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Casimir–Polder potential for a metallic cylinder in cosmic string spacetime

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ABSTRACT

Casimir–Polder potential is investigated for a polarizable microparticle in the geometry of a straight cosmic string with a metallic cylindrical shell. The electromagnetic field Green tensor is evaluated on the imaginary frequency axis. The expressions for the Casimir–Polder potential is derived in the general case of anisotropic polarizability for the both interior and exterior regions of the shell. The potential is decomposed into pure string and shell-induced parts. The latter dominates for points near the shell, whereas the pure string part is dominant near the string and at large distances from the shell. For the isotropic case and in the region inside the shell the both pure string and shell-induced parts in the Casimir–Polder force are repulsive with respect to the string. In the exterior region the shell-induced part of the force is directed toward the cylinder whereas the pure string part remains repulsive with respect to the string. At large distances from the shell the total force is repulsive.

1. Introduction

The production of cosmic strings in symmetry breaking phase transitions during the evolution of the early universe is predicted by a wide class of particle physics models [1]. The considerable attention attracted by this class of topological defects was motivated by the fact that the cosmic strings are candidates for the generation of a variety of interesting physical effects. The latter include gravitational lensing, anisotropies in the cosmic microwave background radiation, the generation of gravitational waves, high-energy cosmic rays, and gamma ray bursts. More recently it has been shown that cosmic strings form in brane inflation models as a by product of the annihilation of the branes (for a review see [2]).

In quantum field theory, the non-trivial topology of space around a cosmic string results in the distortion of the vacuum fluctuations of quantized fields. This induces non-zero vacuum expectation values for physical observables such as the field squared and the energy–momentum tensor. In a previous paper [3], we have shown that the distortion of the vacuum fluctuations spectrum by the cosmic string also gives rise to Casimir–Polder forces acting on a polarizable microparticle (see also [4] for the force in the static limit). These forces have attracted a great deal of attention because of their important role in many areas of science, including material sciences, physical chemistry, nanotechnology, and atom optics (for reviews see [5]). In [3] it has been shown that, in dependence on the eigenvalues for the polarizability tensor and of the orientation of its principal axes, the Casimir–Polder force induced by the string can be either repulsive or attractive. For an isotropic polarizability tensor the force is always repulsive.

Another source for the vacuum polarization is the presence of material boundaries. The boundary conditions imposed on a quantum field alter the zero-point oscillations spectrum and lead to additional shifts in the vacuum expectation values. Combined effects of topology and boundaries on the quantum vacuum in the geometry of a cosmic string have been investigated previously for scalar [6], electromagnetic [7,8] and fermionic fields [9], constrained on a cylindrical boundary coaxial with the cosmic string. The analysis of the vacuum energy for massless scalar fields subject to Dirichlet, Neumann and hybrid boundary conditions in the setting of the conical piston has been recently developed in [10]. The vacuum polarization effects in a cosmic string spacetime induced by a scalar field obeying Dirichlet or Neumann boundary conditions on a surface orthogonal to the string are considered in [11].

In the present Letter we derive the exact Casimir–Polder (CP) potential for the general case of frequency dependent anisotropic polarizability of a microparticle in the geometry of straight cosmic string with a coaxial conducting cylindrical shell. From the point of view of the physics in the region outside the string, this geometry can be considered as a simplified model for the non-trivial core. This model presents a framework in which the influence of the finite core effects on physical processes in the vicinity of the cosmic string can be investigated. The corresponding results may shed light upon features of finite core effects in more realistic models. In addition, the problem considered here is of interest as an example with combined topological and boundary-induced quantum effects

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in which the CP forces can be found in closed form. The CP interaction potential of a microparticle with an ideal metal cylindrical shell in background of Minkowski spacetime has been investigated in a number of papers, in particular, related to applications to carbon nanotubes (see, for instance, [12–16] and references therein). Recently, the exact potential for a microparticle outside a cylindrical shell has been found in [13] using the Hamiltonian approach. The CP potential for both regions inside and outside an ideal metal cylindrical shell is investigated in [15] using the Green function method. In this Letter the exact quantum field theoretical result is compared with that obtained using the proximity force approximation and a very good agreement is demonstrated. In [14] it was shown that for a particle out of thermal equilibrium with its environment inside a cylindrical cavity the CP potential can be enhanced by fine-tuning the cavity radius to resonate with the particle's internal transition wavelength.

We have organized the Letter as follows. In the next section we evaluate the Green tensor in the frequency domain in the region inside a conducting cylindrical shell in the geometry of a cosmic string. By using the generalized Abel–Plana summation formula, the Green tensor is decomposed into the boundary-free and boundary-induced parts. The corresponding CP potential is investigated in Section 3 for the general case of anisotropic polarizability. The Green tensor and the CP potential for the region outside a cylindrical shell are considered in Section 4. Similar to the interior region, these quantities are presented as the sum of pure string and shell-induced parts. Section 5 summarizes the main results of the Letter.

2. Electromagnetic field Green tensor inside a cylindrical shell

In the cylindrical coordinates $(x^1, x^2, x^3) = (r, \phi, z)$, the geometry of an idealized infinitely long straight cosmic string is described by the line element

$$ds^2 = dt^2 - dr^2 - r^2\,d\phi^2 - dz^2, \tag{1}$$

where $0 \leqslant r < \infty$, $-\infty < z < +\infty$, $0 \leqslant \phi \leqslant \phi_0$ and the spatial points (r, ϕ, z) and $(r, \phi + \phi_0, z)$ are identified. The planar angle deficit is related to the mass μ_0 per unit length of the string by $2\pi - \phi_0 = 8\pi G\mu_0$, with G being the Newton gravitational constant. (Effective metric with a planar angle deficit also arises in a number of condensed matter systems (see, for instance, [17]).) In addition, we shall assume the presence of a coaxial metallic cylindrical shell of radius a.

The non-trivial topology due to the cosmic string and the boundary conditions imposed for the electric and magnetic fields on the cylindrical shell change the structure of the zero-point fluctuations of the electromagnetic field. In particular, a neutral polarizable microparticle experiences a dispersion force, the CP force. For a particle with the polarizability tensor $\alpha_{jl}(\omega)$, the corresponding interaction potential is expressed in terms of the subtracted Green tensor as (see [5])

$$U(\mathbf{r}) = \frac{1}{2\pi}\int_0^\infty d\xi\,\alpha_{jl}(i\xi)G^{(s)}_{jl}(\mathbf{r}, \mathbf{r}; i\xi), \tag{2}$$

where $\mathbf{r}$ is the location of the microparticle and summation is understood over the indices $j, l = 1, 2, 3$. In (2),

$$G^{(s)}_{jl}(\mathbf{r}, \mathbf{r}'; \omega) = \int_{-\infty}^{+\infty} d\tau\,\big[G_{jl}(x, x') - G^{(\mathrm{M})}_{jl}(x, x')\big]e^{i\omega\tau}, \tag{3}$$

where $G_{jl}(x, x')$, with $x = (t, \mathbf{r})$, $x' = (t', \mathbf{r}')$, $\tau = t - t'$, is the retarded Green tensor for the electromagnetic field in the geometry of a cosmic string with the cylindrical shell and $G^{(\mathrm{M})}_{jl}(x, x')$ is the corresponding tensor in the boundary-free Minkowski spacetime. The geometry of a cosmic string is flat outside the string core and the renormalization procedure is reduced to the subtraction of the Minkowskian part.

For the evaluation of the Green tensor in (3) we use the direct mode summation method. Let $\{\mathbf{E}_\alpha(x), \mathbf{E}^*_\alpha(x)\}$ be a complete set of normalized mode functions for the electric field, specified by a collective index α. For the Green tensor we have the following mode sum formula:

$$G_{jl}(x, x') = -i\theta(\tau)\sum_\alpha\big[E_{\alpha j}(x)E^*_{\alpha l}(x') - E_{\alpha l}(x')E^*_{\alpha j}(x)\big], \tag{4}$$

where $\theta(\tau)$ is the unit-step function and the indices $j, l = 1, 2, 3$ correspond to the coordinates r, ϕ, z, respectively.

First we consider the region inside the cylindrical shell. In the problem under consideration we have two classes of mode functions corresponding to the cylindrical waves of the transverse magnetic (TM, $\lambda = 0$) and transverse electric (TE, $\lambda = 1$) types. The mode functions for the electric field are obtained from the corresponding functions for the vector potential given in Ref. [8] and they have the form

$$\mathbf{E}^{(\lambda)}_\alpha(x) = \beta_\alpha\mathbf{E}^{(\lambda)}(r)e^{iqm\phi+ikz-i\omega t}, \tag{5}$$

where $m = 0, \pm1, \pm2, \ldots$, $-\infty < k < +\infty$, $\omega = \sqrt{\gamma^2 + k^2}$, and

$$q = 2\pi/\phi_0. \tag{6}$$

The radial functions $E^{(\lambda)}_l(r)$ in (5) are given by the expressions

$$\begin{aligned}
&E^{(0)}_1(r) = ik\gamma\,J'_{q|m|}(\gamma r), \qquad E^{(0)}_2(r) = -\frac{kqm}{r}J_{q|m|}(\gamma r),\\
&E^{(0)}_3(r) = \gamma^2 J_{q|m|}(\gamma r),\\
&E^{(1)}_1(r) = -\frac{\omega qm}{r}J_{q|m|}(\gamma r), \qquad E^{(1)}_2(r) = -i\omega\gamma\,J'_{q|m|}(\gamma r),\\
&E^{(1)}_3(r) = 0,
\end{aligned} \tag{7}$$

where $J_\nu(x)$ is the Bessel function, the prime means the derivative with respect to the argument of the function. From the standard boundary conditions for the electric and magnetic fields on the cylindrical boundary with radius a, we can see that the eigenvalues for the quantum number γ are roots of the equations

$$J^{(\lambda)}_{q|m|}(\gamma a) = 0, \quad \lambda = 0, 1, \tag{8}$$

where $J^{(0)}_\nu(x) = J_\nu(x)$ and $J^{(1)}_\nu(x) = J'_\nu(x)$. In the discussion below the corresponding eigenmodes are denoted by $j^{(\lambda)}_{m,n} = \gamma a$, $n = 1, 2, \ldots$. As a result the set of quantum numbers specifying the eigenfunctions is given by $\alpha = (k, m, \lambda, n)$. The normalization coefficient in (5) is given by the expression [8]

$$\beta^2_\alpha = \frac{qT_{q|m|}(\gamma a)}{\pi\omega a\gamma}, \qquad T_\nu(x) = \frac{x}{J'^2_\nu(x) + (1 - \nu^2/x^2)J^2_\nu(x)}. \tag{9}$$

Substituting the mode functions (5) into the mode sum formula (4), the following representation is obtained for the Green tensor on the imaginary frequency axis:

$$G_{jl}(\mathbf{r},\mathbf{r}';i\xi)=-\frac{q}{\pi}\sum_{m=-\infty}^{+\infty}\int_{-\infty}^{+\infty}dk\sum_{\lambda=0,1}\sum_{n=1}^{\infty}\frac{T_{q|m|}(j_{m,n}^{(\lambda)})}{j_{m,n}^{(\lambda)}\omega_{m,n}^{(\lambda)}}\times\left[E_j^{(\lambda)}(r)E_l^{(\lambda)*}(r')\frac{e^{iqm\Delta\phi+ik\Delta z}}{\omega_{m,n}^{(\lambda)}-i\xi}+E_l^{(\lambda)}(r')E_j^{(\lambda)*}(r)\frac{e^{-iqm\Delta\phi-ik\Delta z}}{\omega_{m,n}^{(\lambda)}+i\xi}\right],\tag{10}$$

where $\Delta\phi=\phi-\phi'$ and $\Delta z=z-z'$, $\omega_{m,n}^{(\lambda)}=\sqrt{j_{m,n}^{(\lambda)2}/a^2+k^2}$. For the summation of the series over n we apply the formula [18]

$$\sum_{n=1}^{\infty}T_{q|m|}(j_{m,n}^{(\lambda)})f(j_{m,n}^{(\lambda)})=\frac{1}{2}\int_0^{\infty}dx\,f(x)-\frac{\pi i}{2}\sum_p\operatorname*{Res}_{z=iy_p}f(z)\frac{H_{q|m|}^{(1,\lambda)}(z)}{J_{q|m|}^{(\lambda)}(z)},\tag{11}$$

where $z=\pm iy_p$, $y_p>0$, $p=1,2,\ldots,$ are poles of the function $f(z)$ and $H_\nu^{(s,0)}(z)=H_\nu^{(s)}(z)$, $H_\nu^{(s,1)}(z)=H_\nu^{(s)\prime}(z)$, with $H_\nu^{(s)}(z)$, $s=1,2$, being the Hankel functions. In (11), it is assumed that $f(z)$ is an analytic function for $\operatorname{Re}z>0$ and obeys the conditions $f(ye^{\pi i/2})=-e^{2q|m|\pi i}f(ye^{-\pi i/2})$ and $|f(x+iy)|<\varepsilon(x)e^{by}$, $b<2$, for $y\to\infty$, with $\varepsilon(x)\to 0$ for $x\to\infty$. For the poles of the function $f(z)$, corresponding to the series in (10), one has $y_p=a\sqrt{k^2+\xi^2}$. The part of the Green tensor corresponding to the first term on the right-hand side of (11) is the Green tensor in the boundary-free cosmic string geometry. The latter will be denoted by $G_{jl}^{(0)}(\mathbf{r},\mathbf{r}';i\xi)$. The boundary-induced part of the Green tensor corresponds to the second term.

As a result, by using (11), the Green tensor is presented in the decomposed form

$$G_{jl}(\mathbf{r},\mathbf{r}';i\xi)=G_{jl}^{(0)}(\mathbf{r},\mathbf{r}';i\xi)+G_{jl}^{(\mathrm{b})}(\mathbf{r},\mathbf{r}';i\xi).\tag{12}$$

The boundary-induced part is given by the formula

$$G_{jl}^{(\mathrm{b})}(\mathbf{r},\mathbf{r}';i\xi)=-\frac{q}{\pi}\sum_{m=-\infty}^{\infty}e^{iqm\Delta\phi}\sum_{\lambda=0,1}(-\xi^2)^{\lambda}\int_{-\infty}^{\infty}dk\,e^{ik\Delta z}\times k^{2(1-\lambda)}\frac{K_{q|m|}^{(\lambda)}(a\gamma)}{I_{q|m|}^{(\lambda)}(a\gamma)}i_j^{(\lambda)}(\gamma r,\gamma/k)i_l^{(\lambda)*}(\gamma r',\gamma/k),\tag{13}$$

where in the integrand $\gamma=\sqrt{k^2+\xi^2}$. In the last expression, $I_\nu(x)$ and $K_\nu(x)$ are the modified Bessel functions, $F_\nu^{(0)}(x)=F_\nu(x)$, $F_\nu^{(1)}(x)=F_\nu'(x)$ for $F=I,K$, and the functions $i_l^{(\lambda)}(x,y)$ are defined as

$$\begin{aligned}&i_1^{(0)}(x,y)=I_{q|m|}'(x),\qquad i_2^{(0)}(x,y)=i\frac{qm}{x}I_{q|m|}(x),\\&i_3^{(0)}(x,y)=iyI_{q|m|}(x),\\&i_1^{(1)}(x,y)=\frac{qm}{x}I_{q|m|}(x),\qquad i_2^{(1)}(x,y)=-iI_{q|m|}'(x),\\&i_3^{(1)}(x,y)=0.\end{aligned}\tag{14}$$

In accordance with (2), for the evaluation of the CP potential we need the expression of the boundary-induced part in the coincidence limit. In this limit the off-diagonal components vanish and for the diagonal components we have

$$G_{ll}^{(\mathrm{b})}(\mathbf{r},\mathbf{r};i\xi)=-\frac{4q}{\pi}\sideset{}{'}\sum_{m=0}^{\infty}\sum_{\lambda=0,1}(-\xi^2)^{\lambda}\times\int_{\xi}^{\infty}d\gamma\,\gamma\frac{K_{qm}^{(\lambda)}(a\gamma)}{I_{qm}^{(\lambda)}(a\gamma)}\frac{|i_l^{(\lambda)}(\gamma r,\gamma/\sqrt{\gamma^2-\xi^2})|^2}{(\gamma^2-\xi^2)^{\lambda-1/2}},\tag{15}$$

where the prime on the summation sign means that the term $m=0$ should be taken with the coefficient $1/2$.

3. Casimir–Polder potential

On the base of Eq. (12), the CP potential in the presence of the cylindrical shell is decomposed as:

$$U(r)=U_0(r)+U_{\mathrm{b}}(r),\tag{16}$$

where $U_0(r)$ is the potential for the geometry of a cosmic string without boundaries and the term $U_{\mathrm{b}}(r)$ is due the presence of the cylindrical shell. The pure string part is investigated in [3] and here we will be mainly concerned with the boundary-induced part. The latter is given by the expression

$$U_{\mathrm{b}}(r)=\frac{1}{2\pi}\int_0^{\infty}d\xi\,\alpha_{jl}(i\xi)G_{jl}^{(\mathrm{b})}(\mathbf{r},\mathbf{r};i\xi).\tag{17}$$

As it has been mentioned before, the off-diagonal components of the boundary induced part of the Green tensor in (17) vanish. As a result, by taking into account the expression (15), we get

$$U_{\mathrm{b}}(r)=-\frac{2q}{\pi^2}\sideset{}{'}\sum_{m=0}^{\infty}\sum_{\lambda=0,1}\sum_{l=1}^{3}\int_0^{\infty}d\xi\,\alpha_{ll}(i\xi)(-\xi^2)^{\lambda}\times\int_{\xi}^{\infty}d\gamma\,\gamma\frac{K_{qm}^{(\lambda)}(a\gamma)}{I_{qm}^{(\lambda)}(a\gamma)}\frac{|i_l^{(\lambda)}(\gamma r,\gamma/\sqrt{\gamma^2-\xi^2})|^2}{(\gamma^2-\xi^2)^{\lambda-1/2}}.\tag{18}$$

In the special case $q=1$ this formula reduces to the result of [15] for the CP interaction potential for a cylindrical shell in Minkowski spacetime.

In (18), $\alpha_{ll}(i\xi)$ are the physical components of the polarizability tensor in the cylindrical coordinates corresponding to line element (1). These components depend on the orientation of the polarizability tensor principal axes. As a consequence, the CP potential depends on the distance of the microparticle from the string and on the angles determining the orientation of the principal axes. Let us introduce Cartesian coordinates $x''^l=(x'',y'',z'')$ with the z''-axis along the string and with the particle location at $(r,0,0)$ and let β_{ln} be the cosine of the angle between x''^l and the nth principal axis of the polarizability tensor. One has $\sum_{n=1}^3\beta_{ln}^2=1$. Now we can write $\alpha_{ll}(\omega)=\sum_{n=1}^3\beta_{ln}^2\alpha_n(\omega)$, where $\alpha_n(\omega)$ are the principal values of the polarizability tensor. The coefficients β_{ln} can be expressed in terms of the Euler angles determining the orientation of the principal axes with respect to the coordinate system x''^l. In the isotropic case $\alpha_n(\omega)\equiv\alpha(\omega)$ and we have $\alpha_{ll}(\omega)=\alpha(\omega)$.

The boundary-induced part of the potential is finite on the string. Assuming that $q>1$, we can see that only the $m=0$ term contributes and

$$U_{\mathrm{b}}(0)=-\frac{q}{\pi^2}\int_0^{\infty}d\xi\,\alpha_{33}(i\xi)\int_{\xi}^{\infty}d\gamma\frac{\gamma^3}{\sqrt{\gamma^2-\xi^2}}\frac{K_0(a\gamma)}{I_0(a\gamma)}.\tag{19}$$

For the evaluation of the CP force we need also the next-to-leading order term near the string. For $q>2$, the dominant contribution comes from the $m=0$ term and the potential is given by the formula

$$U_b(r)\approx U_b(0)-\frac{qr^2}{2\pi^2}\int_0^\infty d\xi\,\alpha_{33}(i\xi)\int_\xi^\infty d\gamma\,\frac{\gamma^5}{\sqrt{\gamma^2-\xi^2}}\frac{K_0(a\gamma)}{I_0(a\gamma)}. \tag{20}$$

The corresponding CP force linearly vanishes on the string. In the case $1<q<2$, the dominant contribution to the next-to-leading order term comes from the $m=1$ term and for the potential near the string one has

$$U_b(r)\approx U_b(0)-\frac{qr^{2q-2}}{2^{2q-1}\pi^2\Gamma^2(q)}\int_0^\infty d\xi\left[\alpha_{11}(i\xi)+\alpha_{22}(i\xi)\right]\\ \times\int_\xi^\infty d\gamma\,\frac{\gamma^{2q-1}}{\sqrt{\gamma^2-\xi^2}}\left[\frac{K_q(a\gamma)}{I_q(a\gamma)}\left(\gamma^2-\xi^2\right)-\xi^2\frac{K_q'(a\gamma)}{I_q'(a\gamma)}\right]. \tag{21}$$

The corresponding CP force vanishes on the string for $q>1.5$ and diverges for $q<1.5$. The boundary-free part in the potential near the string behaves as r^{-3} and it dominates.

For the isotropic polarizability tensor the general expression (18) for the boundary-induced part in the CP potential takes the form

$$U_b(r)=-\frac{2q}{\pi^2}\sideset{}{'}\sum_{m=0}^\infty\int_0^\infty d\xi\,\alpha(i\xi)\int_\xi^\infty\frac{\gamma d\gamma}{\sqrt{\gamma^2-\xi^2}}\\ \times\left\{\frac{K_{qm}(a\gamma)}{I_{qm}(a\gamma)}\left[\left(\gamma^2-\xi^2\right)F_{qm}(\gamma r)+\gamma^2I_{qm}^2(\gamma r)\right]\right.\\ \left.-\xi^2\frac{K_{qm}'(a\gamma)}{I_{qm}'(a\gamma)}F_{qm}(\gamma r)\right\} \tag{22}$$

with the notation $F_{qm}(x)=I_{qm}'^2(x)+(qm/x)^2I_{qm}^2(x)$.

The CP potential diverges on the cylindrical shell. For points near the shell the dominant contribution comes from large values of m and we can use the uniform asymptotic expansions for the modified Bessel functions [19]. For the isotropic case, from (22), to the leading order one finds:

$$U_b(r)\approx-\frac{(r-a)^{-4}}{16\pi}\\ \times\int_0^\infty d\zeta\,\alpha\left(i\zeta/\left[2(r-a)\right]\right)e^{-\zeta}\left(\zeta^2+2\zeta+2\right). \tag{23}$$

The expression in the right-hand side coincides with the CP potential for a metallic plate in Minkowski spacetime, with $r-a$ being the distance from the plate.

For the further transformation of the CP potential the polarizability tensor should be specified. We use the anisotropic oscillator model:

$$\alpha_n(i\xi)=\sum_j\frac{g_j^{(n)}}{\omega_j^{(n)2}+\xi^2}, \tag{24}$$

where $\omega_j^{(n)}$ and $g_j^{(n)}$ are the oscillator frequencies and strengths, respectively. With this model, performing the integration in (18) we find

$$U_b(r)=-\frac{q}{\pi}\sideset{}{'}\sum_{m=0}^\infty\sum_{n=1}^3\sum_j g_j^{(n)}\sum_{\lambda=0,1}\int_0^\infty d\gamma\,\gamma\,\frac{K_{qm}^{(\lambda)}(a\gamma)}{I_{qm}^{(\lambda)}(a\gamma)}\\ \times\left[\sqrt{1+\gamma^2/\omega_j^{(n)2}}-1\right]f_{\lambda,qm}\left(\gamma r,\sqrt{1+\gamma^2/\omega_j^{(n)2}}\right), \tag{25}$$

where we have introduced the notations

$$f_{0,qm}(x,y)=\beta_{1n}^2I_{qm}'^2(x)+\beta_{2n}^2\left(\frac{qm}{x}\right)^2I_{qm}^2(x)\\ +\left(1+\frac{1}{y}\right)\beta_{3n}^2I_{qm}^2(x),\\ f_{1,qm}(x,y)=-\frac{1}{y}\left[\beta_{1n}^2\left(\frac{qm}{x}\right)^2I_{qm}^2(x)+\beta_{2n}^2I_{qm}'^2(x)\right]. \tag{26}$$

The coefficients β_{ln} depend on the orientation of the polarizability tensor principal axes with respect to the string.

In the isotropic case

$$U_b(r)=\frac{q}{\pi}\sideset{}{'}\sum_{m=0}^\infty\sum_j\frac{g_j}{\omega_j^2}\int_0^\infty d\gamma\,\gamma^3\left\{\frac{K_{qm}'(a\gamma)}{I_{qm}'(a\gamma)}\frac{F_{qm}(\gamma r)}{s_j(\gamma)[s_j(\gamma)+1]}\right.\\ \left.-\frac{K_{qm}(a\gamma)}{I_{qm}(a\gamma)}\left[\frac{F_{qm}(\gamma r)}{s_j(\gamma)+1}+\frac{I_{qm}^2(\gamma r)}{s_j(\gamma)}\right]\right\}, \tag{27}$$

with the notation

$$s_j(\gamma)=\sqrt{1+\gamma^2/\omega_j^2}. \tag{28}$$

For the boundary-induced part in the CP force we have $\mathbf{F}_b=F_{b,r}\mathbf{n}_r$, where $\mathbf{n}_r$ is the unit vector along the radial coordinate r and $F_{b,r}=-\partial_rU_b(r)$. Now, by using the inequality $I_{qm}'^2(x)\leqslant[1+(qm/x)^2]I_{qm}^2(x)$, from (27) it can be seen that $\partial_rU_b(r)<0$. Hence, in the isotropic case one has $F_{b,r}>0$ and the boundary-induced part in the CP force inside the cylindrical shell is directed toward the shell. The pure string part of the force has the same direction and the total force in the isotropic case is repulsive with respect to the string and attractive with respect to the shell.

4. Green tensor and the Casimir–Polder potential in the exterior region

In this section we consider the region outside the cylindrical boundary. The corresponding mode functions for the electric field are given by formulas (5) and (7) with the replacement (see [8])

$$J_{q|m|}(\gamma r)\to g_{q|m|}^{(\lambda)}(\gamma a,\gamma r)\\ =J_{q|m|}(\gamma r)Y_{q|m|}^{(\lambda)}(\gamma a)-Y_{q|m|}(\gamma r)J_{q|m|}^{(\lambda)}(\gamma a), \tag{29}$$

and with the normalization coefficient

$$\beta_\alpha^{-2}=\frac{2\pi}{q}\gamma\omega\left[J_{q|m|}^{(\lambda)2}(\gamma a)+Y_{q|m|}^{(\lambda)2}(\gamma a)\right]. \tag{30}$$

Here, as before, $\lambda=0,1$ correspond to the waves of the electric and magnetic types, respectively. Substituting the eigenfunctions into the corresponding mode-sum formula, for the retarded Green tensor we find:

$$G_{jl}(x,x')\\ =-i\frac{q\theta(\tau)}{2\pi}\sum_{m=-\infty}^{+\infty}\sum_{\lambda=0,1}\int_{-\infty}^{+\infty}dk\int_0^\infty d\gamma\,\frac{(\gamma\omega)^{-1}}{J_{q|m|}^{(\lambda)2}(\gamma a)+Y_{q|m|}^{(\lambda)2}(\gamma a)}$$

$$\times \left[e^{iqm\Delta\phi+ik\Delta z-i\omega\Delta t}\bar{E}_j^{(\lambda)}(r)\bar{E}_l^{(\lambda)*}(r') - e^{-iqm\Delta\phi-ik\Delta z+i\omega\Delta t}\bar{E}_l^{(\lambda)}(r')\bar{E}_j^{(\lambda)*}(r)\right], \tag{31}$$

where the expressions for the functions $\bar{E}_l^{(\lambda)}(r)$ are given by (7) with the replacement (29).

For the further transformation of the expression for the Green tensor, we use the identity

$$\frac{\bar{E}_j^{(\lambda)}(r)\bar{E}_l^{(\lambda)*}(r')}{J_{q|m|}^{(\lambda)2}(\gamma a)+Y_{q|m|}^{(\lambda)2}(\gamma a)} = E_j^{(\lambda)}(r)E_l^{(\lambda)*}(r') - \frac{1}{2}\sum_{s=1}^{2}\frac{J_{q|m|}^{(\lambda)}(\gamma a)}{H_{q|m|}^{(s,\lambda)}(\gamma a)}E_{(s)j}^{(\lambda)}(r)\tilde{E}_{(s)l}^{(\lambda)}(r'), \tag{32}$$

where the expression for the functions $E_{(s)l}^{(\lambda)}(r)$ and $\tilde{E}_{(s)l}^{(\lambda)}(r)$ are obtained from the expressions for $E_l^{(\lambda)}(r)$ and $E_l^{(\lambda)*}(r)$ from (7), respectively, by the replacement $J_{q|m|}(\gamma r)\to H_{q|m|}^{(s)}(\gamma r)$. The part of the Green tensor corresponding to the first term in the right-hand side of (32) coincides with the Green tensor in the boundary-free geometry, $G_{jl}^{(0)}(x,x')$. As a result, the Green tensor is decomposed as

$$G_{jl}(x,x') = G_{jl}^{(0)}(x,x') + G_{jl}^{(\mathrm{b})}(x,x'), \tag{33}$$

where the expression for the shell-induced part $G_{jl}^{(\mathrm{b})}(x,x')$ is directly obtained from (31) and (32). For the corresponding spectral component we find

$$G_{jl}^{(\mathrm{b})}(\mathbf{r},\mathbf{r}';i\xi) = \frac{q}{4\pi}\sum_{m=-\infty}^{+\infty}\sum_{\lambda=0,1}\int_{-\infty}^{+\infty}dk\sum_{s=1}^{2}\int_0^\infty d\gamma\frac{1}{\gamma\omega}\frac{J_{q|m|}^{(\lambda)}(\gamma a)}{H_{q|m|}^{(s)(\lambda)}(\gamma a)} \times\left[E_{(s)j}^{(\lambda)}(r)\tilde{E}_{(s)l}^{(\lambda)}(r')\frac{e^{iqm\Delta\phi+ik\Delta z}}{\omega-i\xi} + E_{(s)l}^{(\lambda)}(r')\tilde{E}_{(s)j}^{(\lambda)}(r)\frac{e^{-iqm\Delta\phi-ik\Delta z}}{\omega+i\xi}\right]. \tag{34}$$

For the term with $s=1$ ($s=2$) we rotate the contour of the integration over γ by $\pi/2$ ($-\pi/2$). After introducing the modified Bessel functions, this leads to the final expression

$$G_{jl}^{(\mathrm{b})}(\mathbf{r},\mathbf{r}';i\xi) = -\frac{q}{\pi}\sum_{m=-\infty}^{\infty}e^{iqm\Delta\phi}\sum_{\lambda=0,1}(-\xi^2)^\lambda\int_{-\infty}^{\infty}dk\,e^{ik\Delta z} \times k^{2(1-\lambda)}\frac{I_{q|m|}^{(\lambda)}(a\gamma)}{K_{q|m|}^{(\lambda)}(a\gamma)}e_j^{(\lambda)}(\gamma r,\gamma/k)e_l^{(\lambda)*}(\gamma r',\gamma/k), \tag{35}$$

where $\gamma=\sqrt{k^2+\xi^2}$ and we have defined the functions

$$e_1^{(0)}(x,y) = K'_{q|m|}(x),\qquad e_2^{(0)}(x,y) = i\frac{qm}{x}K_{q|m|}(x),$$
$$e_3^{(0)}(x,y) = iyK_{q|m|}(x),$$
$$e_1^{(1)}(x,y) = \frac{qm}{x}K_{q|m|}(x),\qquad e_2^{(1)}(x,y) = -iK'_{q|m|}(x),$$
$$e_3^{(1)}(x,y) = 0. \tag{36}$$

In the coincidence limit the off-diagonal components of the Green tensor vanish.

By taking into account the expression (35) of the Green tensor, for the CP potential outside a cylindrical shell we get

$$U_{\mathrm{b}}(r) = -\frac{2q}{\pi^2}\sum_{m=0}^{\infty}{}'\sum_{\lambda=0,1}\sum_{l=1}^{3}\int_0^\infty d\xi\,\alpha_{ll}(i\xi)(-\xi^2)^\lambda \times\int_\xi^\infty d\gamma\,\gamma\frac{I_{qm}^{(\lambda)}(a\gamma)}{K_{qm}^{(\lambda)}(a\gamma)}\frac{|e_l^{(\lambda)}(\gamma r,\gamma/\sqrt{\gamma^2-\xi^2})|^2}{(\gamma^2-\xi^2)^{\lambda-1/2}}. \tag{37}$$

In the special case $q=1$, from this formula we obtain the result of [13,15] for the interaction potential with a cylindrical shell in Minkowski spacetime. For the isotropic polarizability tensor the explicit expression has the form

$$U_{\mathrm{b}}(r) = -\frac{2q}{\pi^2}\sum_{m=0}^{\infty}{}'\int_0^\infty d\xi\,\alpha(i\xi)\int_\xi^\infty\frac{\gamma d\gamma}{\sqrt{\gamma^2-\xi^2}} \times\left\{\frac{I_{qm}(a\gamma)}{K_{qm}(a\gamma)}\left[(\gamma^2-\xi^2)G_{qm}(\gamma r)+\gamma^2K_{qm}^2(\gamma r)\right] - \xi^2\frac{I'_{qm}(a\gamma)}{K'_{qm}(a\gamma)}G_{qm}(\gamma r)\right\} \tag{38}$$

with the notation $G_{qm}(x) = K_{qm}'^2(z)+(qm/z)^2K_{qm}^2(\gamma r)$. These formulas are obtained from the corresponding formulas in the interior region by the interchange $I_{qm}\rightleftarrows K_{qm}$. Near the boundary the leading term in the corresponding asymptotic expansion over the distance from the shell coincides with (23).

At large distances from the cylinder the dominant contribution comes from the lower limit of the integration. Expanding the modified Bessel functions for small values of the argument we can see that the dominant contribution comes from the term $m=0$, $\lambda=0$ and to the leading order we find

$$U_{\mathrm{b}}(r) \approx -\frac{q\alpha_{11}(0)}{6\pi r^4\ln(r/a)}. \tag{39}$$

Note that at large distances the leading term in the pure string part is given by the expression

$$U_0(r) \approx \frac{(q^2-1)(q^2+11)}{360\pi r^4}\left[\alpha_{11}(0)-\alpha_{22}(0)+\alpha_{33}(0)\right]. \tag{40}$$

Hence, at large distances the Casimir–Polder potential is dominated by the pure string part and the corresponding force is repulsive. As it follows from (39), at large distances the relative contribution of the boundary-induced effects in the CP potential decays logarithmically. Considering the cylindrical boundary as a simple model for string's core, we see that the internal structure of the string may have non-negligible effects even at large distances (see also the discussion in [20]).

In the oscillator model for the polarizability tensor the expression for the CP potential in the exterior region is obtained from (25) by the replacements $I_{qm}\rightleftarrows K_{qm}$. In particular, in the isotropic case we have

$$U_{\mathrm{b}}(r) = \frac{q}{\pi}\sum_{m=0}^{\infty}{}'\sum_j\frac{g_j}{\omega_j^2}\int_0^\infty d\gamma\,\gamma^3\left\{\frac{I'_{qm}(a\gamma)}{K'_{qm}(a\gamma)}\frac{G_{qm}(\gamma r)}{s_j(\gamma)[s_j(\gamma)+1]} - \frac{I_{qm}(a\gamma)}{K_{qm}(a\gamma)}\left[\frac{G_{qm}(\gamma r)}{s_j(\gamma)+1}+\frac{K_{qm}^2(\gamma r)}{s_j(\gamma)}\right]\right\}, \tag{41}$$

with $s_j(\gamma)$ defined by the relation (28). By using the inequality $K_{qm}'^2(x)\geqslant[1+(qm/x)^2]K_{qm}^2(x)$, from (41) it can be seen that in the exterior region $\partial_r U_{\mathrm{b}}(r)>0$. Consequently, the radial component of the boundary-induced part in the CP force is negative, $F_{\mathrm{b},r} = -\partial_r U_{\mathrm{b}}(r)<0$, and this force is attractive with respect to the cylinder. For the isotropic case the radial component of the pure

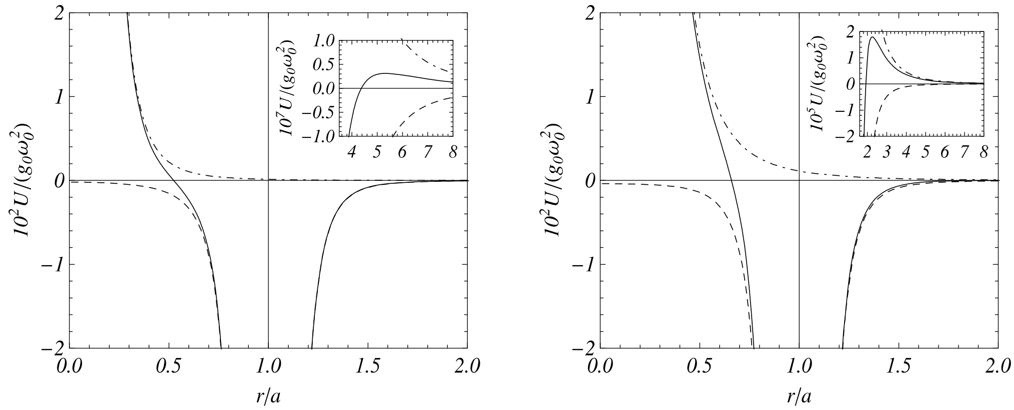

Fig. 1. Total CP potential (full curve), the pure string part (dot-dashed curve) and the boundary-induced part (dashed curve) as functions of r/a for $a/\lambda_0 = 1$. The left and right panels are plotted for $q = 2$ and $q = 4$, respectively.

string part in the force is positive and it has an opposite direction with respect to the boundary-induced part. Near the cylindrical shell the boundary-induced part dominates and the total force in the exterior region is directed toward the cylinder. At large distances from the shell the pure string part is dominant and the total force is repulsive with respect to the cylinder.

In Fig. 1 we display the total CP potential $U(r)$ (full curve), pure string part $U_0(r)$ (dot-dashed curve), and the boundary-induced part $U_b(r)$ (dashed curve) as functions of the ratio r/a for $a/\lambda_0 = 1$, with $\lambda_0 = 2\pi/\omega_0$. The single oscillator model is used with isotropic polarizability and with the parameters $g_j = g_0$ and $\omega_j = \omega_0$. The left and right panels correspond to $q = 2$ and $q = 4$, respectively. As it has been explained before, the potential is dominated by boundary-induced part near the cylindrical shell and by the pure string part for points near the string and at large distances from the cylindrical shell.

In the discussion above we have considered the idealized geometry with a zero thickness cosmic string. A realistic cosmic string has a structure on a length scale defined by the phase transition at which it is formed. In the presence of a conducting cylindrical boundary, the CP potential in the exterior region is uniquely defined by the boundary conditions and the bulk geometry. From here it follows that if we consider a non-trivial core model with finite thickness $b < a$ and with the line element (1) in the region $r > b$, the results in the region outside the cylindrical shell will not be changed.

5. Conclusion

We have investigated the CP potential for a polarizable microparticle in the geometry of a straight cosmic string with a coaxial conducting cylindrical shell. Both regions inside and outside the shell are considered. We start the consideration from the evaluation of the Green tensor inside the shell. In this region the mode sum contains the summation over the corresponding eigenmodes which are expressed in terms of the zeros of the Bessel function and its derivative. For the summation of the series over these zeros we have employed the formula (11). This allowed to extract from the Green tensor the part corresponding to the cosmic string geometry without boundaries. The latter was previously investigated in [3] and here we are mainly interested in the boundary-induced part. This part is presented in the form (13). For the evaluation of the CP potential we need the Green tensor in the coincidence limit of the arguments. In this limit the off-diagonal components vanish and the diagonal components are given by (15).

Similar to the case of the Green tensor, the CP potential is decomposed into the pure string and boundary-induced parts. In the region inside the shell, the latter is given by the expression (18). The CP potential depends on the distance from the string and on the angles determining the orientation of the principal axes of the polarizability tensor with respect to the cosmic string. For the isotropic polarizability the general expression is simplified to (22). Unlike to the pure string part, the boundary-induced part in the CP potential is finite on the string. The corresponding asymptotic behavior near the string is given by expressions (20) and (21) for $q > 2$ and $1 < q < 2$, respectively. The boundary-induced part in the CP potential diverges on the cylindrical shell. The leading term in the asymptotic expansion over the distance from the shell coincides with the CP potential for a metallic plate in Minkowski spacetime. As a model for a polarizability tensor we have used the anisotropic oscillator model. The expressions for the CP potential with this model are given by (25) and (27) for anisotropic and isotropic cases respectively. In the isotropic case, the boundary-induced part in the CP force inside the cylindrical shell is directed toward the shell. The pure string part of the force has the same direction and the total force in the isotropic case is repulsive with respect to the string and attractive with respect to the shell.

The electromagnetic field Green tensor and the CP potential outside a cylindrical shell have been discussed in Section 4. By making use of the identity (32), the Green tensor is presented as the sum of boundary-free and boundary-induced parts. The latter is given by the expression (35). The corresponding expressions for the CP potential have the form (37) and (38) for the anisotropic and isotropic polarizabilities, respectively. At large distances from the cylinder the leading term in the boundary-induced CP potential is given by the expression (39). The leading term in the pure string part is given by the expression (40) and it dominates at large distances. The corresponding force is repulsive. For the oscillator model, the expression for the boundary-induced CP potential takes the form (41). In the isotropic case the corresponding force is attractive with respect to the cylinder. For the isotropic case the pure string part in the force has an opposite direction with respect to the boundary-induced part. Near the cylindrical shell the boundary-induced part dominates and the total force in the exterior region is directed toward the cylinder. At large distances from the shell the pure string part is dominant and the total force is repulsive with respect to the cylinder. From the point of view of the physics in the exterior region the conducting cylindrical surface can be considered as a simple model of superconducting string core. Superconducting strings are predicted in a wide class of field theories and they are sources of a number of interesting astrophysical effects such as generation of synchrotron radiation, cosmic rays, and relativistic jets.

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References

[1] A. Vilenkin, E.P.S. Shellard, Cosmic Strings and Other Topological Defects, Cambridge University Press, Cambridge, England, 1994.
[2] E.J. Copeland, T.W.B. Kibble, Proc. Roy. Soc. Lond. A 466 (2010) 623.
[3] A.A. Saharian, A.S. Kotanjyan, Eur. Phys. J. C 71 (2011) 1765.
[4] V.M. Bardeghyan, A.A. Saharian, J. Contemp. Phys. 45 (2010) 1.
[5] V.A. Parsegian, Van der Waals forces: A Handbook for Biologists, Chemists, Engineers, and Physicists, Cambridge University Press, Cambridge, 2005;
S.Y. Buhmann, D.-G. Welsch, Prog. Quantum Electron. 31 (2007) 51;
S. Scheel, S.Y. Buhmann, Acta Phys. Slov. 58 (2008) 675;
S. Scheel, S.Y. Buhmann, in: Diego Dalvit, Peter Milonni, David Roberts, Felipe da Rosa (Eds.), Lecture Notes in Physics: Casimir Physics, vol. 834, Springer, Berlin, 2011.
[6] E.R. Bezerra de Mello, V.B. Bezerra, A.A. Saharian, A.S. Tarloyan, Phys. Rev. D 74 (2006) 025017.
[7] I. Brevik, T. Toverud, Class. Quantum Grav. 12 (1995) 1229.
[8] E.R. Bezerra de Mello, V.B. Bezerra, A.A. Saharian, Phys. Lett. B 645 (2007) 245.
[9] E.R. Bezerra de Mello, V.B. Bezerra, A.A. Saharian, A.S. Tarloyan, Phys. Rev. D 78 (2008) 105007;
E.R. Bezerra de Mello, V.B. Bezerra, A.A. Saharian, V.M. Bardeghyan, Phys. Rev. D 82 (2010) 085033;
S. Bellucci, E.R. Bezerra de Mello, A.A. Saharian, Phys. Rev. D 83 (2011) 085017.
[10] G. Fucci, K. Kirsten, JHEP 1103 (2011) 016;
G. Fucci, K. Kirsten, J. Phys. A 44 (2011) 295403.
[11] E.R. Bezerra de Mello, A.A. Saharian, Class. Quantum Grav. 28 (2011) 145008.
[12] E.V. Blagov, G.L. Klimchitskaya, V.M. Mostepanenko, Phys. Rev. B 71 (2005) 235401;
M. Bordag, B. Geyer, G.L. Klimchitskaya, V.M. Mostepanenko, Phys. Rev. B 74 (2006) 205431;
R. Fermani, S. Scheel, P.L. Knight, Phys. Rev. A 75 (2007) 062905.
[13] C. Eberlein, R. Zietal, Phys. Rev. A 80 (2009) 012504.
[14] S.A. Ellingsen, S.Y. Buhmann, S. Scheel, Phys. Rev. A 82 (2010) 032516.
[15] V.B. Bezerra, E.R. Bezerra de Mello, G.L. Klimchitskaya, V.M. Mostepanenko, A.A. Saharian, Eur. Phys. J. C 71 (2011) 1614.
[16] K.A. Milton, P. Parashar, N. Pourtolami, I. Brevik, arXiv:1111.4224.
[17] G.E. Volovik, JETP Lett. 67 (1998) 698;
U.R. Fischer, M. Visser, Phys. Rev. Lett. 88 (2002) 110201;
C. Sátiro, F. Moraes, Mod. Phys. Lett. A 20 (2005) 2561.
[18] A.A. Saharian, The Generalized Abel–Plana Formula with Applications to Bessel Functions and Casimir Effect, Yerevan State University Publishing House, Yerevan, 2008, Preprint ICTP/2007/082, arXiv:0708.1187.
[19] M. Abramowitz, I.A. Stegun (Eds.), Handbook of Mathematical Functions, Dover, New York, 1972.
[20] B. Allen, A.C. Ottewill, Phys. Rev. D 42 (1990) 2669;
B. Allen, J.G. McLaughlin, A.C. Ottewill, Phys. Rev. D 45 (1992) 4486;
B. Allen, B.S. Kay, A.C. Ottewill, Phys. Rev. D 53 (1996) 6829.