Comput. Math. Applic. Vol. 16, No. 10/11, pp. 903–914, 1988
Printed in Great Britain.
0097-4943/88 $3.00 + 0.00

SOLVING THE NONLINEAR EQUATIONS OF PHYSICS

G. ADOMIAN
Center for Applied Mathematics, University of Georgia, Athens, GA 30602, U.S.A.

Abstract—Application of the decomposition method and of the asymptotic decomposition method are considered for solution of nonlinear and/or stochastic partial differential equations in space and time. Examples are given to show the potential for solving systems of equations even with strongly coupled boundary conditions.

INTRODUCTION

The frontier applications of science lead naturally to the study of partial differential equations in space and time. Current developments in mathematical physics, engineering and other areas have given impetus to such research and to linearization techniques. The latter assumes, essentially, that a nonlinear system is "almost linear" in order to take advantage of well-known methods. Often, unfortunately, the assumption has little *physical* justification. Advances are vital, not only to mathematics but to the areas of application. Fluid mechanics, soliton physics, quantum field theory, nonlinear evolution equations are all areas which can benefit. In fluid mechanics, for example, the usual analyses are far indeed from any physical reality when they deal with a "mathematized" ocean bearing no resemblance to a real ocean. Strong nonlinearities and strong stochasticity would clearly be involved in a reasonable model.

A methodology with a potential for the needed advances and solutions to the above problems will be discussed in this paper and proposed for rigorous study.

Let us begin with the general form $Fu = g$, where g may be a function of space variables x, y, z and time t. Consider, for example,

$$[L_x + L_y + L_z + L_t]u + Nu = g(x, y, z, t),$$

where L_x, L_y, L_z, L_t represent linear differentiations in x, y, z, and t, respectively, and where $N(u)$ is a nonlinear (possibly stochastic) term. It is written $\mathsf{N}u$ if deterministic. We will assume here that $Nu = f(u)$. Let each $L = \mathsf{L} + R$, where L is an invertible operator, simply the highest-ordered differential operator, and R is the "remainder" operator.† Then

$$(\mathsf{L}_x + \mathsf{L}_y + \mathsf{L}_z + \mathsf{L}_t)u + (R_x + R_y + R_z + R_t)u + Nu = g.$$

The operators R_x, R_y, R_z and R_t, as well as the g, may be stochastic. Or, they may simply be a part of an entirely deterministic operator and be chosen only to make the remaining part easily invertible. We solve for $\mathsf{L}_x u$, $\mathsf{L}_y u$, $\mathsf{L}_z u$, $\mathsf{L}_t u$ in turn, and since the inverses L_x^{-1}, L_y^{-1}, L_z^{-1}, L_t^{-1} exist‡

$$\mathsf{L}_x^{-1}\mathsf{L}_x u = \mathsf{L}_x^{-1}[g - \mathsf{L}_y u - \mathsf{L}_z u - \mathsf{L}_t u] - \mathsf{L}_x^{-1}[R_x + R_y + R_z + R_t]u - \mathsf{L}_x^{-1}Nu$$

$$\mathsf{L}_y^{-1}\mathsf{L}_y u = \mathsf{L}_y^{-1}[g - \mathsf{L}_x u - \mathsf{L}_z u - \mathsf{L}_t u] - \mathsf{L}_y^{-1}[R_x + R_y + R_z + R_t]u - \mathsf{L}_y^{-1}Nu$$

$$\mathsf{L}_z^{-1}\mathsf{L}_z u = \mathsf{L}_z^{-1}[g - \mathsf{L}_x u - \mathsf{L}_y u - \mathsf{L}_t u] - \mathsf{L}_z^{-1}[R_x + R_y + R_z + R_t]u - \mathsf{L}_z^{-1}Nu$$

$$\mathsf{L}_t^{-1}\mathsf{L}_t u = \mathsf{L}_t^{-1}[g - \mathsf{L}_x u - \mathsf{L}_y u - \mathsf{L}_z u] - \mathsf{L}_t^{-1}[R_x + R_y + R_z + R_t]u - \mathsf{L}_t^{-1}Nu.$$

†In stochastic cases, we can also let R be the stochastic part of the operator, i.e. $L = \mathsf{L} + R$, where $\mathsf{L} = \langle L \rangle$. Finally, we can let $L = \mathsf{L} + \mathsf{R} + R$, where L is invertible, R is the remainder operator, and R is stochastic.

‡For example, $\nabla^2 u - u_t$ is written $\mathsf{L}_x u + \mathsf{L}_y u + \mathsf{L}_z u + \mathsf{L}_t u$, where $\mathsf{L}_x, \mathsf{L}_y, \mathsf{L}_z$ are second-order differentiations and $\mathsf{L}_t u = -\partial u/\partial t$.

Solving for u in all four equations, we obtain

$$u = \theta_x + \mathsf{L}_x^{-1}[g - \mathsf{L}_y u + \mathsf{L}_z u - \mathsf{L}_t u] - \mathsf{L}_x^{-1}[R_x + R_y + R_z + R_t]u - \mathsf{L}_x^{-1}\mathsf{N}u$$

$$u = \theta_y + \mathsf{L}_y^{-1}[g - \mathsf{L}_x u - \mathsf{L}_z u - \mathsf{L}_t u] - \mathsf{L}_y^{-1}[R_x + R_y + R_z + R_t]u - \mathsf{L}_y^{-1}\mathsf{N}u$$

$$u = \theta_z + \mathsf{L}_z^{-1}[g - \mathsf{L}_x u - \mathsf{L}_y u - \mathsf{L}_t u] - \mathsf{L}_z^{-1}[R_x + R_y + R_z + R_t]u - \mathsf{L}_z^{-1}\mathsf{N}u$$

$$u = \theta_t + \mathsf{L}_t^{-1}[g - \mathsf{L}_x u - \mathsf{L}_y u - \mathsf{L}_z u] - \mathsf{L}_t^{-1}[R_x + R_y + R_z + R_t]u - \mathsf{L}_t^{-1}\mathsf{N}u$$

where

$$\mathsf{L}_x^{-1}\theta_x = \mathsf{L}_y^{-1}\theta_y = \mathsf{L}_z^{-1}\theta_z = \mathsf{L}_t^{-1}\theta_t = 0.$$

A linear combination of these solutions is necessary. Therefore, adding and dividing by four, we write

$$\begin{aligned} u = u_0 &- (1/4)\{(\mathsf{L}_x^{-1}\mathsf{L}_y + \mathsf{L}_y^{-1}\mathsf{L}_x) + (\mathsf{L}_x^{-1}\mathsf{L}_z + \mathsf{L}_z^{-1}\mathsf{L}_x) + (\mathsf{L}_x^{-1}\mathsf{L}_t + \mathsf{L}_t^{-1}\mathsf{L}_x) \\ &+ (\mathsf{L}_y^{-1}\mathsf{L}_z + \mathsf{L}_z^{-1}\mathsf{L}_y) + (\mathsf{L}_t^{-1}\mathsf{L}_y + \mathsf{L}_y^{-1}\mathsf{L}_t) + (\mathsf{L}_z^{-1}\mathsf{L}_t + \mathsf{L}_t^{-1}\mathsf{L}_z)\}u \\ &- (1/4)[\mathsf{L}_x^{-1} + \mathsf{L}_y^{-1} + \mathsf{L}_z^{-1} + \mathsf{L}_t^{-1}][R_x + R_y + R_z + R_t]u \\ &- (1/4)[\mathsf{L}_x^{-1} + \mathsf{L}_y^{-1} + \mathsf{L}_z^{-1} + \mathsf{L}_t^{-1}]\mathsf{N}u, \end{aligned}$$

where the term u_0 includes

$$(1/4)[\mathsf{L}_x^{-1} + \mathsf{L}_y^{-1} + \mathsf{L}_z^{-1} + \mathsf{L}_t^{-1}]g$$

and also includes the terms arising from the initial conditions which depend on the number of integrations involved in the inverse operators. We have $\mathsf{L}_x^{-1}\mathsf{L}_x u = u(x, y, z, t) - \theta_x$, where $\mathsf{L}_x\theta_x = 0$. Now, $\mathsf{L}_x^{-1}\mathsf{L}_x u = u(x, y, z, t) - u(0, y, z, t)$ if L_x involves a single differentiation. $\mathsf{L}_x^{-1}\mathsf{L}_x u = u(x, y, z, t) - u(0, y, z, t) - x, \partial u(0, y, z, t)/\partial x$ for a second order operator, etc. Similarly $\mathsf{L}_y^{-1}\mathsf{L}_y u = u - \theta_y$, where $\theta_y = u(x, 0, z, t)$ for a single differentiation in L_y, etc. Thus, we have the partial homogeneous solutions $\theta_x, \theta_y, \theta_z, \theta_t$ analogous to the one-dimensional problems considered in earlier work where we wrote

$$\mathsf{L}_t^{-1}\mathsf{L}_t u(t) = \int_0^t (\mathrm{d}u/\mathrm{d}t)\mathrm{d}t = u(t) - u(0)$$

when $\mathsf{L}_t \equiv \mathrm{d}/\mathrm{d}t$. The resulting u_0 is given by:

$$u_0 = (1/4)[\theta_x + \theta_y + \theta_z + \theta_t) + (1/4)[\mathsf{L}_x^{-1} + \mathsf{L}_y^{-1} + \mathsf{L}_z^{-1} + \mathsf{L}_t^{-1}]g.$$

A key element in the solution methodology is the expansion of nonlinear terms in a special set of polynomials denoted by A_n which have been defined by the author [1–3] and are generated for the specific nonlinearity involved. The second key element is decomposition of the solution into components to be determined [1–3]. A final point is not to invert the entire linear operator but only the highest-ordered derivative term to avoid cumbersome integrations. If the solution u is written $\sum_{n=0}^{\infty} u_n$, the $A_n = A_n(u_0, u_1, \ldots, u_n)$ and each component u_n will depend only on $u_0, u_1, \ldots, u_n$ and is therefore calculable. Since these matters have now been sufficiently discussed in several books and numerous papers [5–12], we can simply proceed with their use.

We now write $\mathsf{N}u$, the nonlinear term, as

$$\mathsf{N}u = \sum_{n=0}^{\infty} A_n$$

and assume our usual decomposition of u into

$$\sum_{n=0}^{\infty} u_n,$$

to determine the individual components.

For m-dimensional problems, we can write in a more condensed form,

$$u = u_0 - (1/m) \sum_{\substack{j=i+1 \\ (i \neq j)}}^{m} \sum_{i=1}^{m-1} [\mathsf{L}_{x_i}^{-1}\mathsf{L}_{x_j} + \mathsf{L}_{x_j}^{-1}\mathsf{L}_{x_i}]u - (1/m)\left[\sum_{i=1}^{m} \mathsf{L}_{x_i}^{-1}\right]\left[\sum_{i=1}^{m} R_{xi}\right]u - (1/m)\left[\sum_{i=1}^{m} \mathsf{L}_{x_i}^{-1}\right]\sum_{n=0}^{\infty} A_n,$$

where

$$u_0 = (1/m)\left\{\sum_{i=1}^{m} \theta_i + \sum_{i=1}^{m} \mathsf{L}_{x_i}^{-1}g\right\}.$$

Thus u_0 is easily calculated. The following components of the decomposition follow in terms of u_0. (There are no statistical separability problems in the stochastic case.) Now for $n \geqslant 1$ we can write

$$u_n = -(1/m) \sum_{\substack{i=1 \\ (i \neq j)}}^{m} \sum_{j=1}^{m-1} [\mathsf{L}_{x_i}^{-1}\mathsf{L}_{x_j} + \mathsf{L}_{x_j}^{-1}\mathsf{L}_{x_i}]u_{n-1} - (1/m)\left[\sum_{i=1}^{m} \mathsf{L}_{x_i}^{-1}\right]\left[\sum_{i=1}^{m} R_{x_i}\right]u_{n-1} - (1/m)\left[\sum_{i=1}^{m} \mathsf{L}_{x_i}^{-1}\right]A_{n-1}$$

which allows us to determine $u_1, u_2, \ldots$; the complete solution is

$$u = \sum_{n=0}^{\infty} u_n \quad \text{and our } n \text{ term approximation } \phi_n \text{ is given by} \quad \phi_n = \sum_{i=0}^{n-1} u_i.$$

For the particular problem here,

$$\begin{aligned}
u_0 &= (1/4)[\theta_x + \theta_y + \theta_z + \theta_t] + (1/4)[\mathsf{L}_x^{-1} + \mathsf{L}_y^{-1} + \mathsf{L}_z^{-1} + \mathsf{L}_t^{-1}]g \\
&\vdots \\
u_n &= -(1/4)\{(\mathsf{L}_x^{-1}\mathsf{L}_y + \mathsf{L}_y^{-1}\mathsf{L}_x) + (\mathsf{L}_x^{-1}\mathsf{L}_z + \mathsf{L}_z^{-1}\mathsf{L}_x) + (\mathsf{L}_x^{-1}\mathsf{L}_t + \mathsf{L}_t^{-1}\mathsf{L}_x) \\
&\quad + (\mathsf{L}_y^{-1}\mathsf{L}_z + \mathsf{L}_z^{-1}\mathsf{L}_y) + (\mathsf{L}_t^{-1}\mathsf{L}_y + \mathsf{L}_y^{-1}\mathsf{L}_t) + (\mathsf{L}_z^{-1}\mathsf{L}_t + \mathsf{L}_t^{-1}\mathsf{L}_z)\}u_{n-1} \\
&\quad - (1/4)[\mathsf{L}_x^{-1} + \mathsf{L}_y^{-1} + \mathsf{L}_z^{-1} + \mathsf{L}_t^{-1}][R_x + R_y + R_z + R_t]u_{n-1} \\
&\quad - (1/4)[\mathsf{L}_x^{-1} + \mathsf{L}_y^{-1} + \mathsf{L}_z^{-1} + \mathsf{L}_t^{-1}]A_{n-1}.
\end{aligned}$$

In the one-dimensional ($m = 1$) case, the general solution reduces to the previous result for an ordinary differential equation:

$$\begin{aligned}
u_0 &= \theta_t + \mathsf{L}_t^{-1}g \\
u_1 &= -\mathsf{L}_t^{-1}R_t u_0 - \mathsf{L}_t^{-1}A_0,
\end{aligned}$$

etc. For simplicity in writing, define

$$\frac{\mathsf{L}_x^{-1} + \mathsf{L}_y^{-1} + \mathsf{L}_z^{-1} + \mathsf{L}_t^{-1}}{4} \equiv \mathsf{L}^{-1}$$

and

$$(1/4)[(\mathsf{L}_x^{-1}\mathsf{L}_y + \mathsf{L}_y^{-1}\mathsf{L}_x) + (\mathsf{L}_x^{-1}\mathsf{L}_z + \mathsf{L}_z^{-1}\mathsf{L}_x) + (\mathsf{L}_x^{-1}\mathsf{L}_t + \mathsf{L}_t^{-1}\mathsf{L}_x) + (\mathsf{L}_y^{-1}\mathsf{L}_z + \mathsf{L}_z^{-1}\mathsf{L}_y) + (\mathsf{L}_t^{-1}\mathsf{L}_y + \mathsf{L}_y^{-1}\mathsf{L}_t) + (\mathsf{L}_z^{-1}\mathsf{L}_t + \mathsf{L}_t^{-1}\mathsf{L}_z)] \equiv G$$

and

$$\frac{R_x + R_y + R_z + R_t}{4} = R, \quad \text{(then for } m = 4\text{)}$$

$$u = u_0 - Gu - \mathsf{L}^{-1}Ru - \mathsf{L}^{-1}Nu.$$

In a one-dimensional case, Gu vanishes and the 1/4 or $1/m$ factor is, of course, equal to one and we have

$$u = u_0 - \mathsf{L}^{-1}Ru - \mathsf{L}^{-1}\mathsf{N}u,$$

which is precisely the basis of earlier solutions of nonlinear ordinary differential equations.

For m dimensions, we write:

$$\mathsf{L}^{-1} = (1/m)\sum_{i=1}^{m}\mathsf{L}_{x_i}^{-1}$$

$$R = (1/m)\sum_{i=1}^{m}R_{x_i}.$$

Now

$$u = u_0 - (1/m)\sum_{j=i+1}^{m}\sum_{i=1}^{m-1}[\mathsf{L}_{x_i}^{-1}\mathsf{L}_{x_j} + \mathsf{L}_{x_j}^{-1}\mathsf{L}_{x_i}]u - \mathsf{L}^{-1}Ru - \mathsf{L}^{-1}\mathsf{N}u,$$

which reduces to

$$u = u_0 - \mathsf{L}^{-1}Ru - \mathsf{L}^{-1}\mathsf{N}u,$$

as previously written for ordinary differential equations when $m = 1$, since the second term vanishes.

PARAMETRIZATION AND THE A_n POLYNOMIALS

A parametrization of the equation for u into

$$u = u_0 - (1/m)\lambda\sum_{j=i+1}^{m}\sum_{i=1}^{m-1}[\mathsf{L}_{x_i}^{-1}\mathsf{L}_{x_j} + \mathsf{L}_{x_j}^{-1}\mathsf{L}_{x_i}]u - \lambda\mathsf{L}^{-1}Ru - \lambda\mathsf{L}^{-1}\mathsf{N}u \tag{1}$$

and

$$u = F^{-1}g = \sum_{n=0}^{\infty}u_n$$

into

$$u = \sum_{n=0}^{\infty}\lambda^n F_n^{-1}g = \sum_{n=0}^{\infty}\lambda^n u_n$$

has been *convenient* in determining the components of u and also in finding the A_n polynomials originally. *The λ is not a perturbation parameter.* It is simply an identifier helping us to collect terms in a way which will result in each u_i depending only on $u_{i-1}, u_{i-2}, \ldots, u_0$, for the nonlinear case.

Now $N(u)$ is a nonlinear function and $u = u(\lambda)$. We assume $N(u)$ is analytic and write it as

$$\sum_{n-0}^{\infty}A_n\lambda^n \quad \text{if} \quad N(u) = \mathsf{N}u,$$

i.e. if N is deterministic. (if the nonlinear *stochastic* term Mu appears, we simply carry that along as a second "stochastically analytic" expansion $\Sigma_{n=0}^{\infty}B_n\lambda^n$.)

Now equation (1) becomes

$$\sum_{n=0}^{\infty}\lambda^n F_n^{-1}g = u_0 - (1/m)\lambda\sum_{j=i+1}^{m}\sum_{i=1}^{m-1}[\mathsf{L}_{x_i}^{-1}\mathsf{L}_{x_j} + \mathsf{L}_{x_j}^{-1}\mathsf{L}_{x_i}]\sum_{n=0}^{\infty}\lambda^n F_n^{-1}g$$
$$- \lambda\mathsf{L}^{-1}R\sum_{n=0}^{\infty}\lambda^n F_n^{-1}g - \lambda\mathsf{L}^{-1}\sum_{n=0}^{\infty}\lambda^n A_n.$$

Equating powers of λ:

$$F_0^{-1}g = u_0$$

$$F_1^{-1}g = -(1/m)\sum_{j=i+1}^{m}\sum_{i=1}^{m-1}[\mathsf{L}_{x_i}^{-1}\mathsf{L}_{x_j} + \mathsf{L}_{x_j}^{-1}\mathsf{L}_{x_i}](F_0^{-1}g) - \mathsf{L}^{-1}R(F_0^{-1}g) - \mathsf{L}^{-1}A_0$$

$$\vdots$$

$$F_n^{-1}g = -(1/m)\sum_{j=i+1}^{m}\sum_{i=1}^{m-1}[\mathsf{L}_{x_i}^{-1}\mathsf{L}_{x_j} + \mathsf{L}_{x_j}^{-1}\mathsf{L}_{x_i}](F_{n-1}^{-1}g) - \mathsf{L}^{-1}R(F_{n-1}^{-1}g) - \mathsf{L}^{-1}A_{n-1}.$$

Hence, all terms are calculable. If there are both deterministic and stochastic terms which are nonlinear, i.e. $Nu = \mathsf{N}u + Mu$, we calculate both in the same way, but the second involves randomness. If randomness is involved anywhere in any part of the equation, we will then calculate the statistical measures, e.g., the expectation and covariance of the solution process.

Thus, each $F_{n+1}^{-1}g$ depends on $F_n^{-1}g$ and ultimately on $F_0^{-1}g$. Hence, F^{-1}, the stochastic nonlinear inverse, has been determined. The quantities A_n and B_n have been calculated for general classes of nonlinearities, and explicit formulas have been developed. Their calculation is as simple as writing down a set of Hermite or Legendre polynomials. They depend, of course, on the particular nonlinearity.

If stochastic quantities are involved, the above series, i.e. the n-term approximation ϕ_n, then involves processes and can be averaged for $\langle u \rangle$ or multiplied and averaged to form the correlation $\langle u(t_1)^* u(t_2)\rangle = R_u(t_1, t_2)$ as discussed in the author's previous works. Thus, the solution statistics or *statistical measures* are obtained when appropriate statistical knowledge of the random quantities is available.

Summarizing, we have decomposed the solution process for the output of a physical system into additive components—the first being the solution of a simplified linear deterministic system which takes account of initial conditions. Each of the other components is then found in terms of a *preceding* component, and thus ultimately in terms of the first.

The usual statistical separability problems requiring closure approximations are eliminated with the reasonable assumption of statistical independence of the system *input* and the system itself!† Quasimonochromaticity assumptions are *unnecessary* and processes can be assumed to be general physical processes rather than white noise. White noise is not a physical process. Physical inputs are neither unbounded nor do they have zero correlation times. In any event, the results can be obtained as a special case. If fluctuations are small, the results of perturbation theory are exactly obtained [1] but again this is a special case, as are the diagrammatic methods of physicists.

Just as spectral spreading terms are lost by a quasimonochromatic approximation when a random or scattering medium is involved, or terms are lost in the use of closure approximations, Boussinesq approximations, or replacement of stochastic quantities by their expectations, significant terms may be lost by the usual linearizations, unless, of course, the behavior is actually close to linear.

One hopes, therefore, that physically more realistic and accurate results and predictions will be obtained in many physical problems by this method of solution, as well as interesting new mathematics from the study of such operators and relevant analysis.

The author's approach to these problems began with linear *stochastic operator* equations and has evolved since 1976 to nonlinear stochastic operator equations. Consider, for example

$$\frac{\partial u}{\partial t} + b(t, x)\frac{\partial^2 u}{\partial x^2} = g(t, x),$$

which is rewritten in terms of operators as

$$\mathsf{L}_t u + \mathsf{L}_x u = g(t, x),$$

where

$$\mathsf{L}_t = \frac{\partial}{\partial t} \quad \text{and} \quad \mathsf{L}_x = b\frac{\partial^2}{\partial x^2}.$$

†This assumption can be modified in some particular cases.

In the *stochastic* equation

$$\frac{\partial u(x,t,\omega)}{\partial t}+A(x,t)u(x,t,\omega)+B(x,t,\omega)u(x,t,\omega)=f(x,t,\omega),$$

where $\omega \in (\Omega, F, \mu)$ is a probability space, f is a stochastic process, A is a deterministic coefficient and B is a stochastic process. We can now write this as:

$$\mathsf{L}u + Ru = f,$$

where

$$\mathsf{L} = \frac{\partial}{\partial t} + A(x,t)$$

is a deterministic operator and $R = B(x, t, \omega)$ is a *stochastic*† operator, or

$$Lu = f,$$

where L is a *stochastic operator* with deterministic and random parts, L being $\langle L \rangle$ if R is zero-mean.

Let us consider then the operator equation

$$Fu = g,$$

where F represents a differential operator which may be ordinary or partial, linear or nonlinear, deterministic or stochastic. We suppose F has linear and nonlinear parts, i.e. $Fu = Lu + Nu$, where L is a linear (stochastic) operator and N is a nonlinear (stochastic) operator. We may, of course, have a nonlinear term which depends upon derivatives of u as well as u. Such nonlinear terms are considered elsewhere.

Since L may have deterministic and stochastic components, let $L = \mathsf{L} + R$ where conveniently $\mathsf{L} = \langle L \rangle$ and $R = L - \mathsf{L}$. This is not a limitation on the method but a convenience in explanation. It is necessary that L be invertible. If the above choice makes this difficult, we choose a simpler L and let R incorporate the remainder. Let $Nu = \mathsf{N}u + Mu$, where $\mathsf{N}u$ indicates a deterministic part and Mu indicates a stochastic nonlinear term.

F may involve derivatives with respect to x, y, z, t or mixed derivatives. To avoid difficulties in notation which tend to obscure rather than clarify, we will assume the same probability space for each process and let $\mathsf{L} = \mathsf{L}_x + \mathsf{L}_y + \mathsf{L}_z + \mathsf{L}_t$, where the operators indicate quantities like $\partial^2/\partial x^2$, $\partial/\partial y$, etc., but, for now, no mixed derivatives. Similarly, R is written as $R_x + R_y + R_z + R_t$. Mixed derivatives and product nonlinearities such as $u^2u'^3$, uu'', $f[u, u', \ldots, u^{(m)}]$ can also be handled as shown elsewhere [2].

A simple Langevin equation is written $\mathsf{L}u = g$, where $\mathsf{L} = (\mathrm{d}/\mathrm{d}t) + \beta$ and g is a white noise process. Langevin equations, as used for modeling complex nonlinear phenomena in physics of the form $\dot{\Psi} = f(\Psi) + \xi$, can be represented by $\mathsf{L}u + \mathsf{N}u = g$. *We will not make any Markovian or white noise restrictions.* All processes will be physical processes without restriction to being Gaussian or stationary. In the KdV equation, for example, Fu would become $\mathsf{L}_t u + \mathsf{L}_x u + \mathsf{N}u$, where $\mathsf{N}u$ is of the form uu_x (again a product nonlinearity). In equations of the Satsuma–Kaup type for soliton behavior, we have also such products as uu_x, uu_{xxx}, $u_x u_{xx}$. Stochastic transport equations will fit nicely into our format since $\nabla^2 = \mathsf{L}_x + \mathsf{L}_y + \mathsf{L}_z$ and stochastic behavior in coefficients or inputs are easily included. For example, instead of $\mathsf{L}y(\bar{r}, t) = \xi(\bar{r}, t, \omega)$, where ξ is a random source and $\mathsf{L} = (\partial/\partial t) - \alpha\nabla^2$ or $(\partial/\partial t) - A_{xyz}$, we can include nonlinear terms or stochastic behavior in the operator. In the double sine–Gordon equation we have $u_{tt} - u_{xx} - \sin u + \sin 2u = 0$ or $\mathsf{L}_t + \mathsf{L}_x + \mathsf{N}(u)$, where $\mathsf{N}(u)$ includes the trigonometric nonlinearities. We can allow trigonometric terms, polynomials, exponentials, or products, sums of products, etc. Or, as in the LAX theorem, $\mathsf{N}(u) = f(u, u_x, u_{xx}, \ldots)$. We remark that the Îto equation $\mathrm{d}y = f(t, y)\,\mathrm{d}t + g(t, y)\,\mathrm{d}z$, where z is the Wiener process, which can be written $\mathrm{d}y/\mathrm{d}t = f(t, y) + g(t, y)u(t)$ so that we can write $\mathrm{d}z/\mathrm{d}t = u$ if we do not insist that z is a Wiener process. This equation can, consequently, be put into the author's standard form. The nondifferentiability of the Wiener process is, of course, a *mathematical*

†We prefer script letters to indicate stochasticity. Where script letters have been unavailable, italic letters have been used.

property. We are interested in *physical* solutions. In the Îto integral $\int f\,dz$, the z process is not of bounded variation; however, a Lipschitz condition on z is reasonable for physical processes, so that the integral will be a well-defined Riemann–Stieltjes integral. Physically reasonable models and mathematically tractable models are not necessarily the same. This point of view offers interesting and different mathematics for systems characterized by linear or nonlinear stochastic operator equations in which the operator may be an ordinary or partial differential operator, where the operator may be stochastic. As an example, in a differential operator of nth order, one or more coefficients may be stochastic processes.

Now consider the inhomogeneous heat equation as an example:

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial t} = g(x, t)$$

or

$$\mathsf{L}_x u - \mathsf{L}_t u = g.$$

Assume g is given along with appropriate conditions on u. We write now

$$\mathsf{L}_x u = g + \mathsf{L}_t u, \tag{2}$$

$$\mathsf{L}_t u = -g + \mathsf{L}_x u. \tag{3}$$

Remembering $\mathsf{L}_x = \partial^2/\partial x^2$ and $\mathsf{L}_t = \partial/\partial t$, equations (2) and (3) become

$$u = a + bx + \mathsf{L}_x^{-1} g + \mathsf{L}_x^{-1}\mathsf{L}_t u, \tag{4}$$

$$u = c - \mathsf{L}_t^{-1} g + \mathsf{L}_t^{-1}\mathsf{L}_x u, \tag{5}$$

where a, b and c must be evaluated from the given conditions on u. (They arise from the solutions of $\mathsf{L}_x u = 0$ and $\mathsf{L}_t u = 0$.) Adding equations (4) and (5) we obtain

$$u = (1/2)(a + bx + \mathsf{L}_x^{-1} g - \mathsf{L}_t^{-1} g) + (1/2)(\mathsf{L}_x^{-1}\mathsf{L}_t + \mathsf{L}_t^{-1}\mathsf{L}_x)u.$$

We rewrite this as

$$u = u_0 + (1/2)\{\mathsf{L}_x^{-1}\mathsf{L}_t + \mathsf{L}_t^{-1}\mathsf{L}_x\}u$$

with

$$u_0 = (1/2)(a + bx + \mathsf{L}_x^{-1} g - \mathsf{L}_t^{-1} g).$$

Let

$$u = \sum_{n=0}^{\infty} u_n.$$

Then

$$u = u_0 + (1/2)\{\mathsf{L}_x^{-1}\mathsf{L}_t + \mathsf{L}_t^{-1}\mathsf{L}_x\} \sum_{n=0}^{\infty} u_n.$$

Since u_0 is known when the conditions on u are specified, we have

$$\begin{aligned} u_1 &= (1/2)\{\mathsf{L}_x^{-1}\mathsf{L}_t + \mathsf{L}_t^{-1}\mathsf{L}_x\}u_0 \\ u_2 &= (1/2)\{\mathsf{L}_x^{-1}\mathsf{L}_t + \mathsf{L}_t^{-1}\mathsf{L}_x\}u_1 \\ &\vdots \\ u_{n+1} &= (1/2)\{\mathsf{L}_x^{-1}\mathsf{L}_t + \mathsf{L}_t^{-1}\mathsf{L}_x\}u_n, \quad n \geqslant 0, \end{aligned}$$

so all terms of u are easily evaluated once u_0 is determined. If, for example, initial conditions are specified, we have

$$\begin{aligned} a &= u(x, 0) \\ b &= u(0, t) \\ c &= \partial u(0, t)/\partial x. \end{aligned}$$

At any stage of approximation, we write

$$\phi_n = u_0 + u_1 + \cdots + u_{n-1}$$

and can easily verify that

$$\frac{\partial^2 \phi_n}{\partial x^2} - \frac{\partial \phi_n}{\partial t} = g.$$

ASYMPTOTIC DECOMPOSITION FOR PARTIAL DIFFERENTIAL EQUATIONS

In an equation such as $[\mathsf{L}_x + \mathsf{L}_y]u + \mathsf{N}u = g$, write

$$\sum_{n=0}^{\infty} A_n \quad \text{for} \quad \mathsf{N}u,$$

then solve for the A_n (or for u if $\mathsf{N}u = u$)

$$\sum_{n=0}^{\infty} A_n = g - \mathsf{L}_x u - \mathsf{L}_y u.$$

For example, if $\mathsf{N}u = u^2$, we get $u_0 = g^{1/2}$ and $u_1 = -\mathsf{L}_x u_0 - \mathsf{L}_y u_0$, etc.

Example

Consider

$$u_{xx} + u_{yy} + u = g,$$

where

$$g = x^2y^2 + 2x^2 + 2y^2.$$

Write

$$u = g - u_{xx} - u_{yy}$$

$$\sum_{n=0}^{\infty} u_n = g - \frac{\partial^2}{\partial x^2}\sum_{n=0}^{\infty} u_n - \frac{\partial^2}{\partial y^2}\sum_{n=0}^{\infty} u_n$$

$$u_0 = g = x^2y^2 + 2x^2 + 2y^2$$

$$u_1 = -\frac{\partial^2}{\partial x^2}u_0 - \frac{\partial^2}{\partial y^2}u_0 = -2y^2 - 2x^2 - 8$$

$$u_2 = -\frac{\partial^2}{\partial x^2}u_1 - \frac{\partial^2}{\partial y^2}u_1 = 4 + 4 = 8$$

$$u_3 = 0 \quad \text{and} \quad u_n = 0 \quad \text{for } n > 3 \text{ as well.}$$

Thus we have a terminating series which is the solution.

$$u = u_0 + u_1 + u_2 = x^2y^2.$$

If the last term on the left side of the example is nonlinear, i.e. if we have, say, $Nu = u^2$ instead of u, the A_0 term is u_0^2. Then $u_0 = g^{1/2}$,

$$A_1 = 2u_0u_1 = -\frac{\partial^2}{\partial x^2}u_0 - \frac{\partial^2}{\partial y^2}u_0$$

and we can solve for u_1, etc.

Example

Consider the equation

$$u_{xx}+u_{yy}+u^2=x^2y^2$$

$$\sum_{n=0}^{\infty} A_n = x^2y^2-(\partial^2/\partial x^2)\sum_{n=0}^{\infty} u_n-(\partial^2/\partial y^2)\sum_{n=0}^{\infty} u_n$$

$$A_0=u_0^2=x^2y^2$$

$$u_0=xy$$

$$A_1=2u_0u_1=-(\partial^2/\partial x^2)(xy)-(\partial^2/\partial y^2)(xy)$$

$$u_1=u_2=\cdots=0.$$

Thus, $u=xy$.

Thus, both forms—the decomposition and asymptotic decomposition—offer a new and powerful way of gaining insight into the behavior of very complicated nonlinear equations.

SYSTEMS OF NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS

The decomposition method [1] has been demonstrated to solve a wide class of equations [2]. These have included differential equations and systems of differential equations and also partial differential equations. Consider now a system of nonlinear partial differential equations given by:

$$u_t=uu_x+vu_y$$

$$u_v=uv_x+vv_y \tag{6}$$

$$u(x,y,0)=f(x,y)$$

$$v(x,y,0)=g(x,y). \tag{7}$$

We wish to investigate the solution by the decomposition technique. Let $\mathsf{L}_t=\partial/\partial t$, $\mathsf{L}_x=\partial/\partial_x$, $\mathsf{L}_y=\partial/\partial y$ and write the above equations in the form:

$$\mathsf{L}_t u=u\,\mathsf{L}_x u+v\,\mathsf{L}_y u$$

$$\mathsf{L}_t v=u\,\mathsf{L}_x v+v\,\mathsf{L}_y v. \tag{8}$$

Let

$$\mathsf{L}_t^{-1}=\int_0^t[\cdot]\,\mathrm{d}t,$$

remembering we solve only for linear operator terms:†

$$u=u(x,y,0)+\mathsf{L}_t^{-1}u\,\mathsf{L}_x u+\mathsf{L}_t^{-1}v\,\mathsf{L}_y u$$

$$v=v(x,y,0)+\mathsf{L}_t^{-1}u\,\mathsf{L}_x v+\mathsf{L}_t^{-1}v\,\mathsf{L}_y v.$$

Let

$$u=\sum_{n=0}^{\infty}u_n \quad \text{and} \quad v=\sum_{n=0}^{\infty}v_n$$

and using equations (8) let

$$u_0=u(x,y,0)=f(x,y)$$

$$v_0=v(x,y,0)=g(x,y)$$

†Writing L_x, L_y in the nonlinear terms avoids double subscripts when we decompose u and v.

so that the first term of u and of v are known. We now have:

$$u = u_0 + \mathsf{L}_t^{-1} u\,\mathsf{L}_x u + \mathsf{L}_t^{-1} v\,\mathsf{L}_y u$$
$$v = v_0 + \mathsf{L}_t^{-1} u\,\mathsf{L}_x v + \mathsf{L}_t^{-1} v\,\mathsf{L}_y v.$$

We can use the A_n polynomials for the nonlinear terms, that

$$u = u_0 + \mathsf{L}_t^{-1} \sum_{n=0}^{\infty} A_n(u\,\mathsf{L}_x u) + \mathsf{L}_t^{-1} \sum_{n=0}^{\infty} A_n(v\,\mathsf{L}_y u)$$

$$v = v_0 + \mathsf{L}_t^{-1} \sum_{n=0}^{\infty} A_n(u\,\mathsf{L}_x v) + \mathsf{L}_t^{-1} \sum_{n=0}^{\infty} A_n(v\,\mathsf{L}_y v).$$

(The notation $A_n(u\,\mathsf{L}_x u)$ means the A_n generated for uu_x.)

$$A_0(u\,\mathsf{L}_x u) = u_0 \mathsf{L}_x u_0$$
$$A_1(u\,\mathsf{L}_x u) = u_0 \mathsf{L}_x u_1 + u_1 \mathsf{L}_x u_0$$
$$A_2(u\,\mathsf{L}_x u) = u_0 \mathsf{L}_x u_2 + u_1 \mathsf{L}_x u_1 + u_2 \mathsf{L}_x u_0$$
$$\vdots$$

etc., for the other A_n. A simple rule here is that the sum of the subscripts of each term is the same as the subscript of A. Consequently, we can write:

$$u_1 = \mathsf{L}_t^{-1} u_0 \mathsf{L}_x u_0 + \mathsf{L}_t^{-1} v_0 \mathsf{L}_y u_0$$
$$v_1 = \mathsf{L}_t^{-1} u_0 \mathsf{L}_x v_0 + \mathsf{L}_t^{-1} v_0 \mathsf{L}_y v_0,$$

which yields the next component of u and of v. Then

$$u_2 = \mathsf{L}_t^{-1}[u_0 \mathsf{L}_x u_1 + u_1 \mathsf{L}_x u_0] + \mathsf{L}_t^{-1}[v_0 \mathsf{L}_y u_1 + v_1 \mathsf{L}_y u_0]$$
$$v_2 = \mathsf{L}_t^{-1}[u_0 \mathsf{L}_x v_1 + u_1 \mathsf{L}_x v_0] + \mathsf{L}_t^{-1}[v_0 \mathsf{L}_y v_1 + v_1 \mathsf{L}_y v_0]$$
$$u_3 = \mathsf{L}_t^{-1}[u_0 \mathsf{L}_x u_2 + u_1 \mathsf{L}_x u_1 + u_2 \mathsf{L}_x u_0] + \mathsf{L}_t^{-1}[v_0 \mathsf{L}_y u_2 + v_1 \mathsf{L}_y u_1 + v_2 \mathsf{L}_y u_0]$$
$$v_3 = \mathsf{L}_t^{-1}[u_0 \mathsf{L}_x v_2 + u_1 \mathsf{L}_x v_1 + u_2 \mathsf{L}_x v_0] + \mathsf{L}_t^{-1}[v_0 \mathsf{L}_y v_2 + v_1 \mathsf{L}_y v_1 + v_2 \mathsf{L}_y v_0],$$

etc., up to some u_n, v_n; then we have the n-term approximations

$$\sum_{i=0}^{n-1} u_i \quad \text{for } u \text{ and } \quad \sum_{i=0}^{n-1} v_i \quad \text{for } v$$

as our approximate solutions.

Since the solution of equations (6) and (7) can exhibit a shock phenomenon for finite t, we select f, g such that the shock occurs for a value of t far from our region of interest. Let $f(x, y) = g(x, y) = x + y$. Therefore,

$$u_0 = v_0 = x + y,$$

then u_1, v_1 can be calculated as

$$\begin{aligned} u_1 &= \mathsf{L}_t^{-1} u_0 \mathsf{L}_x u_0 + \mathsf{L}_t^{-1} v_0 \mathsf{L}_y u_0 \\ &= \mathsf{L}_t^{-1}(x + y)\mathsf{L}_x(x + y) + \mathsf{L}_t^{-1}(x + y)\mathsf{L}_y(x + y) \\ &= xt + yt + xt + yt = 2xt + 2yt \\ v_1 &= \mathsf{L}_t^{-1} u_0 \mathsf{L}_x v_0 + \mathsf{L}_t^{-1} v_0 \mathsf{L}_y v_0 = 2xt + 2yt \end{aligned}$$

and u_2, v_2 are calculated as

$$\begin{aligned} u_2 &= \mathsf{L}_t^{-1}[(x + y)\mathsf{L}_x(2xt + 2yt) + (2xt + 2yt)\mathsf{L}_x(x + y)] \\ &\quad + \mathsf{L}_t^{-1}[(x + y)\mathsf{L}_y(2xt + 2yt) + (2xt + 2yt)\mathsf{L}_y(x + y)] \\ &= 4t^2(x + y) \\ v_2 &= 4t^2(x + y). \end{aligned}$$

Thus,

$$u = (x + y) + 2t(x + y) + 4t^2(x + y) + \cdots$$
$$v = (x + y) + 2t(x + y) + 4t^2(x + y) + \cdots,$$

which we can also write as

$$u(x, y) = (x + y)/(1 - 2t)$$
$$v(x, y) = (x + y)/(1 - 2t),$$

which is identical to the solution obtained by Bellman using the method of differential quadrature [3].

Using the same procedure, it is shown in Ref. [2] that we can consider linear, nonlinear, stochastic, or coupled boundary conditions. Adomian and Rach [4] give the the example:

$$\mathrm{d}^2u/\mathrm{d}t^2 + v = 0$$
$$\mathrm{d}^2v/\mathrm{d}t^2 + u = 0,$$

on the interval $[\pi/2, \pi/2]$. Let's assume the boundary conditions:

$$B_1(u, v) = \frac{\mathrm{d}}{\mathrm{d}t} u(\pi/4) + \tfrac{1}{2}u(\pi/4) + 2v(\pi/4) = 3$$

$$B_2(u, v) = \frac{\mathrm{d}}{\mathrm{d}t} u(\pi/2) + \tfrac{1}{3}u(\pi/2) + 4v(\pi/2) = 5$$

$$B_3(u, v) = \frac{\mathrm{d}}{\mathrm{d}t} v(\pi/4) + \tfrac{1}{4}v(\pi/4) + 8u(\pi/4) = 7$$

$$B_4(u, v) = \frac{\mathrm{d}}{\mathrm{d}t} v(\pi/2) + \tfrac{1}{5}v(\pi/2) + 16u(\pi/2) = 11.$$

If $L = \mathrm{d}^2/\mathrm{d}t^2$, we have $Lu = -v$ and $Lv = -u$. Then

$$u = c_1 + c_2t - \mathrm{L}^{-1}v$$
$$v = k_1 + k_2t - \mathrm{L}^{-1}u,$$

so that

$$u_0 = c_1 + c_2t$$
$$v_0 = k_1 + k_2t$$
$$u_1 = -\mathrm{L}^{-1}[k_1 + k_2t] = -k_1t_2/2 - k_2t^3/3!$$
$$v_1 = -\mathrm{L}^{-1}[c_1 + c_2t] = -c_1t^2/2 - c_2t^3/3!$$

Continuing in this way, we can write the n-term approximations $\phi_n(t)$ and $\phi_n(t)$. Using only three terms, i.e. ϕ_3 and θ_3 substituted in the given conditions, we find $c_1 = 0.20300$, $c_2 = 1.00769$, $k_1 = 0.49390$ and $k_2 = 0.98480$. Verification by substitution yields three decimal place accuracy.

REFERENCES

1. G. Adomian, *Stochastic Systems*. Academic Press, New York (1983).
2. G. Adomian, *Nonlinear Stochastic Operator Equations*. Academic Press, New York (1986).
3. R. E. Bellman and G. Adomian, *Partial Differential Equations—New Methods for their Treatment and Application*. Reidel, Dordrecht, Holland (1985).
4. G. Adomian and R. Rach, Coupled differential equations and coupled boundary conditions. *J. math. Anaysis Applic.* **112** (1), 129–135 (1985).
5. N. Bellomo, R. Cafaro and G. Rizzi, On the mathematical modelling of physical systems by ordinary differential stochastic equations. *Maths Comput. Simulation* **4**, 361–367 (1984).
6. R. Riganti, Transient behaviour of semilinear stochastic systems with random parameters. *J. math. Analysis Applic.* **98**, 314–327 (1984).

7. N. Bellomo and R. Monaco, A comparison between Adomian's decomposition method and perturbation techniques for nonlinear random differential equations. *J. math. Analysis Applic.* **110,** 495–502 (1985).
8. D. Bigi and R. Riganti, Solutions of nonlinear boundary value problems by the decomposition method. *Appl. Math. Modelling* **10,** 48–52 (1986).
9. I. Bonzani, Analysis of stochastic Van der Pol oscillators using the decomposition method, in *Complex and Distributed Systems—Analysis, Simulation and Control* (Eds S. Tzafestas and P. Borne), pp. 163–168. North Holland, Amsterdam (1986).
10. N. Bellomo and R. Riganti, *Nonlinear Stochastic Systems in Physics and Mechanics*. World Scientific Publ., Singapore (1987).
11. R. Riganti, On a class of nonlinear dynamical systems: the structure of a differential operator in the application of the decomposition method. *J. math. Analysis Applic.* **123** (1987).
12. I. Bonzani and R. Riganti, Soluzioni periodiche di sistemi dinamici non lineari applicando il metodo di decomposizione. *Atti VIII Congr. naz. AIMETA* **II,** 525–530 (1986).