1, v) < \varepsilon
\]

or equivalently

\[
\text{dist}(v, b^\perp_1) < \varepsilon \quad \text{and} \quad \text{dist}(v_1, v) < \varepsilon. \quad (8)
\]

Let \( p_s = b_0 + s \cdot b_0 \) be the parabolic subalgebra of \( g_0 \) associated to the simple root \( \alpha \). Both \( \exp(\mu(v) s \cdot b_0) \) and \( h \cdot b_0 \) in (6) depend only on \( v \) mod the nilradical \( p^\perp_\alpha \) of \( p_s \). Furthermore, since

\[
\exp \left( -\frac{1}{\lambda_s} v \right) \cdot \lambda \equiv \lambda + v \mod p^\perp_\alpha
\]

with \( \lambda_s = \langle \lambda, \check{\alpha} \rangle \), \( \check{\alpha} \) the coroot of the simple root \( \alpha \) belonging to \( s \), we obtain from (7) that

\[
h \cdot (s \cdot \lambda + \eta) \equiv \exp \left( -\frac{1}{\lambda_s} v \right) \cdot \lambda \mod p^\perp_\alpha,
\]
which may be written as
\[ hbs \cdot \hat{\lambda} = \exp \left( -\frac{1}{\lambda_x} v \right) \cdot \hat{\lambda} \]
for some \( h \in B_0 \); hence
\[ h \cdot b_0 = \exp \left( -\frac{1}{\lambda_x} v \right) s \cdot b_0. \]

We shall verify presently

For any \( \varepsilon > 0 \) there is a constant \( R \) so that for all \( v \in b_0^\perp \)
\[ \text{dist}(v, \exp(tv) s \cdot b_0^\perp) < \varepsilon \] for \( |t| > R. \) \( \tag{9} \)

If one applies this with \( R = 1/|\lambda_x| \) (\( \lambda \) sufficiently close to 0), one sees that we may take
\[ \mu(v) = -\frac{1}{\lambda_x} \text{const} \] \( \tag{10} \)
in Eq. (1). With this choice of \( \mu(v) \) we can take
\[ b_t \equiv \exp \left( -\frac{1}{\lambda_x} v \right) s \cdot b_0 = k \cdot b_0, \] \( \tag{11} \)
independent of \( t \), to satisfy Eqs. (6) in the \( b \)-component.

To insure that (6) also holds in the \( v \)-component, choose \( v \), continuous in \((t, v) \in [0, 1] \times \mathcal{N}_0\) with
\[ v_t = \begin{cases} v & \text{for } t = 0 \\ h \cdot \eta & \text{for } t = 1 \end{cases} \]
and so that
\[ \text{dist}(v_t, v) < \varepsilon \] for all \( t \in [0, 1] \). \( \tag{12} \)

This is possible (for \( \lambda \) close to 0) because the given endpoints of the path \( v \), satisfy
\[ |h \cdot v - v| \leq \text{const} |\lambda| \]
in view of (7). Then both inequalities in (8) are satisfied, as required.

It only remains to check the assertion (9). This assertion concerns only the subalgebra of \( g \) generated by the root vectors for \( \pm \alpha \), so that we may assume \( g = \mathfrak{sl}(2, \mathbb{C}) \) in order to prove (9). With this assumption, rewrite (9):
\[ \frac{1}{|t|} \text{dist}(tv, \exp(tv) s \cdot b_0^\perp) < \varepsilon \] for \( |t| > R. \)
Replacing \( tv \) by \( v \) the condition (9) may be replaced by

\[
\text{dist}(v, \exp(v) s \cdot b_0^\perp) < |t|e \quad \text{for} \quad |t| > R.
\]

which says simply that

\[
\text{dist}(v, \exp(v) s \cdot b_0^\perp) < \text{const}
\]

with "const" independent of \( v \in b_0^\perp \).

That (13) holds as long as \( |v| \) remains bounded is obvious. On the other hand,

\[
\text{dist}(v, \exp(v) s \cdot b_0^\perp) \to 0 \quad \text{as} \quad |v| \to \infty
\]

because

\[
\exp(v) s \cdot b_0 \to b_0 \quad \text{as} \quad |v| \to \infty.
\]

This last assertion says that on \( \mathbb{CP}^1 \) the point \([0 : 1]\) approaches \([1 : 0]\) under the right action of \( \left[ \begin{array}{cc} 1 & 0 \\ 1 & 1 \end{array} \right] \) as \( |c| \to \infty \) (as does every other point)—we mention this triviality only because ultimately the whole construction of the \( x \), and the \( a(s, 1) \) comes down to this.

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