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On Linear Groups of Degrees at Most $|P|-1$

ZHANG JIPING*

Institute of Mathematics, Beijing University, Beijing 100871, China

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In this paper, we determine the structure of complex linear groups G of degree at most $|P|-1$, where P is a T.I. Sylow p-subgroup of G. © 1991 Academic Press, Inc.

Let G be a finite complex linear group of degree m and P a Sylow p-subgroup of G. If the order of P is p and $m \leqslant (2p+1)/3$, Brauer and Tuan determined the structure of G in the 1940s; see [1, 3, 4] for references. Since then, many group theorists have been involved in studying the subject and various beautiful structure theorems on linear groups have been given. The classification of finite simple groups made the study develop quickly. In [20], Ferguson classified the linear groups of degree $\leqslant p-3$ and the author [11] determined the structure of linear groups G of degree at most $p-1$. Under the assumption that P is a cyclic or T.I. subgroup, Blau [7] established some best-possible lower bounds for the degrees. The aim of the present paper is to generalize and extend the results of [7].

Throughout the paper, all groups are assumed to be finite and the notation and terminology are standard and follow those of [13] and [15].

LEMMA 1 (Weir [19]). *Let G be a classical group of order $q^t \prod_{i=1}^{n} (q^{m(i)} - d_i)$, where $m(i)$ is an integer, $d_i = \pm 1$, $q = r^f$, and r is the characteristic of G. If G has a cyclic Sylow p-subgroup P for some $p \neq r$, then there exists only one i such that $|p|' \mid (q^{m(i)} - d_i)$.*

LEMMA 2. *Let G be p-nilpotent with a T.I. Sylow p-subgroup P. If G has a faithful complex character χ of degree $\leqslant |P|-1$, then one of the following must hold:*

(1) *P is normal in G.*

(2) *P is generalized quaternion and $\chi(1) \geqslant |P|/2 - 1$.*

(3) *P is cyclic; $O_{p'}(G) = Z(G)\, Q_0$ for some extra-special q-subgroup Q_0 and $Q_0 O_q(Z(G)) \in \mathrm{Syl}_q(O_{p'}(G))$; χ is irreducible of degree $|P|-1 =$*

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$|Q/Z(Q)|^{1/2}$; *and one of the following holds*: $p=2$ *and* $|P|=q+1$, *or* $p=3$ *and* $|P|=9$, *or* $p=2^n+1$ *and* $|P|=p$.

Proof. Suppose that the lemma is not true and let G be a counterexample such that $|G|+\chi(1)$ is minimal. Since P is a T.I. set of G, $O_p(G)=1$ and P acts non-trivially on $O_{p'}(G)$. Let Q be a P-invariant Sylow q-subgroup of $O_{p'}(G)$ such that $O_p(PQ)=1$. By [13, Theorem 5.3.11], there exists a characteristic subgroup A of Q such that

(1) $A/Z(A)$ is elementary abelian,

(2) $[Q, A] \leqslant Z(A)$,

(3) $C_Q(A) \leqslant A$,

(4) P acts on A nontrivially.

Set $\bar{A}=A/\mathrm{Fr}(A)$, where $\mathrm{Fr}(A)$ is the Frattini subgroup of A. Then P acts also nontrivially on $\bar{A}$. By Maschke's theorem, $\bar{A}=A_1+A_2$, where $A_1=C_{\bar{A}}(P)$ and A_2 is P-invariant. Since P is a T.I. set of $P\bar{A}$, P acts on A_2 with no fixed points. Thus PA_2 is a Frobenius group and therefore P is either cyclic or generalized quaternion. If P is generalized quaternion, then $\chi(1) \geqslant |P|/2-1$ by [7]. If P is cyclic, then $\chi|_{PA}$ is irreducible by the minimality of G and $\chi(1)=|P|-1$ by Theorem 2 of [7]. Since P is cyclic and $p \nmid \chi(1)$, $\chi|_A$ is irreducible. It follows that $A' \neq 1$, $C_G(A)=Z(G)$ is cyclic, and $\chi(1)^2=|A/Z(A)|$ by [14, Theorem 2.31]. Suppose $|A/Z(A)|=q^{2r}$; then $|P|-1=q^r$. If $p=2$, then $r=1$. If $p \neq 2$, then either $|P|=9$ or $|P|=p$; see [10] for reference. Since $Z(A) \leqslant Z(G)$ is cyclic, $A=Q$ by [2, Lemma 3A]. Now $|P|-1=q^r$, P acts trivially on any other P-invariant Sylow t-subgroups of $O_{p'}(G)$, $t \neq q$. So $O_{p'}(G)=QC_G(P)$. Since P acts irreducibly on $A/Z(Q)$ and $Z(Q) \leqslant Z(G)$, $Q_0 := [P, Q]$ is extra-special and $Q=Z(Q)\,Q_0$.

Next we show that $C_G(P)=P \times Z(G)$. Since $O_{p'}(G)=QC_G(P)$, this establishes $O_{p'}(G)=Z(G)Q_0$.

Let B be the p-block of χ. G p-nilpotent and P a T.I. set imply that $N := N_G(P)=N_G(\Omega_1(P))=C_G(\Omega_1(P))=C_G(P)$ and hence that the inertial index of B is 1. Note that $N=P \times V$ for some p'-group V such that $Z(G) \leqslant V$. By [5, VII. 1.5, VII. 2.4, VII. 2.14], as a Brauer character $\chi|_N=d\varphi$, where φ is some irreducible Brauer character of N (and so an ordinary irreducible character of V), and $d=1$ or $|P|-1$. If $d=1$, then $\chi|_N=\lambda\varphi$, where λ is a linear character of P and both λ and φ are inflated to characters of N. So $\chi|_p=\chi(1)\lambda$. Then χ faithful implies $P \leqslant Z(G)$, a contradiction. So $d=|P|-1$, $\varphi(1)=1$, and χ faithful yields $V=Z(G)$. The lemma is now proved.

Since the linear groups of degree less than 8 have been determined, in the following, we assume that P is of order at least 8 for convenience only.

THEOREM 3. *Let G be a finite group with a T.I. Sylow p-subgroup P of order at least 8. Set $H=O^{p'}(G)$. If G has a faithful complex character χ of degree at most $|P|-1$, then one of the following is true*:

(1) *P is normal in G.*

(2) *P is generalized quaternion, G is solvable, and $\chi(1)\geqslant |P|/2-1$.*

(3) *P is cyclic; χ is irreducible of degree $|P|-1$; $O_{p'}(G)=Z(G)Q$, where Q is an extra-special q-subgroup of order q^{2r+1} and $Z(Q)=Q\cap Z(G)$; and $|P|=1+q^r$ (hence $r=1$ if $p=2$ and $|P|=9$ or p if $q=2$). Furthermore, if G is not solvable, then $q=2$, $|P|=1+2^r$, and $H/O_{p'}(H)\approx PSP(2n,2^m)$, $PSO^-(2n,2^m)'$, or $PSL(2,p)$, where $r=nm$.*

(4) *$|P|=p=1+2^n$, $n\geqslant 3$; $G\approx SL(2,2^n)\times Z(G)$; and $\chi(1)\geqslant p-2$.*

(5) *χ is irreducible of degree $|P|-1$; $H/Z(H)$ is isomorphic to $PSL(3,4)$ for $|P|=9$ or* Aut($Sz(32)$) *for $|P|=125$.*

(6) *χ is irreducible of degree $|P|-1$; P is cyclic; $H/Z(H)$ is isomorphic to $PSL(n,q)$ with $n\geqslant 3$ and $|P|=(q^n-1)/(q-1)$; $PSU(n,q)$ with n odd; and $|P|=(q^n+1)/(q+1)$ or $PSP(2n,q)$ with $|P|=(q^n+1)/2$.*

(7) *$H/Z(H)$ is isomorphic to $PSL(2,q)$ with $|P|=(q+1)/(2,q-1)$, q, or $q+1$ and $\chi(1)\geqslant(q-1)/(2,q-1)$; $PSU(3,q)$ with $|P|=q^3$, $q>2$, and $\chi(1)\geqslant q(q-1)$; ${}^2G_2(q)$ with $|P|=q^3$ and $\chi(1)\geqslant q^2-q+1$; ${}^2B_2(q)$ with $|P|=q^2$, $q=2^{2m+1}$, and $\chi(1)\geqslant\sqrt{q/2}\,(q-1)$ or Mc with $|P|=125$ and $\chi(1)\geqslant 22$.*

(8) *$|P|=p$; χ is irreducible of degree $p-1$; $H/Z(H)$ is isomorphic to M_{11}, M_{12}, or M_{22} for $p=11$; $G_2(4)$ or Suz for $p=13$; J_3 for $p=19$; M_{23} for $p=23$; and Ru for $p=29$ or A_p.*

Proof. Suppose the theorem is not true and let G be a counterexample of minimal order. We argue toward a contradiction.

(I) $O_p(G)=1$. Obviously.

(II) P is not generalized quaternion and $PO_{p'}(G)=P\times O_{p'}(G)$. Set $T=PO_{p'}(G)$. If P is not normal in T then $O_p(T)=1$. Now Lemma 2 holds for T. If P is generalized quaternion, then $\chi(1)\geqslant|P|/2-1$. By the Feit–Thompson theorem and Suzuki's theorem [22, Theorem 3], G is solvable. This shows that case (2) of the theorem is true for G, contrary to the assumption on G. If G is solvable, then case (3) of the theorem holds for G by Lemma 2, which is a contradiction. Therefore G is not solvable, $|P|=1+2^r$, and P is cyclic. It follows that $D:=H/O_{p'}(H)$ is nonabelian simple. Set $R(D)=\min\{m\mid D$ is contained in $PGL(m,\mathbb{C})\}$ and $M(D)=\min\{m\mid a$ finite extension of D is contained in $PGL(m,\mathbb{C})\}$, where $\mathbb{C}$ is the field of complex numbers. Let Q be a Sylow 2-subgroup of $O_{p'}(G)$; then P acts irreducibly on $Q/Z(Q)$ and $Q/Z(Q)$ is a symplectic space of dimension $2r$ over the field of two elements. Hence D is contained in $SP(2r, 2)$ and $M(D) \leqslant 2^r$ by [23]. If $M(D) = R(D)$, $R(D) \leqslant 2^r \leqslant |P| - 1$. Thus D is isomorphic to a group listed in (4)–(8) of Theorem 3 (see the proof of (VI) below). Let v be a power of a prime. It is easy to verify that $(v^k - 1)/(v - 1) \neq 1 + 2^r$ $(k \geqslant 3)$, $(v^k + 1)/(v + 1) \neq 1 + 2^r$ $(k \geqslant 3)$, and $(v^k + 1)/2 = 1 + 2^r$ implies that $k = 1$ and $v = 17$ (note that $r \geqslant 3$ and $|P| = 1 + 2^r$). Since P is cyclic, $D \approx SL(2, 2^r)$, $PSL(2, p)$, A_p, $PSL(2, 17)$ $(r = 3)$, or $PSL(3, 4)$ $(r = 3)$. By the assumption on G and since $SL(2, 2^r) \approx PSP(2, 2^r)$, D is not isomorphic to $SL(2, 2^r)$ or $PSL(2, p)$. Since $PSL(2, 17)$ and $PSL(3, 4)$ are not contained in $SP(6, 2)$, $D \approx A_p$. Hence $((1 + 2^r)!)_{2'} = (p!)_{2'} = |A_p|_{2'} \leqslant |SP(2r, 2)|_{2'} = \prod_{i=1}^{r} (2^i - 1)(2^i + 1) < ((1 + 2^r)!)_{2'}$, a contradiction. So $M(D) < R(D)$. Then D is isomorphic to $PSP(2n, 2^m)$ or $PSO^-(2n, 2^m)'$ with $r = nm$ by [23], which is again a contradiction.

(III) $G = H = P^G := \langle P^x \mid x \in G \rangle$. It is clear that $H = P^G$. If H is a proper subgroup of G, then the theorem is true for H by the minimality of G. By (I) and (II), $O_{p'}(H) = Z(H)$. Since $O^{p'}(H) = H$, $H/Z(H)$ is isomorphic to a group listed in (4)–(8) of the theorem. This implies that Theorem 3 is true for G, contrary to the assumption on G.

(IV) $G/Z(G)$ is nonabelian simple. Now $O_{p'}(G) = Z(G)$. Set $\bar{G} = G/Z(G)$. Then $\bar{P}$ is a T.I. set of $\bar{G}$ with $O_p(\bar{G}) = 1$. Let M be a normal subgroup of G such that $Z(G) \leqslant M$ and $\bar{M}$ is a minimal normal subgroup of $\bar{G}$. It follows that $\bar{M}$ is a direct product of isomorphic nonabelian simple groups. Clearly $p \mid |\bar{M}|$ and $\bar{P} \cap \bar{M}$ is a T.I. set of $\bar{M}$. So $\bar{M}$ is nonabelian simple. By the Frattini argument, $\bar{G} = N_{\bar{G}}(\bar{P} \cap \bar{M})\bar{M}$. Since $\bar{P}$ is normalized by $N_{\bar{G}}(\bar{P} \cap \bar{M})$, $\bar{P}\bar{M}$ is normal in $\bar{G}$. By (III), $G = PM$. If $P \leqslant M$ then $\bar{G}$ is nonabelian simple. If P is not cointained in M, then $p = 3$ and $\bar{G} \approx \mathrm{Aut}(PSL(2, 8))$ or $p = 5$ and $\bar{G} \approx \mathrm{Aut}(Sz(32))$ by [12]. By the atlas [15], (5) of the theorem is true for G if $p = 5$. So $p = 3$. Since $\mathrm{Aut}(PSL(2, 8)) \approx {}^2G_2(3)$, (7) of the theorem is true for G, which contradicts the assumption on G.

(V) P is cyclic. If P is not cyclic, by [9], $\bar{G}$ is isomorphic to one of the following groups:

(a) $PSL(2, p^n)$, $n \geqslant 2$, $PSU(3, p^n)$.

(b) ${}^2B_2(q)$, $q = 2^{2m+1}$, $p = 2$.

(c) ${}^2G_2(q)$, $q = 3^{2m+1}$, $m \geqslant 1$, $PSL(3, 4)$, or M_{11} for $p = 3$.

(d) ${}^2F_4(2)'$ or Mc for $p = 5$.

(e) J_4 for $p = 11$.

If $\bar{G} \approx PSL(2, p^n)$, then $|P| = p^n$, $\chi(1) \geqslant (|P| - 1)/2$. If $\bar{G} \approx PSU(3, p^n)$, then $|P| = p^{3n}$, $\chi(1) \geqslant p^n(p^n - 1)$ [16]. If $\bar{G} \approx {}^2B_2(q)$, $|P| = q^2$, $\chi(1) \geqslant$

$\sqrt{q/2}\,(q-1)$. If $\bar{G}\approx {}^2G_2(q)$, then $|P|=q^3$, $\chi(1)\geqslant q(q-1)+1$. If $\bar{G}\approx Mc$, then $|P|=125$, $\chi(1)\geqslant 22$. If $\bar{G}\approx PSL(3,4)$, then $|P|=9$, χ is irreducible of degree 8. These show that the theorem is true for G, contrary to the assumption on G. If $\bar{G}\approx M_{11}$ with $|P|=9$, by the atlas [15], $\chi(1)\geqslant 10$. Then $8=9-1\geqslant\chi(1)\geqslant 10$. It is absurd. If $\bar{G}\approx {}^2F_4(2)'$ with $p=5$, then $|P|=25$, $\chi(1)\geqslant 26$. But $|P|>\chi(1)$; this is absurd. Therefore $\bar{G}\approx J_4$ with $p=11$, $|P|=11^3$. By the atlas, $\chi(1)\geqslant 1333$, $11^3-1=1330\geqslant\chi(1)\geqslant 1333$, a contradiction. The contradiction proves that P is cyclic.

(VI) Last contradiction. Now P is cyclic. If χ is reducible or of degree less than $|P|-1$, then $\chi=\sum_i \chi_i$, where $\chi_i\in \mathrm{Irr}(G)$, $\chi_i(1)<|P|-1$. Since $G/Z(G)$ is nonabelian simple, there exists χ_i such that $\ker\chi_i\leqslant Z(G)$. By [7], $\tilde{G}\approx SL(2,2^n)\times Z(\tilde{G})$ with $|P|=p=2^n+1$ or $SL(2,8)\times Z(\tilde{G})$ with $|P|=9$ or $PSL(2,p)$, where $\tilde{G}=G/\ker\chi_i$. This shows that (4) or (7) of the theorem holds for G, a contradiction! Hence we may assume that χ is irreducible of degree $|P|-1$.

In the following, we set $\bar{G}=G/Z(G)$.

If $\bar{G}$ is isomorphic to A_n with $n\geqslant 5$, then $n=p$ by [17] and [18]. If $\bar{G}$ is a sporadic simple group. $\bar{G}\approx M_{11}$, M_{12}, or M_{22} for $p=11$; Suz for $p=13$; J_3 for $p=19$; M_{23} for $p=23$; or Ru for $p=29$ by [15], which is a contradiction. So $\bar{G}$ is isomorphic to a simple group of Lie type by the classification of finite simple groups.

Suppose $\bar{G}$ is of characteristic r. By the atlas [15] and the assumption on G, $\bar{G}$ is not isomorphic to any one of the following groups:

$$PSL(2,4),\ PSL(2,9),\ PSL(3,2),\ PSL(3,4),\ PSP(6,2),\ PSU(4,2),$$
$$PSU(4,3),\ PSO^+(8,2),\ PSO(7,3),\ F_4(2),\ S_z(8),\ G_2(4),\ G_2(3).$$

If $p=r$, then $\bar{G}\approx PSL(2,p)$ since P is cyclic; see [6] for reference. Then our minimality assumption is contradicted. So $p\neq r$. Set $q=r^t$. If $\bar{G}\approx PSL(2,q)$, by [16], $\chi(1)\geqslant q-1/d$, $d=(2,q-1)$. Then $|P|-1\geqslant q-1/d$, $|P|\geqslant q+1/d$. It follows that $|P|\,|\,q+1$ and $|P|=q+1/d$, $\chi(1)\geqslant q-1/2$, contrary to the assumption on G.

If $\bar{G}\approx PSL(n,q)$, $n\geqslant 3$, by [16], $\chi(1)\geqslant q^{n-1}-1$, $|P|\geqslant q^{n-1}$. Then $|P|\,|\,(q^n-1)$ by Lemma 1. It follows that $|P|=q^n-1/(q-1)$, which contradicts the assumption on G.

If $\bar{G}\approx PSP(2n,q)$, then by [16], $\chi(1)\geqslant(q-1)(q^{n-1}-1)\,q^{n-1}/2$ for q even or $q^n-1/2$ for q odd. If q is even, $|P|\geqslant\chi(1)+1>q^n+1$. By Lemma 1, $p\nmid|\bar{G}|$, contrary to the assumption on G and P. If q is odd, $\chi(1)\geqslant q^n-1/2$, $|P|\geqslant q^n+1/2$, $|P|\,|\,(q^n+1)$. Therefore $|P|=q^n+1/2$, another contradiction!

If $\bar{G}\approx PSU(n,q)$ then $\chi(1)\geqslant q(q^{n-1}-1)/(q+1)$ for n odd or $(q^n-1)/(q+1)$ for n even by [16]. If $\chi(1)\geqslant q(q^{n-1}-1)/(q+1)$, then

$|P| \geqslant q(q^{n-1}-1)/(q+1)+1=(q^n+1)/(q+1)$. By Lemma 1, $|P|=(q^n+1)/(q+1)$. We again have a contradiction. If $\chi(1) \geqslant (q^n-1)/(q+1)$ with n even, then $|P| \geqslant q(q^{n-1}+1)/(q+1)$. Since the r'-part of $|PSU(n,q)|$ is $(q^2-1)(q^3+1)\cdots(q^{n-1}+1)(q^n-1)$, $|P| \mid (q^i \pm 1)$ for some integer $i \leqslant n-1$ by Lemma 1. Hence $|P| \leqslant (q^{n-1}+1)/(q+1)$, $q(q^{n-1}+1)/(q+1) \leqslant (q^{n-1}+1)/(q+1)$. It is absurd.

If $\bar{G} \approx PSO^{\pm}(2n,q)'$ $(n \geqslant 4)$ or $PSO(2n+1,q)'$ $(n \geqslant 3$ and q is odd), then $\chi(1) \geqslant q^n+1$ by [16]. So $|P| \geqslant q^n+2$. By Lemma 1, $p \nmid |\bar{G}|$, a contradiction.

If $\bar{G} \approx F_4(q)$ with $q>2$, $\chi(1) \geqslant q^4(q^6-1)$ if q is odd or $q^7(q^3-1)(q-1)/2$ if q is even by [16]. So $|P| \geqslant q^9+1$, Since $|F_4(q)|=q^{24}(q^{12}-1)(q^8-1)(q^6-1)(q^2-1)$ and P is cyclic, $p \nmid (q^2-1)$. Let e be the smallest positive integer such that $p \mid (q^e-1)$. Then $e=3, 4, 6, 8$, or 12. If $e=3$ or 6, $|P| \leqslant (q^2+q+1)^2 < q^6$, contrary to that $|P| \geqslant q^9+1$. If $e=4, 8$, or 12, $|P| \leqslant (q^2+1)^2 < q^6$, which is again a contradiction.

If $\bar{G} \approx {}^2E_6(q)$, then $\chi(1) \geqslant q^8(q^4+1)(q^3-1)$ for $q>2$ or $3 \cdot 2^9$ for $q=2$ by [16]. If $q>2$, $|P| \geqslant q^{14}+1$. Since $|{}^2E_6(q)|=q^{36}(q^{12}-1)(q^9+1)(q^8-1)(q^6-1)(q^5+1)\cdot(q^2-1)$ and P is cyclic, $p \nmid (q^2-1)$. Let e be the number defined as above; then $e=3, 4, 6, 8, 10, 12$, or 18. If $e=8, 10, 12$, or 18, $|P| \leqslant q^6+1$. This is impossible since $|P| \geqslant q^{14}+1$. If $e=3, 4, 6$, then $|P| \leqslant (q^2+q+1)^3 < q^9$. This is again impossible. So $q=2$. Then $|P| \leqslant (2+1)^6/(3, 2+1)=3^5$. It follows that $3 \cdot 2^9 \leqslant \chi(1) \leqslant |P|-1 \leqslant 3^5$. It is absurd.

Similar arguments yield that $\bar{G}$ is not isomorphic to $E_i(q)$, $i=6, 7$, or 8.

If $\bar{G} \approx G_2(q)$ with $q \leqslant 4$, then $q=3$ or 4. By the atlas [15], $q=4$ and $p=13$. This contradicts the assumption on G. If $\bar{G} \approx G_2(q)$ with $q>4$, then $\chi(1) \geqslant q(q^2-1)$ by [16]. $|P| \geqslant q^3-q+1$. Since the order of $G_2(q)$ is $q^6(q^6-1)(q^2-1)$, $|P| \leqslant 3(q+1)^2$ if $p \mid (q^2-1)$ and $|P| \leqslant q^2+q+1$ if $p \nmid (q^2-1)$. Then $q^3-q+1 \leqslant 3(q+1)^2$. But this implies that $q \leqslant 4$, contrary to that $q>4$.

If $\bar{G} \approx {}^3D_4(q)$, then $\chi(1) \geqslant q^3(q^2-1)$ by [16]. $|P| \geqslant q^5-q^3+1$. The order of ${}^3D_4(q)$ is $q^{12}(q^8+q^4+1)(q^6-1)(q^2-1)$. Then $|P| \leqslant 9(q+1)^2$ for $p \mid (q^2-1)$, $(q^2+q+1)^2$ for $p \mid (q^2 \pm q+1)$, or q^4-q^2+1 for $p \mid (q^4-q^2+1)$. So $q^5-q^3+1 \leqslant 9(q+1)^2$, $(q^2+q+1)^2$, or q^4-q^2+1. The inequalities imply $q=2$. By the character table of ${}^3D_4(q)$ [15], this is impossible.

If $\bar{G}$ is isomorphic to ${}^2G_2(q)$ $(q=3^{2m+1}$ with $m \geqslant 1)$, then $\chi(1) \geqslant q(q-1)$ by [16]. Therefore $|P| \geqslant q(q-1)+1=q^2-q+1$. Since $|{}^2G_2(q)|=q^3(q^3+1)(q-1)$, $|P| \mid (q^2-q+1)$, $q+1$, or $q-1$. So $|P|=q^2-q+1$. But $q^2-q+1=(q-\sqrt{3q}+1)(q+\sqrt{3q}+1)$, which is a contradiction.

If $\bar{G} \approx S_z(q)$, $q=2^{2m+1}$, $m \geqslant 2$, then $\chi(1) \geqslant \sqrt{q/2}(q-1)$ by [16]. $|P| \geqslant \sqrt{q/2}(q-1)+1$. Since the order of $S_z(q)$ is $q^2(q^2+1)(q-1)$ and $q^2+1=(q-\sqrt{2q}+1)(q+\sqrt{2q}+1)$, $|P| \leqslant q+\sqrt{2q}+1$, $\sqrt{q/2}(q-1)+1 \leqslant$

$q+\sqrt{2q}+1$. $q-1 \leqslant \sqrt{2q}+2$, $2^{2m+1} \leqslant 2^{m+1}+3$. So $2^{m+1}(2^m-1) \leqslant 3$. It follows that $m=0$, which is a contradiction.

If $\bar{G} \approx {}^2F_4(2)'$, then $\chi(1) \geqslant 26$ and $p=13$ by the atlas [15]. It follows that $26 \leqslant |P|-1 \leqslant 12$. It is absurd! If $\bar{G} \approx {}^2F_4(q)$, $q=2^{2m+1}$, $m \geqslant 1$, then $\chi(1) \geqslant \sqrt{q/2}\, q^4(q-1)$. Since the order of ${}^2F_4(q)$ is $q^{12}(q^6+1)(q^4-1)(q^3+1)(q-1)$, $|P| \leqslant (q^2+1)^2$. Then it follows that $\sqrt{q/2}\, q^4(q-1) \leqslant |P|-1 \leqslant (q^2+1)^2-1$. This implies that $3q^2 \leqslant 2$. This is impossible. The contradictions prove the theorem.

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