On the Euler's Factor of an Odd Perfect Number

PAOLO STARNI

Liceo "Virgilio," 39049 Vipiteno, Bolzano, Italy

Communicated by Hans Zassenhaus

Received January 22, 1990; revised March 29, 1990

Let $n = \pi^x \prod_k u_k^{2b_k}$ be an odd perfect number; $\pi^x$, with $\pi \equiv x \equiv 1 \pmod{4}$, is the Euler's factor. It is shown that:

(a) $\pi \equiv x \pmod{8}$ if each prime $u_k \equiv 1 \pmod{4}$.
(b) $\delta = \sigma(\pi^x)/2$ cannot be prime if each prime $u_k \equiv 3 \pmod{4}$.

Let $\sigma(n)$ be the sum of the positive divisors of a natural number $n$; $n$ is said to be perfect if and only if $\sigma(n) = 2n$.

General remarks on the problem of the odd perfect numbers may be found in [5, p. 81–85]. Odd perfect numbers, if any, must be of the form

$$n = \pi^x \prod_k u_k^{2b_k},$$

where $\pi, u_k$ are odd primes and $\pi \equiv x \equiv 1 \pmod{4}$ [1, p. 19]; $\pi^x$ is called the Euler's factor of $n$. Besides, $n$ is $\equiv 1 \pmod{12}$ or $\equiv 9 \pmod{36}$ (Touchard, 1953, cited in [5, p. 83]; a simpler proof is given by Raghavachari [4]).

We shall write also

$$n = \pi^x \prod_j q_j^{2c_j} \prod_i p_i^{2d_i}, \quad (1)$$

where $q_j \equiv 1 \pmod{4}$ and $p_i \equiv 3 \pmod{4}$.

In what follows "$p_i = 1$" ["$q_j = 1$"] is an abbreviation for "$\prod_i p_i^{2d_i} = 1$" ["$\prod_j q_j^{2c_j} = 1$"], which means that the product therein is vacuous.

In the cases $p_i = 1$ and $q_j = 1$, we will obtain nonexistence and conditions on $\sigma(\pi^x)$, respectively. Many conditions regarding the exponents $b_k$ are
known. Some of these are summarized by Hagis and McDaniel, [3]: for example, it cannot be \( b_k = 2 \) (Kanold in [3]). As for the exponents \( b_k \), we will prove, in Sections 2 and 3, that

\[
\text{if } n \text{ of the form (1) is perfect and } p_i = 1, \text{ then the } c_i's \text{ must be even.}
\]

By a result of Ewell [2], we will prove in Section 4

\[\text{the nonexistence of odd perfect numbers decomposable into primes all of the type } \equiv 1 \pmod{4}, \text{ if } \pi \not\equiv x \pmod{8}.\]

An immediate condition on \( \sigma(\pi^a) \) is \( \sigma(\pi^a) = 2\delta \), where \( \delta \) is odd and not a divisor of \( \pi \). Note that \( \delta \) can be prime, as in \( \sigma(13) = 2 \cdot 7 \), or composite, as in \( \sigma(29) = 2 \cdot 15 \).

Moreover, about \( \sigma(\pi^a) \), we will prove, in Section 5, that

\[
\text{if } n \text{ of the form (1) is perfect and } q_j = 1, \text{ then } \delta = \sigma(\pi^a)/2 \text{ must be composite.}
\]

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Remarking that the square of an odd integer is \( \equiv 1 \pmod{8} \), we raise the question: if \( r = 2k + 1 \) is a prime and \( b \geq 2 \), what form does \( \sigma(r^{2b}) \) take?

We have

\[
\sigma(r^{2b}) = 1 + r + r^2 + r^3 + \ldots + r^{2b}
\]

in which

\[
r^2 + r^4 + \ldots + r^{2b} = 8m + b
\]

and

\[
r^3 + r^5 + \ldots + r^{2b-1} = r(r^2 + r^4 + \ldots + r^{2b-2}) = r(8h + b - 1).
\]

Thus, recalling that \( r = 2k + 1 \), \( \sigma(r^{2b}) \) becomes

\[
\sigma(r^{2b}) = 2[4D + b(k + 1)] + 1. \quad (2)
\]

In particular from (2) we obtain immediately the following lemma:

(i) if \( b \) is odd and \( k \) even, i.e., \( r \equiv 1 \pmod{4} \), then \( \sigma(r^{2b}) \equiv 3 \pmod{4} \). In all the other cases \( \sigma(r^{2b}) \equiv 1 \pmod{4} \) (if \( b = 1 \), \( \sigma(r^2) = 1 + r + r^2 \), same result).
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Now we are able to prove that

(ii) if n of the form (1) is perfect and \( p_i = 1 \), then the \( c_j \)'s are even.

**Proof.** Since \( n \) is perfect, i.e., \( \sigma(n) = 2n \), we have

\[
2^\pi \prod_j q_j^{2c_j} = \prod_j \sigma(q_j^{2c_j})
\]

in which the prime odd factors on the left are \( \equiv 1 \) (mod 4).

Hence all the prime divisors of \( \sigma(q_j^{2c_j}) \) must be \( \equiv 1 \) (mod 4), so that

\[
\sigma(q_j^{2c_j}) \equiv 1 \pmod{4}. \tag{3}
\]

From (3) and (i) we obtain that the \( c_j \)'s must be even. \( \blacksquare \)

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In relation to an odd perfect number of the form (1), we set \( \pi = 4\beta + 1 \), \( \alpha = 4\varepsilon + 1 \), and \( q_j = 2\beta_j - 1 \). Ewell, congruence (2) in [2], has proved that

\[
\beta + \varepsilon + \sum_j \beta_j c_j \equiv 0 \pmod{2}. \tag{4}
\]

From this Ewell’s congruence and (ii), it follows:

(iii) if \( n \) of the form (1) is perfect and \( p_i = 1 \), then \( \pi \equiv \alpha \) (mod 8).

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(iv) if \( n \) of the form (1) is perfect, then \( \delta \equiv \prod_j \sigma(q_j^{2c_j}) \pmod{4} \), where \( \delta = \sigma(\pi^2)/2 \).

**Proof.** From the definition of perfect number, setting \( \sigma(\pi^2) = 2\delta \), we have

\[
\pi^2 \prod_j q_j^{2c_j} \prod_i p_i^{2d_i} = \delta \prod_j \sigma(q_j^{2c_j}) \prod_i \sigma(p_i^{2d_i}). \tag{4}
\]

The left hand side of (4) is \( \equiv 1 \) (mod 4); all the factors \( \sigma(p_i^{2d_i}) \) are likewise \( \equiv 1 \) (mod 4), so

\[
\delta \prod_i \sigma(q_j^{2c_j}) \equiv 1 \pmod{4},
\]
thus

\[ \delta \equiv \prod_j \sigma(q_j^{2^\ell}) \quad (\text{mod } 4). \]

When \( q_j = 1 \), from (iv) we obtain

\[ \delta \equiv 1 \quad (\text{mod } 4) \quad (5) \]

and, in particular we can state as a theorem the following:

(v) if \( n \) of the form (1) is perfect and \( q_j = 1 \), then \( \delta = \sigma(\pi^2)/2 \) is composite.

Proof (by Contradiction). We can write

\[ \pi^2 \prod_i p_i^{2d_i} = \delta \prod_i \sigma(p_i^{2d_i}) \]

in which, excluding \( \pi \), the prime factors on the left are \( \equiv 3 \) (mod 4). Let us suppose \( \delta \) prime; since \( \delta \neq \pi \), it must be \( \delta \equiv 3 \) (mod 4) and that is in contradiction with (5).

ACKNOWLEDGMENT

I thank Professor P. Piazzoli (University of Cagliari) for useful comments and advice.

REFERENCES

3. W. L. McDaniel and P. Hagis, Some results concerning the non-existence of odd perfect numbers of the form \( p^2M^{2^\ell} \), Fibonacci Quart. 131 (1975), 25–28.