Approximating an expectation related to cooperative robots

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Abstract

In this paper we develop an approximation to the expectation of a random variable implied in cooperation stability, presented in a previous work. This approximation is obtained by means of a continuous monotonous function that upper bounds the expectation. Finally, we analyze the quality of this approximation.

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1. Introduction

This paper is a continuation of [1]. The model is the following (one can consult [1] for more details and [2–5] for related works in cooperative domains):

A selfish robot interacts with \( n \) cooperative robots. At time 0 none of the cooperative robots have recognised the selfish one. At time 1 the selfish robot interacts with one of the cooperative robots and is recognised by that one with probability 1 and by each of the remaining robots independently with probability \( 1 - q \); \( 0 < q < 1 \); \( p := 1 - q \). This process is iterated until all the robots have recognised the selfish one.

Let \( X_{n,q} \) denote the number of interactions necessary for all the cooperative robots to recognise the selfish one. The law and the expectation of \( X_{n,q} \) were obtained in [1]; namely, the law of \( X_{n,q} \) is...
\[ f_{n,q}(x) = \begin{cases} (1-q)^{n-1} & \text{if } x = 1 \\ q^{\frac{x(x-1)}{2}} \sum_{i=1}^{x} \frac{(1-q)^{n-1}}{\prod_{j=1, j \neq i}^{x} (q^j - q^i)} & \text{otherwise} \end{cases} \] (1)

which is true for \(1 \leq x \leq n\). Moreover, this probability function verifies the following recursive expression:

\[ f_{n,q}(x) = (1-q^x) f_{n-1,q}(x) + q^{x-1} f_{n-1,q}(x-1), \] (2)

if it is assumed that \(f_{1,q}(i) = 0, i \geq 1\).

The following expressions with regard to the expectation of \(X_{n,q}\) were obtained from (2):

\[ E[X_{n,q}] = E[X_{n-1,q}] + E[q^{X_{n-1,q}}] = \sum_{i=0}^{n-1} E[q^{X_{i,q}}], \] (3)

\[ E[X_{n,q}] = \begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ n + \sum_{i=1}^{n-1} \left( \frac{n}{i+1} \right) \prod_{j=1}^{i} (q^j - 1), & \text{otherwise}. \end{cases} \] (4)

2. Approximation to the expectation of \(X_{n,q}\)

Some multi-robotics systems (such as swarm intelligence ones that resemble eusocial insect colonies) have hundreds to thousands of small robots interacting one each other. In such cases, where \(n\) is large, and given the characteristics of the expression of the expectation of \(X_{n,q}\) (4), the implied calculations are very tedious. The aim of this section is to establish an approximation for \(E[X_{n,q}]\) and also to analyze its precision.

To obtain this approximation, we have considered the following difference equation:

\[ y_{k+1} = (y_k - 1)q, \] (5)

with initial condition \(y_0 = n\). From the solution of this equation, that is,

\[ y_k = nq^k - \frac{q(1-q^k)}{1-q} \]

and solving the equation \(y_k = 0\) in relation to \(k\), we define the sequence \(a_n\), which will approximate the expectation \(E[X_{n,q}]\):

\[ a_n = \frac{\ln \left( \frac{q}{np+q} \right)}{\ln q}. \]

We also define \(R_n = a_n - E[X_{n,q}]\), the error sequence for this approximation.

In order to state the main result of this section (Proposition 1), about the quality of the above approximation, we will need to define some sequences. We consider

\[ T_{n+1} = q^{E[X_{n,q}]}, \]
\[ S_{n+1} = E[q^{X_{n,q}}], \]
\[ J_n = a_n - \sum_{i=1}^{n} T_i = \sum_{i=1}^{n} (a_i - a_{i-1}) - \sum_{i=1}^{n} T_i. \]

From (3) we obviously have that \( R_n = \sum_{i=1}^{n} (a_i - a_{i-1}) - \sum_{i=1}^{n} S_i \). On the other hand \( h(x) = q^x \) is a convex function and by Jensen’s inequality (see, for example [6, p. 153]) it results that \( S_{n+1} \geq T_{n+1}, \forall n \), and so
\[ J_{n+1} - J_n \geq R_{n+1} - R_n. \] (6)

**Lemma 1.** Let \( G \) be the concave function for \( x \geq 0 \) defined by
\[ G(x) = (1 - (x - [x]))E(X_{[x],q}) + (x - [x])E(X_{[x]+1,q}), \]
where \([x]\) denotes the integer part of \( x \), which verifies \( G(x) = x, 0 \leq x \leq 1 \), and \( G(r) = E(X_{r,q}) \) if \( r \geq 0 \) is an integer. If \( k \geq 0 \) is an integer such that
\[ k \leq 1 - \frac{\ln(x - (x - 1)q)}{\ln q}. \] (7)
then
\[ G(x) \leq k + G\left(q^k(x - 1) - \frac{q - q^k}{1 - q}\right). \]

**Proof.** First we prove that the result is true for \( k = 1 \) and \( x = n \geq 1 \), integer. We consider the random variable \( Y \) = the number of cooperative robots that do not recognise the selfish robot after the first interaction, whose distribution is obviously binomial with parameter \((n - 1, q)\). Then we can observe that \( X_{n,q} = 1 + X_{Y,q} \). Since \( G \) is concave, we have
\[ E(X_{Y,q}) = E(G(Y)) \leq G(E(Y)) = G(q(n - 1)), \]
so
\[ E(X_{n,q}) = G(n) \leq 1 + G(q(n - 1)). \] (8)

Now, let \( x \geq 1 \) be a real number. From (8) it results that
\[ G(x) = (1 - (x - [x]))G([x]) + (x - [x])G([x] + 1) \leq 1 + (1 - (x - [x]))G(q([x] - 1)) + (x - [x])G(q[x]). \]

On the other hand, since \( G \) is concave we have that \( (G(x_j) - G(x_i))(x_j - x_i) \geq (G(x_i) - G(x_j))(x_j - x_i) \) for \( x_i \leq x_j \leq x_i \). Then, taking \( x_i = q([x] - 1), x_j = q(x - 1) \) and \( x_i = q[x] \), we obtain that
\[ G(q(x - 1)) \geq 1 - (1 - (x - [x]))G(q([x] - 1)) + (x - [x])G(q[x]), \]
so
\[ G(x) \leq 1 + G(q(x - 1)). \] (9)

Proceeding recursively from (9) we have
\[ G(x) \leq 1 + G(q(x - 1)) \leq 2 + G(q^2(x - 1) - q) \leq \cdots \leq k + G\left(q^k(x - 1) - \frac{q - q^k}{1 - q}\right), \]
providing that \( q^k(x - 1) - \frac{q - q^k}{1 - q} \geq 0 \), which is equivalent to (7).
**Proposition 1.** Let \( c_n \) be the increasing sequence defined by \( q^{a_n} = a_{n+1} - a_n \), that is,

\[
c_n = \ln \left( \frac{\frac{q^p}{x^{p+q}} \ln \frac{1}{q}}{\ln (1+ \frac{q}{x^{p+q}})} \right).
\]

Then \( 0 \leq R_{n+1} \leq c_n \), and

\[
\lim_{n \to \infty} c_n = \ln \left( \frac{\frac{q}{p} \ln \frac{1}{q}}{\ln q} \right) \leq 1.
\]

**Proof.** First we will prove by induction that \( R_{n+1} \leq c_n \), taking into account that \( R_1 = c_0 = 0 \). Assume that \( R_{k+1} \leq c_k \), so \( R_{k+1} = c_{k+1} - a \), \( a \geq 0 \). Since

\[
J_{k+2} - J_{k+1} = (a_{k+2} - a_{k+1}) - q^{a_{k+1} - c_{k+1} + a} \leq a,
\]

from (6) we have \( R_{k+2} - R_{k+1} \leq a \), which implies obviously that \( R_{k+2} \leq c_{k+1} \), and then the inequality is true for all \( n \in \mathbb{N} \).

Finally we will show that \( R_n \geq 0 \), for all \( n \in \mathbb{N} \). We define for \( x > -\frac{q}{p} \) the function

\[
H(x) = \frac{\ln \left( \frac{q}{x^{p+q}} \right)}{\ln q},
\]

which verifies \( H(n) = a_n \) if \( n \in \mathbb{N} \). For any integer \( k \geq 0 \) it is easy to see by induction on \( k \) that

\[
H(x) = k + H \left( q^k (x - 1) - \frac{q - q^k}{1 - q} \right).
\]

Let \( k \geq 0 \) be the smaller integer such that

\[
0 \leq c = q^k (x - 1) - \frac{q - q^k}{1 - q} \leq 1,
\]

then \( G(c) = c \), and from (10) and Lemma 1, \( H(x) = k + H(c) \), \( G(x) \leq k + c \).

Finally, since \( H(r) = G(r) \) for \( r = 0, 1 \), and \( H \) is also a concave function, we obviously have \( H(c) \geq c \), \( 0 \leq c \leq 1 \). Therefore \( G(x) \leq H(x) \), \( \forall x \geq 0 \), so

\[
E(X_{n,q}) = G(n) \leq H(n) = a_n,
\]

that is,

\[
R_n = a_n - E(X_{n,q}) \geq 0.
\]

In order to numerically illustrate the quality of the approximation, we present in Table 1 the absolute and relative computed errors (\( R_n \) and \( r_n \) respectively) for different values of the \( n \) and \( q \) parameters. We have only represented examples for high values of \( q \) because large teams of robots typically work on large arenas and then \( q \) takes high values (frequently close to 1).

### 3. Conclusions and remarks

The benefit of cooperation in robotics systems frequently increases as the number of cooperative robots does. Furthermore, new challenges in multi-agent systems (i.e. virtual agents, exploratory tasks)
Table 1
Absolute and relative computed errors ($R_n$ and $r_n$ respectively) for different values of the $n$ and $q$ parameters

<table>
<thead>
<tr>
<th>$n$ = 50, $q$ = 0.90</th>
<th>$n$ = 50, $q$ = 0.95</th>
<th>$n$ = 100, $q$ = 0.90</th>
<th>$n$ = 100, $q$ = 0.95</th>
<th>$n$ = 1000, $q$ = 0.90</th>
<th>$n$ = 1000, $q$ = 0.95</th>
</tr>
</thead>
<tbody>
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<td>$E[X_{n,q}]$</td>
<td>$a_n$</td>
<td>$R_n$</td>
<td>$r_n$</td>
<td>$c_n$</td>
<td>$E[X_{n,q}]$</td>
</tr>
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<td>17.66</td>
<td>17.84</td>
<td>0.179</td>
<td>0.010</td>
<td>0.424</td>
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<td>0.005</td>
<td>0.361</td>
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<td>0.212</td>
<td>0.009</td>
<td>0.461</td>
<td>77.138</td>
</tr>
</tbody>
</table>

$a$ The exact values are not easily calculable but can be bounded with the results of this section.

often involves large teams of agents. In consequence, the analysis of large teams of robots becomes a relevant issue. In this paper we have obtained a high quality approximation to the expectation of $X_{n,q}$ by a continuous monotonous function. The practical value and significance of this approximation can be appreciated in two different ways. First, because it establishes an easy procedure to approximate the expectation of $X_{n,q}$ which is difficult to obtain when $n$ is large. Second, the approximation is more tractable than the exact expression under an analytical point of view. Furthermore, the expectation of $X_{n,q}$ allows us to analyze in advance the feasibility of cooperation, avoiding the shortcoming of analyzing this feasibility by means of a heuristic, a posteriori, approach, such as frequently occurs in intelligent robotics. It is worth noting that the spread of intelligent robotics depends on reliability, and reliability strongly depends on predictability.

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References