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Positive solutions for $(n-1, 1)$ three-point boundary value problems with coefficient that changes sign

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Abstract

In this paper, we establish existence results for positive solutions for the $(n-1,1)$ three-point boundary value problems consisting of the equation

$$u^{(n)} + \lambda a(t) f\big(u(t)\big) = 0, \quad t \in (0,1),$$

with one of the following boundary value conditions:

$$u(0) = \alpha u(\eta), \qquad u(1) = \beta u(\eta),$$
$$u^{(i)}(0) = 0 \quad \text{for } i = 1, 2, \ldots, n-2,$$

and

$$u^{(n-2)}(0) = \alpha u^{(n-2)}(\eta), \qquad u^{(n-2)}(1) = \beta u^{(n-2)}(\eta),$$
$$u^{(i)}(0) = 0 \quad \text{for } i = 0, 1, \ldots, n-3,$$

where $\eta \in (0,1)$, $\alpha \geqslant 0$, $\beta \geqslant 0$, and $a : (0,1) \to \mathbb{R}$ may change sign and $\mathbb{R} = (-\infty, +\infty)$. $f(0) > 0$, $\lambda > 0$ is a parameter. Our approach is based on the Leray–Schauder degree theory. This paper is motivated by Eloe and Henderson (Nonlinear Anal. 28 (1997) 1669–1680).

Keywords: Higher-order differential equation; Positive solution; Cone; Leray–Schauder degree theory

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1. Introduction

Three-point boundary value problems for differential equations were presented by Il'in and Moiseev [10,11]. Motivated by the study of Il'in and Moiseev, Gupta in [1,2] and Ma in [3–5] studied certain three-point boundary value problems for nonlinear second-order ordinary differential equations. The solvability of two-point boundary value problems for higher-order ordinary differential equations has been discussed extensively in the literature in the past ten years; see, for example, monograph [8] and the recent paper [6]. To the best of our knowledge, existence results for positive solutions of three-point boundary value problem of higher-order ordinary differential equations, however, have not been studied previously.

In this paper, we study the existence of positive solutions of the following $(n-1,1)$ three-point boundary value problem consisting of the differential equation

$$u^{(n)} + \lambda a(t) f\big(u(t)\big) = 0, \quad t \in (0,1), \tag{1}$$

with one of the following boundary value conditions:

$$u(0) = \alpha u(\eta), \qquad u(1) = \beta u(\eta),$$
$$u^{(i)}(0) = 0 \quad \text{for } i = 1, 2, \ldots, n-2, \tag{2}$$

and

$$u^{(n-2)}(0) = \alpha u^{(n-2)}(\eta), \qquad u^{(n-2)}(1) = \beta u^{(n-2)}(\eta),$$
$$u^{(i)}(0) = 0 \quad \text{for } i = 0, 1, \ldots, n-3, \tag{3}$$

where $\eta \in (0,1)$, $\alpha \geqslant 0$, $\beta \geqslant 0$, and $a : (0,1) \to \mathbb{R}$ and $\mathbb{R} = (-\infty, +\infty)$. $f(0) > 0$, $\lambda > 0$ is a parameter, $n \geqslant 3$.

For the case where $\alpha = \beta = 0$, (1)–(2) becomes

$$\begin{cases} u^{(n)} + \lambda a(t) f(u) = 0, & 0 < t < 1, \\ u^{(i)}(0) = u(1) = 0, & i = 0, 1, 2, \ldots, n-2. \end{cases} \tag{4}$$

BVP (4) was studied by Eloe and Henderson [6]. In [6], Eloe and Henderson proved that BVP (4) has positive solutions under the following assumptions (A) and (B) or (A) and (C):

(A) $a : [0,1] \to [0, +\infty)$, $f : [0, +\infty) \to [0, +\infty)$ are continuous.
(B) $\lim_{x \to 0} (f(x)/x) = 0$ and $\lim_{x \to +\infty} (f(x)/x) = +\infty$ (super-linear).
(C) $\lim_{x \to 0} (f(x)/x) = +\infty$ and $\lim_{x \to +\infty} (f(x)/x) = 0$ (sub-linear).

BVP (1)–(2) also contains as special case the following BVP:

$$\begin{cases} u''(t) + \lambda f(t,u) = 0, & 0 < t < 1, \\ u(0) = u(1) - \beta u(\eta) = 0. \end{cases} \tag{5}$$

In [5], Ma proved that BVP (5) has positive solutions under the above conditions $0 < \beta < 1/\eta$, (A) and (B) or (A) and (C). Very recently, motivated by [12], the author in [9] proved that it has at least three positive solutions by imposing conditions on f.

In this paper, we make the following assumptions:

(A_1) $M_1 = 1 - \alpha - (\beta - \alpha)\eta^{n-1} > 0$.
(A'_1) $M_2 = 1 - \alpha - (\beta - \alpha)\eta > 0$.
(A_2) $f : [0, +\infty) \to [0, +\infty)$ is continuous and $f(0) > 0$.
(A_3) $a : [0, 1] \to \mathbb{R}$ is continuous and there is $k > 1$ such that

$$\int_0^1 G_i(t,s)a^+(s)\,ds \geqslant k \int_0^1 G_i(t,s)a^-(s)\,ds \quad \text{for } t \in [0,1],\ i = 1, 2,$$

where $a^+(t) = \max\{0, a(t)\}$ and $a^-(t) = \max\{0, -a(t)\}$, $G_i(t,s)$ is defined by

$$G_1(t,s) = \frac{1}{(n-1)!M_1} \begin{cases} (1-s)^{n-1}[\alpha\eta^{n-1} - (\alpha-1)t^{n-1}] \\ \quad -(t-s)^{n-1}[1-\alpha-(\beta-\alpha)\eta^{n-1}] \\ \quad -(\eta-s)^{n-1}[(\beta-\alpha)t^{n-1}+\alpha], \\ \quad 0 \leqslant s \leqslant t \leqslant \eta < 1 \text{ or } 0 \leqslant s \leqslant \eta < t \leqslant 1, \\ (1-s)^{n-1}[\alpha\eta^{n-1} - (\alpha-1)t^{n-1}] \\ \quad -(\eta-s)^{n-1}[(\beta-\alpha)t^{n-1}+\alpha], \\ \quad 0 \leqslant t \leqslant s \leqslant \eta < 1, \\ (1-s)^{n-1}[\alpha\eta^{n-1} - (\alpha-1)t^{n-1}], \\ \quad 0 \leqslant t \leqslant \eta \leqslant s \leqslant 1, \text{ or } 0 < \eta \leqslant t \leqslant s \leqslant 1, \\ (1-s)^{n-1}[\alpha\eta^{n-1} - (\alpha-1)t^{n-1}] \\ \quad -(t-s)^{n-1}[1-\alpha-(\beta-\alpha)\eta^{n-1}], \\ \quad 0 < \eta \leqslant s \leqslant t \leqslant 1, \end{cases}$$

for BVP (1)–(2), and

$$G_2(t,s) = \frac{1}{M} \begin{cases} -(n-2)![1-\alpha-(\beta-\alpha)\eta](t-s)^{n-1} \\ \quad +[(n-2)!(1-\alpha)t^{n-1} + (n-1)!\alpha\eta t^{n-2}](1-s) \\ \quad +[(n-2)!(\alpha-\beta)t^{n-1} - (n-1)!\alpha t^{n-2}](\eta-s), \\ \quad 0 \leqslant s \leqslant \eta \leqslant t < 1 \text{ or } 0 \leqslant s \leqslant t \leqslant \eta < 1, \\ -(n-2)![1-\alpha-(\beta-\alpha)\eta](t-s)^{n-1} \\ \quad +[(n-2)!(1-\alpha)t^{n-1} + (n-1)!\alpha\eta t^{n-2}](1-s), \\ \quad 0 \leqslant \eta \leqslant s \leqslant t \leqslant 1, \\ t^{n-2}(1-s)[(n-2)!(1-\alpha)t + (n-1)!\alpha\eta], \\ \quad 0 \leqslant \eta \leqslant t \leqslant s \leqslant 1 \text{ or } 0 \leqslant t \leqslant \eta \leqslant s \leqslant 1, \\ (1-s)[(n-2)!(1-\alpha)t^{n-1} + (n-1)!\alpha\eta t^{n-2}] \\ \quad +(\eta-s)[(n-2)!(\alpha-\beta)t^{n-1} - (n-1)!\alpha t^{n-2}], \\ \quad 0 \leqslant t \leqslant s \leqslant \eta < 1, \end{cases}$$

for BVP (1)–(3), where $M = (n-1)!(n-2)!M_2$.

Our main result is as follows.

Theorem 1. *Let* (A_1)–(A_3) *hold. Then there is a positive number* λ^* *such that BVP* (1)–(2) *has at least one positive solution for* $\lambda \in (0, \lambda^*)$.

Theorem 2. *Let* (A'_1) *and* (A_2)–(A_3) *hold. Then there is a positive number* λ^* *such that BVP* (1)–(3) *has at least one positive solution for* $\lambda \in (0, \lambda^*)$.

The organization of the paper is as follows. In Section 2, we prove Theorems 1 and 2. We now present an example.

Example 1. Consider $(n-1,1)$ three-point boundary value problem

$$\begin{cases} u'''(t)+\lambda a(t)f(u)=0, & 0<t<1,\\ u(0)=u\left(\frac{1}{2}\right), \quad u(1)=\frac{1}{2}u\left(\frac{1}{2}\right), \quad u'(0)=0, \end{cases} \tag{6}$$

where $a(t)=3/4-t$ for $t\in[0,1]$ and f satisfies (A_2). We see $M_1=1-\alpha-(\beta-\alpha)\eta^{n-1}=1/8>0$. Again, it is easy to check that

$$\int_0^1 G_1(t,s)a^-(s)\,ds=\begin{cases} \frac{1}{4^5}, & 0\leqslant t\leqslant \frac{3}{4},\\ \frac{59}{4^5\times 6}-\frac{1}{24}t^4+\frac{1}{8}t^3+\frac{9}{64}t^2-\frac{9}{128}t, & \frac{3}{4}\leqslant t\leqslant 1, \end{cases}$$

and

$$\int_0^1 G_1(t,s)a^+(s)\,ds=\begin{cases} \frac{11644}{4^6\times 30}+\frac{1}{24}t^4-\frac{1}{8}t^3+\frac{2}{4^4\times 15}t^2, & 0\leqslant t\leqslant \frac{3}{4},\\ \frac{7564}{4^6\times 30}-\frac{58}{4^4\times 15}t^2+\frac{9}{4^3\times 2}t, & \frac{3}{4}\leqslant t\leqslant 1. \end{cases}$$

Hence, one has

$$k=\inf_{t\in[0,1]}\frac{\int_0^1 G_1(t,s)a^+(s)\,ds}{\int_0^1 G_1(t,s)a^-(s)\,ds}>2.$$

Applying Theorem 1, we know that there is a number $\lambda^*>0$ such that (6) has at least one positive solution for $\lambda\in(0,\lambda^*)$. The results in [3–6,9] cannot be applied to this equation. Our theorems are new and different from [3–6,9] and are easy to check. Particularly, we do not need the assumptions that f is either super-linear or sub-linear, which was supposed in [3–6].

By the way, the proofs of the theorems are based on the Leray–Schauder fixed point theorem and motivated by [8]. In [8], Hai studied the existence of positive solutions for elliptic equation

$$\Delta u+\lambda a(t)g(u)=0, \quad u|_{\partial\Omega}=0,$$

where a may change sign. We note that the techniques in our paper are well known for certain nonlinear BVP problems, see [7] and references cited therein.

2. Proofs of theorems

In order to prove Theorem 1, we need the following lemmas.

Lemma 1. *Suppose that* $M_1=1-\alpha-(\beta-\alpha)\eta^{n-1}\neq 0$. *Then for* $y\in C[0,1]$, *the problem*

$$\begin{cases} u^{(n)}+y(t)=0, & t\in(0,1),\\ u(0)=\alpha u(\eta), \quad u(1)=\beta u(\eta), \quad u^{(i)}(0)=0 & \text{for } i=1,2,\dots,n-2, \end{cases} \tag{7}$$

has unique solution

$$u(t)=\int_0^1 G_1(t,s)y(s)\,ds,$$

where $G_1(t,s)$ *is defined in Section* 1.

Proof. To the purpose, we let

$$u(t)=-\int_0^t \frac{(t-s)^{n-1}}{(n-1)!}y(s)\,ds+At^{n-1}+B+\sum_{i=1}^{n-2}A_i t^i. \tag{8}$$

Since $u^{(i)}(0)=0$ for $i=1,2,\dots,n-2$, one gets $A_i=0$ for $i=1,2,\dots,n-2$. Now, we solve for A and B. By $u(0)=\alpha u(\eta)$ and $u(1)=\beta u(\eta)$, it follows that

$$\begin{cases} B=-\alpha\int_0^\eta \frac{(\eta-s)^{n-1}}{(n-1)!}y(s)\,ds+\alpha A\eta^{n-1}+\alpha B,\\ -\int_0^1 \frac{(1-s)^{n-1}}{(n-1)!}y(s)\,ds+A+B=-\beta\int_0^\eta \frac{(\eta-s)^{n-1}}{(n-1)!}y(s)\,ds+\beta A\eta^{n-1}+\beta B. \end{cases}$$

Solving the above equations, we get

$$\begin{cases} A=\frac{1}{M_1}\Big[(1-\alpha)\int_0^1 \frac{(1-s)^{n-1}}{(n-1)!}y(s)\,ds-(\beta-\alpha)\int_0^\eta \frac{(\eta-s)^{n-1}}{(n-1)!}y(s)\,ds\Big],\\ B=\frac{1}{M_1}\Big[-\alpha\int_0^\eta \frac{(\eta-s)^{n-1}}{(n-1)!}y(s)\,ds+\alpha\eta^{n-1}\int_0^1 \frac{(1-s)^{n-1}}{(n-1)!}y(s)\,ds\Big]. \end{cases}$$

Substituting A and B into (8), one has

$$\begin{aligned} u(t)&=-\int_0^t \frac{(t-s)^{n-1}}{(n-1)!}y(s)\,ds\\ &\quad+\frac{1}{M_1}\left[-\alpha\int_0^\eta \frac{(\eta-s)^{n-1}}{(n-1)!}y(s)\,ds+\alpha\eta^{n-1}\int_0^1 \frac{(1-s)^{n-1}}{(n-1)!}y(s)\,ds\right]\\ &\quad+\frac{t^{n-1}}{M_1}\left[(1-\alpha)\int_0^1 \frac{(1-s)^{n-1}}{(n-1)!}y(s)\,ds-(\beta-\alpha)\int_0^\eta \frac{(\eta-s)^{n-1}}{(n-1)!}y(s)\,ds\right]\\ &=\int_0^1 G_1(t,s)y(s)\,ds. \qquad \square \end{aligned}$$

Lemma 2. *Let* $M_1>0$. *If* $y\in C[0,1]$ *and* $y(t)\geqslant 0$, *then the unique solution of* (7) *satisfies* $u(t)\geqslant 0$ *for all* $t\in[0,1]$.

Proof. It suffices to prove that

$$G_1(t,s)\geqslant 0 \quad \text{for } (t,s)\in[0,1]\times[0,1]. \tag{9}$$

We consider four cases.

Case 1: $0 \leqslant s \leqslant t \leqslant \eta < 1$ or $0 \leqslant s \leqslant \eta \leqslant t \leqslant 1$.

$$\begin{aligned}
&(1-s)^{n-1}\big[\alpha\eta^{n-1}-(\alpha-1)t^{n-1}\big]-(t-s)^{n-1}\big[1-\alpha-(\beta-\alpha)\eta^{n-1}\big]\\
&\quad -(\eta-s)^{n-1}\big[(\beta-\alpha)t^{n-1}+\alpha\big]\\
&\quad =(1-s)^{n-1}\left\{\big[\alpha\eta^{n-1}-(\alpha-1)t^{n-1}\big]-\left(\frac{t-s}{1-s}\right)^{n-1}\big[1-\alpha-(\beta-\alpha)\eta^{n-1}\big]\right.\\
&\quad \left.-\left(\frac{\eta-s}{1-s}\right)^{n-1}\big[(\beta-\alpha)t^{n-1}+\alpha\big]\right\}\\
&\quad \geqslant (1-s)^{n-1}\big\{\big[\alpha\eta^{n-1}-(\alpha-1)t^{n-1}\big]-t^{n-1}\big[1-\alpha-(\beta-\alpha)\eta^{n-1}\big]\\
&\quad -\eta^{n-1}\big[(\beta-\alpha)t^{n-1}+\alpha\big]\big\}\geqslant 0.
\end{aligned}$$

Case 2: $0 \leqslant t \leqslant s \leqslant \eta < 1$.

$$\begin{aligned}
&(1-s)^{n-1}\big[\alpha\eta^{n-1}-(\alpha-1)t^{n-1}\big]-(\eta-s)^{n-1}\big[(\beta-\alpha)t^{n-1}+\alpha\big]\\
&\quad =(1-s)^{n-1}\left\{\big[\alpha\eta^{n-1}-(\alpha-1)t^{n-1}\big]-\left(\frac{\eta-s}{1-s}\right)^{n-1}\big[(\beta-\alpha)t^{n-1}+\alpha\big]\right\}\\
&\quad \geqslant (1-s)^{n-1}\big\{\big[\alpha\eta^{n-1}-(\alpha-1)t^{n-1}\big]-\eta^{n-1}\big[(\beta-\alpha)t^{n-1}+\alpha\big]\big\}\\
&\quad =(1-s)^{n-1}t^{n-1}\big[1-\alpha-(\beta-\alpha)\eta^{n-1}\big]\geqslant 0.
\end{aligned}$$

Case 3: $0 \leqslant t \leqslant \eta \leqslant s \leqslant 1$ or $0 < \eta \leqslant t \leqslant s \leqslant 1$.

$$\begin{aligned}
&(1-s)^{n-1}\big[\alpha\eta^{n-1}-(\alpha-1)t^{n-1}\big]\\
&\quad \geqslant \begin{cases} 0, & 0\leqslant\alpha\leqslant 1,\\ \eta^{n-1}(1-s)^{n-1}\geqslant 0, & \alpha>1 \text{ and } t\leqslant\eta,\\ (1-s)^{n-1}[\alpha\eta^{n-1}-\alpha+1\geqslant\beta\eta^{n-1}(1-s)^{n-1}]\geqslant 0, & \alpha>1 \text{ and } 1\geqslant t\geqslant\eta. \end{cases}
\end{aligned}$$

Case 4: $0 < \eta \leqslant s \leqslant t \leqslant 1$.

$$\begin{aligned}
&(1-s)^{n-1}\big[\alpha\eta^{n-1}-(\alpha-1)t^{n-1}\big]-(t-s)^{n-1}\big[1-\alpha-(\beta-\alpha)\eta^{n-1}\big]\\
&\quad =(1-s)^{n-1}\left\{\big[\alpha\eta^{n-1}-(\alpha-1)t^{n-1}\big]-\left(\frac{t-s}{1-s}\right)^{n-1}\big[1-\alpha-(\beta-\alpha)\eta^{n-1}\big]\right\}\\
&\quad \geqslant (1-s)^{n-1}\big\{\big[\alpha\eta^{n-1}-(\alpha-1)t^{n-1}\big]-t^{n-1}\big[1-\alpha-(\beta-\alpha)\eta^{n-1}\big]\big\}\\
&\quad =(1-s)^{n-1}\big[\alpha\eta^{n-1}(1-t^{n-1})+\beta\eta^{n-1}t^{n-1}\big]\geqslant 0.
\end{aligned}$$

The proof is complete. □

Lemma 3. *Suppose that* (A_1)–(A_3) *hold. Then for every* $0<\delta<1$, *there exists a positive number* $\bar\lambda$ *such that, for* $\lambda\in(0,\bar\lambda)$, *the equation*

$$\begin{cases} u^{(n)}+\lambda a^{+}(t)f(u(t))=0, & t\in(0,1),\\ u(0)=\alpha u(\eta), \quad u(1)=\beta u(\eta), \quad u^{(i)}(0)=0 & \textit{for } i=1,2,\dots,n-2, \end{cases} \tag{10}$$

has a positive solution $\bar u_\lambda$ *with* $\|\bar u_\lambda\|\to 0$ *as* $\lambda\to 0$ *and*

$$\bar{u}_\lambda \geqslant \lambda\delta f(0)\big\|p(t)\big\|, \tag{11}$$

where

$$p(t) = \int_0^1 G_1(t,s)a^+(s)\,ds.$$

Proof. We know that $p(t) \geqslant 0$ for $t \in \mathbb{R}$ and (10) is equivalent to the integral equation

$$u(t) = \lambda \int_0^1 G(t,s)a^+(s)f\big(u(s)\big)\,ds := Tu(t), \tag{12}$$

where $u \in X := C[0,1]$. It is easy to prove that T is completely continuous, $TX \subset X$ and the fixed points of T are solutions of (1)–(2). We shall apply the Leray–Schauder fixed point theorem to prove T has at least one fixed point for small λ.

Let $\epsilon > 0$ be such that

$$f(t) \geqslant \delta f(0) \quad \text{for } 0 \leqslant t \leqslant \epsilon. \tag{13}$$

Suppose that

$$0 < \lambda < \frac{\epsilon}{2\|p\|\bar{f}(\epsilon)} := \bar{\lambda},$$

where $\bar{f}(t) = \max_{0\leqslant s\leqslant t} f(s)$, since

$$\lim_{t\to 0^+} \frac{\bar{f}(t)}{t} = +\infty,$$

again $\bar{f}(\epsilon)/\epsilon < 1/(2\|p\|\lambda)$, there is $r_\lambda \in (0,\epsilon)$ such that

$$\frac{\bar{f}(r_\lambda)}{r_\lambda} = \frac{1}{2\lambda\|p\|}.$$

We note that this implies $r_\lambda \to 0$ as $\lambda \to 0$.

Now, consider the homotopy equation

$$u = \theta T u, \quad \theta \in (0,1).$$

Let $u \in X$ and $\theta \in (0,1)$ be such that $u = \theta T u$. We claim that $\|u\| \neq r_\lambda$. In fact,

$$u(t) = \theta\lambda \int_0^1 G_1(t,s)a^+(s)f\big(u(s)\big)\,ds.$$

Set

$$w(t) = \theta\lambda \int_0^1 G_1(t,s)a^+(s)\bar{f}\big(\|u\|\big)\,ds \leqslant \theta\lambda\bar{f}\big(\|u\|\big)p(t).$$

Then by $f(u) \leqslant \bar{f}(\|u\|)$, we know that $u(t) \leqslant w(t)$ for all $t \in \mathbb{R}$. Moreover, we have

$$\|u\| \leqslant \lambda \|p\| \bar{f}\big(\|u\|\big),$$

i.e.,

$$\frac{\bar{f}(\|u\|)}{\|u\|} \geqslant \frac{1}{\lambda \|p\|},$$

which implies that $\|u\| \neq r_\lambda$. Thus by Leray–Schauder fixed point theorem, T has a fixed point $\bar{x}_\lambda$ with

$$\|\bar{u}_\lambda\| \leqslant r_\lambda < \epsilon.$$

Moreover, combining (12) and (13), we get

$$\bar{u}_\lambda \geqslant \lambda \delta f(0) p(t), \quad t \in \mathbb{R}. \tag{14}$$

This completes the proof. □

Proof of Theorem 1. Let

$$q(t) = \int_0^1 G_1(t,s) a^-(s)\, ds. \tag{15}$$

Then $q(t) \geqslant 0$. Since $p(t)/q(t) \geqslant k > 1$. Choosing $d \in (0,1)$ such that $kd > 1$. There is $c > 0$ such that $|f(y)| \leqslant kdf(0)$ for $y \in [0,c]$, then

$$q(t)\big|f(y)\big| \leqslant dp(t)\, f(0), \quad t \in \mathbb{R},\ y \in [0,c].$$

Fix $\delta \in (d,1)$ and let $\lambda^* > 0$ be such that

$$\|\bar{u}_\lambda\| + \lambda \delta f(0) \|p\| \leqslant c, \quad \lambda \in (0,\lambda^*), \tag{16}$$

where $\bar{u}_\lambda$ is given by Lemma 1 and

$$\big|f(x) - f(y)\big| \leqslant f(0) \frac{\delta - d}{2} \tag{17}$$

for $x, y \in [-c,c]$ with $|x-y| \leqslant \lambda^* \delta f(0) \|p\|$.

Let $\lambda \in (0,\lambda^*)$; we look for a solution x_λ of the form $\bar{u}_\lambda + y_\lambda$ such that y_λ solves the following equation:

$$\begin{cases} y^{(n)} + \lambda a^+(t)\big[f(\bar{u}_\lambda + y) - f(\bar{u}_\lambda)\big] - \lambda a^-(t) f(\bar{u}_\lambda + y) = 0, & 0 < t < 1, \\ y(0) = \alpha y(\eta), \quad y(1) = \beta y(\eta), \quad y^{(i)}(0) = 0 & \text{for } i = 1,2,\ldots,n-2. \end{cases} \tag{18}$$

For each $y \in C[0,1]$, let $w = Ty$ be the solution of $\lambda \in (0,\lambda^*)$; we look for a solution x_λ of the form $\bar{u}_\lambda + y_\lambda$ such that y_λ solves the following equation:

$$\begin{cases} w^{(n)} + \lambda a^+(t)\big[f(\bar{u}_\lambda + y) - f(\bar{u}_\lambda)\big] - \lambda a^-(t) f(\bar{u}_\lambda + y) = 0, & 0 < t < 1, \\ w(0) = \alpha w(\eta), \quad w(1) = \beta w(\eta), \quad w^{(i)}(0) = 0 & \text{for } i = 1,2,\ldots,n-2. \end{cases}$$

Then T is completely continuous. Let $y \in X$ and $\theta \in (0,1)$ be such that $y = \theta T y$; then we have

$$y^{(n)} + \lambda\theta a^{+}(t)\big[f(\bar{u}_\lambda + y) - f(\bar{u}_\lambda)\big] - \lambda\theta a^{-}(t) f(\bar{u}_\lambda + v) = 0, \quad 0 < t < 1.$$

We claim that $\|y\| \neq \lambda\delta f(0)\|p\|$. Suppose to the contrary that $\|y\| = \lambda\delta f(0)\|p\|$. Then by (16) and (17), we get

$$\|\bar{u}_\lambda + y\| \leqslant \|\bar{u}_\lambda\| + \|y\| \leqslant c \tag{19}$$

and

$$\big|f(\bar{u}_\lambda + y) - f(\bar{u}_\lambda)\big| \leqslant f(0)\frac{\delta - d}{2}. \tag{20}$$

Using (12) and $q(t)|f(y)| \leqslant dp(t)\, f(0)$, we get

$$\begin{aligned}
\big|y(t)\big| &= \lambda\left|\int_0^1 G_1(t,s)a^{+}(s)\big[f\big(\bar{u}_\lambda(s) + y(s)\big) - f\big(\bar{u}_\lambda(s)\big)\big]\,ds \right.\\
&\qquad \left. + \lambda\int_0^1 G_1(t,s)a^{-}(s) f\big(\bar{u}_\lambda(s) + y(s)\big)\,ds\right|\\
&\leqslant \lambda\left|\int_0^1 G_1(t,s)a^{+}(s) f(0)\frac{\delta - d}{2}\,ds + \lambda\int_0^1 G_1(t,s)a^{-}(s)\frac{p(t)}{q(t)}\,df(0)\,ds\right|\\
&\leqslant \lambda\frac{\delta - d}{2}p(t) + \lambda\, df(0)\,p(t) = \lambda\frac{\delta + d}{2} f(0)p(t).
\end{aligned}$$

In particular,

$$\|y\| \leqslant \lambda\frac{\delta + d}{2} f(0)\|p\| < \lambda\delta f(0)\|p\|, \tag{21}$$

a contradiction and the claim is proved. Thus by Leray–Schauder fixed point theorem, T has a fixed point y_λ with

$$\|y_\lambda\| \leqslant \lambda\delta f(0)\|p\|.$$

Using Lemma 1 and (21), we obtain

$$u_\lambda(t) \geqslant \bar{u}_\lambda - \|y_\lambda\| \geqslant \lambda\delta f(0)p(t) - \lambda\frac{\delta + d}{2} f(0)p(t) = \lambda\frac{\delta - d}{2} f(0)p(t) > 0,$$

i.e., u_λ is a positive T-periodic solution. The proof of Theorem 1 is complete. □

Proof of Theorem 2. Similarly, let $t \in C[0,1]$. The unique solution of the equation

$$\begin{cases} u^{(n)} + y(t) = 0, \quad t \in (0,1),\\ u^{(n-2)}(0) = \alpha u^{(n-2)}(\eta), \qquad u^{(n-2)}(1) = \beta u^{(n-2)}(\eta),\\ u^{(i)}(0) = 0 \quad \text{for } i = 0, 1, 2, \ldots, n-3, \end{cases} \tag{22}$$

has unique solution

$$u(t)=\int_0^1 G_2(t,s)y(s)\,ds,$$

where $G_2(t,s)$ is defined as follows:

$$G_2(t,s)=\frac{1}{M}\begin{cases} -(n-2)![1-\alpha-(\beta-\alpha)\eta](t-s)^{n-1} \\ \quad +[(n-2)!(1-\alpha)t^{n-1}+(n-1)!\alpha\eta t^{n-2}](1-s) \\ \quad +[(n-2)!(\alpha-\beta)t^{n-1}-(n-1)!\alpha t^{n-2}](\eta-s), \\ \quad 0\leqslant s\leqslant \eta\leqslant t<1 \text{ or } 0\leqslant s\leqslant t\leqslant \eta<1, \\ -(n-2)![1-\alpha-(\beta-\alpha)\eta](t-s)^{n-1} \\ \quad +[(n-2)!(1-\alpha)t^{n-1}+(n-1)!\alpha\eta t^{n-2}](1-s), \\ \quad 0\leqslant \eta\leqslant s\leqslant t\leqslant 1, \\ t^{n-2}(1-s)[(n-2)!(1-\alpha)t+(n-1)!\alpha\eta], \\ \quad 0\leqslant \eta\leqslant t\leqslant s\leqslant 1 \text{ or } 0\leqslant t\leqslant \eta\leqslant s\leqslant 1, \\ (1-s)[(n-2)!(1-\alpha)t^{n-1}+(n-1)!\alpha\eta t^{n-2}] \\ \quad +(\eta-s)[(n-2)!(\alpha-\beta)t^{n-1}-(n-1)!\alpha t^{n-2}], \\ \quad 0\leqslant t\leqslant s\leqslant \eta<1, \end{cases}$$

where $M=(n-1)!(n-2)!M_2=(n-1)!(n-2)!(1-\alpha-(\beta-\alpha)\eta)$. It is easy to see that if $y(t)\geqslant 0$, then $u(t)\geqslant 0$ for all $t\in[0,1]$. The proof is similar to that of Theorem 1 and thus is omitted. □

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