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Annihilators of tensor density modules

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Abstract

We describe the two-sided ideals in the universal enveloping algebras of the Lie algebras of vector fields on the line and the circle which annihilate the tensor density modules. Both of these Lie algebras contain the projective subalgebra, a copy of $\mathfrak{sl}_2$. The restrictions of the tensor density modules to this subalgebra are duals of Verma modules (of $\mathfrak{sl}_2$) for $\mathrm{Vec}(\mathbb{R})$ and principal series modules (of $\mathfrak{sl}_2$) for $\mathrm{Vec}(S^1)$. Thus our results are related to the well-known theorem of Duflo describing the annihilating ideals of Verma modules of reductive Lie algebras. We find that, in general, the annihilator of a tensor density module of $\mathrm{Vec}(\mathbb{R})$ or $\mathrm{Vec}(S^1)$ is generated by the Duflo generator of its annihilator over $\mathfrak{sl}_2$ (the Casimir operator minus a scalar) together with one other generator, a cubic element of $\mathfrak{U}(\mathrm{Vec}(\mathbb{R}))$ not contained in $\mathfrak{U}(\mathfrak{sl}_2)$.

Keywords: Annihilating ideals; Tensor density modules

1. Introduction and results over the line

It is a well-known theorem of Duflo's that the ideal in the universal enveloping algebra $\mathfrak{U}(\mathfrak{g})$ of a finite-dimensional complex semisimple Lie algebra $\mathfrak{g}$ which annihilates a Verma module is generated by its intersection with the center of $\mathfrak{U}(\mathfrak{g})$ [Du71]. In this paper we discuss the annihilating ideals of certain modules of the Lie algebra $\mathrm{Vec}(S^1)$, the vector fields on the circle, and its subalgebra $\mathrm{Vec}(\mathbb{R})$, the vector fields on the line.

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$\mathrm{Vec}(S^1)$ has two fundamental types of modules: the tensor density modules, also known as modules of the intermediate series, and the Verma modules. The same holds for its universal central extension, the Virasoro Lie algebra Vir; see [MP91] and [Ma92] for the roles played by these modules in the classification of the irreducible admissible representations of Vir. Mathieu has proven that Duflo's theorem carries over unchanged to the Verma modules of both $\mathrm{Vec}(S^1)$ and Vir [Ma].[2]

Here we shall describe the annihilating ideals of the tensor density modules. The center of Vir acts trivially on these modules, so it is enough to work with $\mathrm{Vec}(S^1)$. It turns out that the results for $\mathrm{Vec}(S^1)$ are essentially the same as those for $\mathrm{Vec}(\mathbb{R})$, and that the proofs for $\mathrm{Vec}(S^1)$ are based on those for $\mathrm{Vec}(\mathbb{R})$. We begin by stating the theorems for $\mathrm{Vec}(\mathbb{R})$; they are stated for $\mathrm{Vec}(S^1)$ in Section 5.

We define $\mathrm{Vec}(\mathbb{R})$ to be the complex polynomial vector fields on $\mathbb{R}$:

$$\mathrm{Vec}(\mathbb{R}) := \big\{ f(x)D\colon\ f : \mathbb{R} \to \mathbb{C} \text{ is a polynomial} \big\},$$

where D denotes d/dx. We will work with the basis

$$\big\{ e_n := x^{n+1}D\colon\ n \geqslant -1 \big\}$$

of $\mathrm{Vec}(\mathbb{R})$, whose brackets are $[e_n, e_m] = (m-n)e_{m+n}$.

We will make frequent use of the *projective subalgebra* $\mathfrak{a}$ of $\mathrm{Vec}(\mathbb{R})$, the infinitesimal linear fractional transformations, a copy of $\mathfrak{sl}_2$. We will also need $\mathfrak{a}$'s *Casimir operator* Q and its *affine subalgebra* $\mathfrak{b}$, a Borel subalgebra:

$$\mathfrak{a} := \mathrm{Span}_{\mathbb{C}}\{e_{-1}, e_0, e_1\}, \qquad \mathfrak{b} := \mathrm{Span}_{\mathbb{C}}\{e_{-1}, e_0\}, \qquad Q := e_0^2 - e_0 - e_1 e_{-1}.$$

As is well known, the center of $\mathfrak{U}(\mathfrak{a})$ is the polynomial algebra $\mathbb{C}[Q]$.

For any $\gamma \in \mathbb{C}$, the space of polynomial tensor densities on $\mathbb{R}$ of degree γ is

$$\mathcal{F}(\gamma) := \big\{ g(x)\, dx^\gamma\colon\ g(x) \text{ a polynomial on } \mathbb{R} \big\}.$$

It is a $\mathrm{Vec}(\mathbb{R})$-module under the action

$$\pi_\gamma(fD)\big(g\, dx^\gamma\big) := (fg' + \gamma f' g)\, dx^\gamma. \tag{1}$$

For $\gamma \neq 0$, $\mathcal{F}(\gamma)$ is irreducible. The module $\mathcal{F}(0)$ is simply the space of polynomials, which is indecomposable and contains the trivial submodule of constants. The corresponding quotient is irreducible and isomorphic to $\mathcal{F}(1)$.

At this point we recall some standard notation pertaining to any Lie algebra $\mathfrak{g}$. The *adjoint action* of $\mathfrak{g}$ on $\mathfrak{U}(\mathfrak{g})$ is $\mathrm{ad}(X)\Omega = X\Omega - \Omega X$. Extend ad to an action of $\mathfrak{U}(\mathfrak{g})$ on itself as usual

[2] In fact, Mathieu has extended Duflo's theorem to the following more general setting [Ma]. Let $\mathfrak{g}$ be a finitely generated $\mathbb{Z}$-graded Lie algebra whose negative part $\mathfrak{g}^- = \bigoplus_{n<0} \mathfrak{g}_n$ is infinite-dimensional. Assume that the center $\mathfrak{z}$ of $\mathfrak{g}$ lies in $\mathfrak{g}_0$ and that $\mathfrak{g}/\mathfrak{z}$ is simple. Let χ be a representation of the Borel subalgebra $\mathfrak{g}_0 \oplus \mathfrak{g}^+$ in which $\mathfrak{z}$ acts by scalars and $\mathfrak{g}^+$ acts trivially, and write M_χ for the generalized Verma module $\mathfrak{U}(\mathfrak{g}) \otimes_{\mathfrak{U}(\mathfrak{g}_0 \oplus \mathfrak{g}^+)} \chi$.

Theorem. *The annihilator of M_χ in $\mathfrak{U}(\mathfrak{g})$ is generated by its intersection with the center of $\mathfrak{U}(\mathfrak{g})$. Moreover, this center is equal to $\mathfrak{U}(\mathfrak{z})$ (see Lemma* A.4 *of this paper).*

(note that in general, $\mathrm{ad}(\Theta)\Omega$ is not $\Theta\Omega - \Omega\Theta$). Let $\mathfrak{U}^k(\mathfrak{g})$ denote the *degree filtration* of $\mathfrak{U}(\mathfrak{g})$, which of course is ad-invariant. Given any subspace J of $\mathfrak{U}(\mathfrak{g})$, define

$$J^k := J \cap \mathfrak{U}^k(\mathfrak{g}).$$

For any module (π, M) of $\mathfrak{g}$, write $\mathrm{Ann}_{\mathfrak{g}} M$ for the two-sided ideal of $\mathfrak{U}(\mathfrak{g})$ annihilating M, the kernel of the extension of π to $\mathfrak{U}(\mathfrak{g})$:

$$\mathrm{Ann}_{\mathfrak{g}} M := \mathrm{kernel}(\pi|_{\mathfrak{U}(\mathfrak{g})}).$$

Given any subset S of $\mathfrak{U}(\mathfrak{g})$, let $\langle S\rangle_{\mathfrak{g}}$ denote the two-sided ideal in $\mathfrak{U}(\mathfrak{g})$ generated by S, that is, $\mathfrak{U}(\mathfrak{g})S\mathfrak{U}(\mathfrak{g})$. Let $\mathfrak{U}^+(\mathfrak{g})$ denote $\langle\mathfrak{g}\rangle_{\mathfrak{g}}$, the annihilator of the trivial module.

Suppose that $\mathfrak{g}$ is a subalgebra of $\mathrm{Vec}(\mathbb{R})$ containing the infinitesimal rotation e_0, and let M be a $\mathfrak{g}$-module. In this case we write M_μ for the μ-eigenspace of e_0, which is called the *μ-weight space* of M. We say that an element of M is *homogeneous* if it lies in some weight space. If $\mathfrak{g}$ contains $\mathfrak{b}$, let $M^{e_{-1}}$ denote the kernel of the action of e_{-1}. We refer to elements of $M^{e_{-1}}$ as *lowest weight vectors.*

Our goal in this paper is to find the simplest possible generating sets for the ideals $\mathrm{Ann}_{\mathrm{Vec}(\mathbb{R})} \mathcal{F}(\gamma)$. We will always seek generators which are homogeneous lowest weight vectors with respect to the adjoint action. For $\gamma \neq 0$ or 1, we will see that it is possible to find two such generators but not one. For $\gamma = 0$ or 1 no collection of lowest weight vectors generates the ideal, but it is possible to find a single homogeneous generator which is almost a lowest weight vector in the sense that it is annihilated by $\mathrm{ad}(e_1e_{-1})$ (this means that it is a lowest weight vector modulo some submodule).

Note that if M is any $\mathfrak{a}$-module, the Casimir operator Q acts on $M_\mu^{e_{-1}}$ as the scalar $\mu^2 - \mu$. For example, $\mathcal{F}(\gamma)^{e_{-1}} = \mathbb{C}\,dx^\gamma$ is the γ-weight space $\mathcal{F}(\gamma)_\gamma$, which for $\gamma \neq 0$ generates $\mathcal{F}(\gamma)$ under the action of $\mathfrak{a}$. It follows that Q acts on $\mathcal{F}(\gamma)$ by the scalar $\gamma^2 - \gamma$, and so $Q - \gamma^2 + \gamma$ is in $\mathrm{Ann}_{\mathfrak{a}} \mathcal{F}(\gamma)$.

We remark that the $\mathcal{F}(\gamma)$ are the duals of the Verma modules of $\mathfrak{a}$, so by Duflo's theorem $Q - \gamma^2 + \gamma$ generates $\mathrm{Ann}_{\mathfrak{a}} \mathcal{F}(\gamma)$. Note that $(1-\gamma)^2 - (1-\gamma) = \gamma^2 - \gamma$, so we have

$$\mathrm{Ann}_{\mathfrak{a}} \mathcal{F}(\gamma) = \langle Q - \gamma^2 + \gamma\rangle_{\mathfrak{a}} = \mathrm{Ann}_{\mathfrak{a}} \mathcal{F}(1-\gamma)$$

(this is explained by the fact that $\mathcal{F}(1-\gamma)$ is closely related to the dual of $\mathcal{F}(\gamma)$). The situation is different for the $\mathrm{Vec}(\mathbb{R})$-annihilators: $Q - \gamma^2 + \gamma$ does not generate $\mathrm{Ann}_{\mathrm{Vec}(\mathbb{R})} \mathcal{F}(\gamma)$ for any γ, and $\mathrm{Ann}_{\mathrm{Vec}(\mathbb{R})} \mathcal{F}(\gamma)$ is not equal to $\mathrm{Ann}_{\mathrm{Vec}(\mathbb{R})} \mathcal{F}(1-\gamma)$ unless $\gamma = 0$, 1, or 1/2.

In order to state our results we define the following elements of $\mathfrak{U}(\mathrm{Vec}(\mathbb{R}))$:

$$Z_1 := (e_1e_0 - e_2e_{-1} - e_1)/2, \qquad Y_0 := Q(e_0 - 1/2) - Z_1e_{-1}.$$

One checks that on $\mathcal{F}(\gamma)$, Z_1 acts as multiplication by $(\gamma^2 - \gamma)x$ and Y_0 acts by the scalar $(\gamma - 1/2)(\gamma^2 - \gamma)$. Therefore $Y_0 - (\gamma - 1/2)(\gamma^2 - \gamma)$ is in $\mathrm{Ann}_{\mathrm{Vec}(\mathbb{R})} \mathcal{F}(\gamma)$.

The following two theorems are our main results for $\mathrm{Vec}(\mathbb{R})$; they will be proven in Section 4. As we mentioned, they carry over essentially unchanged to the tensor density modules of $\mathrm{Vec}(S^1)$; see Section 5.

Theorem 1.1. *For $\gamma \neq 0$ or 1, the ideals $\mathrm{Ann}_{\mathrm{Vec}(\mathbb{R})} \mathcal{F}(\gamma)$ are all distinct. Each of them is generated by its intersection with $\mathfrak{U}_0^3(\mathrm{Vec}(\mathbb{R}))^{e_{-1}}$, the space of lowest weight vectors in $\mathfrak{U}(\mathrm{Vec}(\mathbb{R}))$*

of weight 0 *and degree* $\leqslant 3$. *This intersection is* 2*-dimensional and is spanned by* $Q - (\gamma^2 - \gamma)$ *and* $Y_0 - (\gamma - 1/2)(\gamma^2 - \gamma)$. *Therefore*

$$\operatorname{Ann}_{\operatorname{Vec}(\mathbb{R})} \mathcal{F}(\gamma) = \bigl\langle Q - \bigl(\gamma^2 - \gamma\bigr),\ Y_0 - (\gamma - 1/2)\bigl(\gamma^2 - \gamma\bigr)\bigr\rangle_{\operatorname{Vec}(\mathbb{R})}.$$

These two homogeneous lowest weight generators may be replaced by a single inhomogeneous cubic generator, a single homogeneous quartic generator of weight 0*, or a single inhomogeneous quartic lowest weight generator, but there is no single homogeneous lowest weight generator.*

$\operatorname{Ann}_{\operatorname{Vec}(\mathbb{R})} \mathcal{F}(\gamma)$ *is not generated either by its intersection with* $\mathfrak{U}(\mathfrak{a})$ *or with* $\mathfrak{U}^2(\operatorname{Vec}(\mathbb{R}))$*, both of which generate only the proper subideal* $\langle Q - \gamma^2 + \gamma\rangle_{\operatorname{Vec}(\mathbb{R})}$. *It is not contained in* $\mathfrak{U}^+(\operatorname{Vec}(\mathbb{R}))$; *in particular, it is not generated by the sum of all of its weight spaces of non-zero weight.*

Theorem 1.2. *For* $\gamma = 0$ *or* 1 *we have*

$$\operatorname{Ann}_{\operatorname{Vec}(\mathbb{R})} \mathcal{F}(0) = \operatorname{Ann}_{\operatorname{Vec}(\mathbb{R})} \mathcal{F}(1) = \langle Z_1\rangle_{\operatorname{Vec}(\mathbb{R})}.$$

The generator Z_1 *is of weight* 1 *and is not a lowest weight vector, although it is annihilated by both* $\operatorname{ad}(e_1 e_{-1})$ *and* $\operatorname{ad}(Q)$ *(in fact,* $\operatorname{ad}(e_{-1})Z_1 = Q$*).*

This ideal is not generated by its lowest weight vectors, nor by its intersection with $\mathfrak{U}(\mathfrak{a})$*, nor by the sum of all of its weight spaces of weight* $\leqslant 0$. *It is contained in* $\mathfrak{U}^+(\operatorname{Vec}(\mathbb{R}))$*, and it is generated by each of its positive weight spaces.*

Let us point out some directions for further research, which we will elaborate on in Section 7. Another important class of modules is provided by differential operators between tensor density modules. These modules have been the focus of numerous works, such as [FF80,CMZ97,LO99, Ga00,CS04], and they will be an important tool in this paper. It is natural to ask what their annihilators are, but we cannot yet say much about this question beyond the following remarks.

For any complex scalars γ and p, the space of differential operators from $\mathcal{F}(\gamma)$ to $\mathcal{F}(\gamma + p)$ is defined to be

$$\operatorname{Diff}(\gamma, p) := \operatorname{Span}_{\mathbb{C}}\bigl\{dx^p\, h(x) D^k\colon\ k \in \mathbb{N},\ h(x) \text{ a polynomial on } \mathbb{R}\bigr\},$$

where $dx^p\, hD^k$ maps an element $g(x)\, dx^\gamma$ of $\mathcal{F}(\gamma)$ to $hg^{(k)}\, dx^{\gamma+p}$. (Throughout this paper, $\mathbb{N}$ will denote the non-negative integers, including zero.) This space is a $\operatorname{Vec}(\mathbb{R})$-module under the two-sided action

$$\sigma_{\gamma,p}\bigl(f(x)D\bigr)(T) := \pi_{\gamma+p}(fD) \circ T - T \circ \pi_\gamma(fD).$$

Write $\operatorname{Diff}^k(\gamma, p)$ for the natural order filtration of $\operatorname{Diff}(\gamma, p)$. This filtration is $\sigma_{\gamma,p}$-invariant, and it is not hard to check that $\operatorname{Diff}^k(\gamma, p)/\operatorname{Diff}^{k-1}(\gamma, p)$ is naturally isomorphic to $\mathcal{F}(p - k)$.

It may be that $\operatorname{Ann}_{\operatorname{Vec}(\mathbb{R})} \operatorname{Diff}(\gamma, p)$ is trivial, although we have not proven this. However, for each positive integer m one has the two-sided ideal

$$I(\gamma, p, m) := \bigl\{\Omega \in \mathfrak{U}\bigl(\operatorname{Vec}(\mathbb{R})\bigr)\colon\ \sigma_{\gamma,p}(\Omega) \operatorname{Diff}^k(\gamma, p) \subseteq \operatorname{Diff}^{k-m}(\gamma, p)\ \forall k\bigr\}.$$

These ideals are non-trivial, as it can be shown that $I(\gamma, p, m)$ contains all lowest weight vectors in $\mathfrak{U}(\operatorname{Vec}(\mathbb{R}))$ of weight $\geqslant m$ (see Section 7).

The ideals $I(\gamma, p, 1)$ and $I(\gamma, p, 2)$ both turn out to be equal to the intersection of the annihilators of all of the tensor density modules, independent of γ and p (again, see Section 7). The proofs of Theorems 1.1 and 1.2 are based on the following theorem concerning this ideal, which is proven in Section 3.

Theorem 1.3. *The ideal $\bigcap_{\gamma\in\mathbb{C}} \mathrm{Ann}_{\mathrm{Vec}(\mathbb{R})} \mathcal{F}(\gamma)$ is generated by the single lowest weight vector $\mathrm{ad}(e_2)Q$ of weight 2, and also by its intersection with $\mathfrak{U}^2_n(\mathrm{Vec}(\mathbb{R}))$ for any $n \geqslant 2$. It is contained in $\mathfrak{U}^+(\mathrm{Vec}(\mathbb{R}))$, and it contains all homogeneous lowest weight vectors of $\mathfrak{U}(\mathrm{Vec}(\mathbb{R}))$ of weight $\geqslant 1$.*

This ideal is not generated by any lowest weight vector of weight $\neq 2$, nor by the sum of its weight spaces of weight $\leqslant 1$. Its intersection with $\mathfrak{U}(\mathfrak{a})$ is zero.

In light of this theorem and the preceding claims, we have

$$I(\gamma, p, 1) = I(\gamma, p, 2) = \bigcap_{\nu\in\mathbb{C}} \mathrm{Ann}_{\mathrm{Vec}(\mathbb{R})} \mathcal{F}(\nu) = \big\langle \mathrm{ad}(e_2)Q \big\rangle_{\mathrm{Vec}(\mathbb{R})}$$

for all γ and p. It would be interesting to generalize these results to $I(\gamma, p, m)$ for $m \geqslant 3$. The higher $I(\gamma, p, m)$'s are related to the annihilators of the $\mathrm{Vec}(\mathbb{R})$-modules of finite length composed of tensor density modules (they are intersections of such annihilators). Modules of this type were studied in [FF80,MP92,BO98,Co01]; subquotients of the modules $\mathrm{Diff}^k(\gamma, p)$ furnish many examples of them [Co05].

In a different direction, it seems to us that it would be quite interesting to consider this circle of problems for $\mathrm{Vec}(\mathbb{R}^m)$. There the projective subalgebra $\mathfrak{a}$ is $\mathfrak{sl}_{m+1}$, and under it the tensor density modules are the duals of the Verma modules relative to the geometric subalgebra $\mathfrak{gl}_m$. Thus we expect a more subtle interplay with Duflo's theorem. One would start by considering the intersection of the annihilators of all the tensor density modules, as in Theorem 1.3 above. This intersection is interesting in its own right: it is the annihilator of what is known as the *universal Verma module,* of which the usual Verma modules are specializations.

This paper is organized as follows. In the next section we collect various facts which will be useful during the proofs of our main theorems over $\mathbb{R}$. Sections 3 and 4 contain the proofs themselves. In Section 5 we state our results over S^1 and in Section 6 we prove them. Section 7 elaborates on the above directions for further research, and Appendix A contains some useful results on $\mathfrak{sl}_2$-modules.

2. The $\mathfrak{a}$-structure of $\mathfrak{U}(\mathrm{Vec}(\mathbb{R}))$ and the π_γ

In this section we give several definitions, lemmas and remarks concerning the behavior of $\mathfrak{U}(\mathrm{Vec}(\mathbb{R}))$ and the actions π_γ with respect to $\mathfrak{a}$, culminating in a complete description of the $\mathfrak{a}$-decomposition of the intersections of the ideals $\mathrm{Ann}_{\mathrm{Vec}(\mathbb{R})} \mathcal{F}(\gamma)$ with $\mathfrak{U}^3(\mathrm{Vec}(\mathbb{R}))$. Some of these results will be used in the proofs of Theorems 1.1–1.3, and the rest help to clarify the significance of our generators of the ideals. For brevity let us define

$$\mathcal{W} := \mathrm{Vec}(\mathbb{R}), \qquad \mathrm{Diff}(\gamma) := \mathrm{Diff}(\gamma, 0), \qquad I(\gamma) := \mathrm{Ann}_{\mathrm{Vec}(\mathbb{R})} \mathcal{F}(\gamma).$$

Recall that for any $J \subseteq \mathfrak{U}(\mathcal{W})$ we write J^k for $J \cap \mathfrak{U}^k(\mathcal{W})$. Thus for example $I^k(\gamma)$ denotes $I(\gamma) \cap \mathfrak{U}^k(\mathcal{W})$ and $\mathfrak{U}^{+,k}(\mathcal{W})$ denotes $\mathfrak{U}^+(\mathcal{W}) \cap \mathfrak{U}^k(\mathcal{W})$. Some lemmas will be stated for $\mathfrak{U}^+(\mathcal{W})$ rather than for $\mathfrak{U}(\mathcal{W})$, as they are sharper in this form. Our starting point is the following lemma describing the actions π_γ of $\mathcal{W}$ on $\mathcal{F}(\gamma)$.

Lemma 2.1. *For all γ, π_γ maps $\mathfrak{U}(\mathcal{W})$ into $\mathrm{Diff}(\gamma)$ so as to carry the degree filtration to the order filtration*: $\pi_\gamma(\mathfrak{U}^k(\mathcal{W})) \subseteq \mathrm{Diff}^k(\gamma)$. *The image of $\mathfrak{U}^{+,k}(\mathcal{W})$ is as follows* (*for the image of $\mathfrak{U}^k(\mathcal{W})$, add $\mathbb{C}1$*):

$$\pi_\gamma\big(\mathfrak{U}^{+,k}(\mathcal{W})\big) = \mathrm{Diff}^k(\gamma) \quad \textit{for all } \gamma \neq 0 \textit{ or } 1 \textit{ and } k \geqslant 2,$$

$$\pi_0\big(\mathfrak{U}^{+,k}(\mathcal{W})\big) = \big\{T \circ D\colon\ T \in \mathrm{Diff}^{k-1}(0)\big\} \quad \textit{for all } k \geqslant 1,$$

$$\pi_1\big(\mathfrak{U}^{+,k}(\mathcal{W})\big) = \big\{D \circ T\colon\ T \in \mathrm{Diff}^{k-1}(1)\big\} \quad \textit{for all } k \geqslant 1.$$

Proof. By Eq. (1), $\pi_\gamma(fD) = fD + \gamma f'$, so π_γ maps $\mathcal{W}$ to the first order differential operators $\mathrm{Diff}^1(\gamma)$. This implies the first sentence of the lemma.

To prove the statement about π_γ for $\gamma \neq 0$ or 1, note that if $\pi_\gamma(\mathfrak{U}^{+,k}(\mathcal{W}))$ contains $\mathrm{Diff}^0(\gamma)$ then it contains all of $\mathrm{Diff}^k(\gamma)$ by an obvious symbol calculation. For $k \geqslant 2$ it does contain $\mathrm{Diff}^0(\gamma)$, as by Lemma 2.6, proven independently below, there are elements Z_n of $\mathfrak{U}^{+,2}(\mathcal{W})$ such that $\pi_\gamma(Z_n) = (\gamma^2 - \gamma)x^n$.

To prove the statement about π_0, it will suffice to prove that π_0 maps $\mathfrak{U}^+(\mathcal{W})$ onto the subalgebra $\mathrm{Diff}(0) \circ D$ of $\mathrm{Diff}(0)$. It maps $\mathfrak{U}^+(\mathcal{W})$ into this subalgebra because $\pi_0(fD) = fD$, and surjectivity is trivial by another obvious symbol calculation. The statement about π_1 follows similarly from $\pi_1(fD) = Df$. $\square$

It turns out that the projective subalgebra $\mathfrak{a}$ acts nearly semisimply on $\mathfrak{U}(\mathcal{W})$ and $\mathrm{Diff}(\gamma)$. Therefore the $\mathfrak{a}$-decomposition of $I^k(\gamma)$ can be obtained by computing the $\mathfrak{a}$-decompositions of $\mathfrak{U}^k(\mathcal{W})$ and its image $\pi_\gamma(\mathfrak{U}^k(\mathcal{W}))$, and then essentially deleting from the former those terms occurring in the latter. This procedure is not hard to carry out explicitly for $k = 2$ and 3, and this was the way in which we originally discovered our generators of the $I(\gamma)$. We will now describe the process in detail.

The next lemma gives the $\mathfrak{a}$-structure of $\mathrm{Diff}(\gamma)$. It is essentially classical; modern proofs may be found in [CMZ97] and [LO99]. A proof using notation close to ours may be obtained by taking $p = 0$ in Lemma 2.4 of [CS04], so we will not give a proof here.

For any Lie algebra $\mathfrak{g}$, we will use the symbol $\stackrel{\mathfrak{g}}{\cong}$ to denote equivalence of $\mathfrak{g}$-modules.

Lemma 2.2. $\mathrm{Diff}^k(\gamma) \stackrel{\mathfrak{a}}{\cong} \mathcal{F}(0) \oplus \mathcal{F}(-1) \oplus \cdots \oplus \mathcal{F}(-k)$ *for all γ. At $\gamma = 0$ and* 1,

$$\pi_0\big(\mathfrak{U}^{+,k}(\mathcal{W})\big) \stackrel{\mathfrak{a}}{\cong} \pi_1\big(\mathfrak{U}^{+,k}(\mathcal{W})\big) \stackrel{\mathfrak{a}}{\cong} \mathcal{F}(-1) \oplus \cdots \oplus \mathcal{F}(-k).$$

To compute the $\mathfrak{a}$-decomposition of $\mathfrak{U}(\mathcal{W})$ we must recall the symmetrizer map. For any Lie algebra $\mathfrak{g}$, let $\mathcal{S}^k(\mathfrak{g})$ be the kth symmetric power of $\mathfrak{g}$ and write $\mathcal{S}(\mathfrak{g})$ for the symmetric algebra $\bigoplus_0^\infty \mathcal{S}^k(\mathfrak{g})$. The symmetrizer map $\mathrm{Sym}\colon \mathcal{S}(\mathfrak{g}) \to \mathfrak{U}(\mathfrak{g})$ is an ad-equivalence from $\bigoplus_0^k \mathcal{S}^i(\mathfrak{g})$ to $\mathfrak{U}^k(\mathfrak{g})$. It will be useful to note that it carries $\bigoplus_1^k \mathcal{S}^i(\mathfrak{g})$ to $\mathfrak{U}^{+,k}(\mathfrak{g})$.

The adjoint action of $\mathcal{W}$ on itself is equivalent to $\mathcal{F}(-1)$, so we have

$$\mathfrak{U}^k(\mathcal{W}) \stackrel{\mathcal{W}}{\cong} \bigoplus_{i=0}^k \mathcal{S}^i\big(\mathcal{F}(-1)\big), \qquad \mathfrak{U}^{+,k}(\mathcal{W}) \stackrel{\mathcal{W}}{\cong} \bigoplus_{i=1}^k \mathcal{S}^i\big(\mathcal{F}(-1)\big). \tag{2}$$

Thus in order calculate the $\mathfrak{a}$-decompositions of $I^2(\gamma)$ and $I^3(\gamma)$ we need the $\mathfrak{a}$-decomposition of $\mathcal{S}^k\mathcal{F}(-1)$ at $k=2$ and 3. Henceforth we will rely heavily on the elementary results on $\mathfrak{a}$-modules collected in Appendix A.

Lemma 2.3. *The $\mathfrak{b}$-decompositions of $\mathcal{S}^2\mathcal{F}(-1)$ and $\mathcal{S}^3\mathcal{F}(-1)$ are $\bigoplus_{i=0}^{\infty}\mathcal{F}(2i-2)$ and $\bigoplus_{i,j=0}^{\infty}\mathcal{F}(2i+3j-3)$, respectively. Their $\mathfrak{a}$-decompositions are the same, except that under $\mathfrak{a}$, the $\mathfrak{b}$-submodule $\mathcal{F}(0)\oplus\mathcal{F}(1)$ of $\mathcal{S}^3\mathcal{F}(-1)$ is indecomposable with submodule $\mathcal{F}(1)$ and quotient $\mathcal{F}(0)$ (it is a copy of the module $\mathcal{G}(1)$ of Appendix* A*). Thus, writing $\mathcal{F}(0)\oplus_{\mathfrak{b}}\mathcal{F}(1)$ for the $\mathfrak{a}$-indecomposable summand, we have*

$$\mathcal{S}^2\mathcal{F}(-1) \overset{\mathfrak{a}}{\cong} \mathcal{F}(-2)\oplus\mathcal{F}(0)\oplus\mathcal{F}(2)\oplus\mathcal{F}(4)\oplus\cdots,$$
$$\mathcal{S}^3\mathcal{F}(-1) \overset{\mathfrak{a}}{\cong} \mathcal{F}(-3)\oplus\mathcal{F}(-1)\oplus\big(\mathcal{F}(0)\oplus_{\mathfrak{b}}\mathcal{F}(1)\big)\oplus\mathcal{F}(2)\oplus 2\mathcal{F}(3)\oplus\cdots.$$

Proof. Lemma A.6 gives the $\mathfrak{b}$-decompositions and also shows that $\mathcal{S}^k\mathcal{F}(-1)$ is a "good" $\mathfrak{a}$-module in the sense of Appendix A. Hence Proposition A.2 and Corollary A.3 show that the $\mathfrak{a}$-decomposition of $\mathcal{S}^2\mathcal{F}(-1)$ must be the same as its $\mathfrak{b}$-decomposition.

For $\mathcal{S}^3\mathcal{F}(-1)$ we use Corollary A.5, which tells us that the sum of all the finite-dimensional $\mathfrak{a}$-submodules of $\mathcal{S}^k\mathcal{F}(-1)$ is $\mathcal{S}^kL(1)$, where $L(1)$ denotes the irreducible 3-dimensional $\mathfrak{a}$-submodule of $\mathcal{F}(-1)$ of weights -1, 0, and 1. It is well known that

$$\mathcal{S}^kL(1) \overset{\mathfrak{a}}{\cong} L(k)\oplus L(k-2)\oplus L(k-4)\oplus\cdots\oplus\big(L(1)\text{ or }L(0)\big), \tag{3}$$

where $L(k)$ is the irreducible $(2k+1)$-dimensional $\mathfrak{a}$-module of highest weight k (which occurs as a submodule of $\mathcal{F}(-k)$). Thus $\mathcal{S}^3L(1)$ is $L(3)\oplus L(1)$, so by Corollary A.3, $\mathcal{S}^3\mathcal{F}(-1)$ contains $\mathfrak{a}$-copies of $\mathcal{F}(-3)$ and $\mathcal{F}(-1)$ but not of $\mathcal{F}(0)$. The result now follows from a further application of Corollary A.3. $\square$

Remark. By Corollary A.5 applied to $\gamma=-1$, the sum of all the finite-dimensional $\mathfrak{a}$-submodules of $\mathfrak{U}(\mathcal{W})$ is $\mathfrak{U}(\mathfrak{a})$. In particular, the commutant of $\mathfrak{a}$ in $\mathfrak{U}(\mathcal{W})$ is $\mathbb{C}[Q]$. One can use Eq. (3) and Corollary A.3 to deduce the $\mathfrak{a}$-structure of $\mathcal{S}^k\mathcal{F}(-1)$ for all k, and hence that of $\mathfrak{U}(\mathcal{W})$.

Corollary 2.4. *$\mathfrak{U}^2(\mathcal{W})$ and $\mathfrak{U}^3(\mathcal{W})$ have the following $\mathfrak{a}$-decompositions. The lowest weight vectors of the first few summands are written beneath them. The $\mathfrak{a}$-decompositions and lowest weight vectors of $\mathfrak{U}^{+,2}(\mathcal{W})$ and $\mathfrak{U}^{+,3}(\mathcal{W})$ are the same, but without the summand $\mathbb{C}$. We write Q^{e_2} for* $\mathrm{ad}(e_2)Q$:

$$\begin{array}{lcccccccccc}
\mathfrak{U}^2(\mathcal{W}) & \overset{\mathfrak{a}}{\cong} & & \mathbb{C}\oplus & \mathcal{F}(-1) & \oplus & \mathcal{F}(-2) & \oplus & \mathcal{F}(0) & \oplus & \mathcal{F}(2)\ \oplus\cdots, \\
 & & & 1 & e_{-1} & & e_{-1}^2 & & Q & & Q^{e_2} \\
\mathfrak{U}^3(\mathcal{W}) & \overset{\mathfrak{a}}{\cong} & \mathfrak{U}^2(\mathcal{W}) & \oplus & \mathcal{F}(-3) & \oplus & \mathcal{F}(-1) & \oplus & \big(\mathcal{F}(0) & \oplus_{\mathfrak{b}} & \mathcal{F}(1)\big)\ \oplus\cdots. \\
 & & & & e_{-1}^3 & & Qe_{-1} & & Y_0 & & Q^{e_2}e_{-1}
\end{array}$$

Proof. The $\mathfrak{a}$-decompositions are immediate from Eq. (2) and Lemma 2.3. It is easy to use the definitions of Q and Y_0 to verify that the given vectors are indeed lowest weight vectors of the appropriate weight. In light of the $\mathfrak{a}$-decompositions, in most cases this is enough to prove that they are the correct lowest weight vectors. For example, the $\mathfrak{a}$-decompositions show that $\mathfrak{U}_0^{+,2}(\mathcal{W})^{e_{-1}}$ is 1-dimensional and is the lowest weight space of the $\mathfrak{a}$-copy of $\mathcal{F}(0)$. Since it contains Q, Q is correct. (We remark that one must be more careful if one uses $\mathfrak{U}(\mathcal{W})$ instead of $\mathfrak{U}^+(\mathcal{W})$: there $\mathfrak{U}_0^2(\mathcal{W})^{e_{-1}}$ is 2-dimensional, and the copy of $\mathcal{F}(0)_0$ is the unique line in the image of $\mathrm{ad}(e_{-1})$.)

We must say a few words about Y_0. The 0-generalized eigenspace of $\mathrm{ad}(Q)$ in $\mathfrak{U}^{+,3}(\mathcal{W})$ is the $\mathfrak{a}$-copy of $\mathcal{F}(0) \oplus (\mathcal{F}(0) \oplus_{\mathfrak{b}} \mathcal{F}(1))$, i.e., $\mathcal{F}(0) \oplus \mathcal{G}(1)$ (see Appendix A). The $\mathfrak{a}$-submodule $\mathcal{F}(1)$ here is unique and $\mathfrak{U}_1^{+,3}(\mathcal{W})^{e_{-1}}$ is 1-dimensional, so $Q^{e_2}e_{-1}$ must be correct. But the $\mathfrak{a}$-quotient $\mathcal{F}(0)$ is not unique: $\mathfrak{U}_0^{+,3}(\mathcal{W})^{e_{-1}}$ is 2-dimensional, and using the methods of Appendix A one checks that any element of it not in $\mathbb{C}Q$ may be taken as the lowest weight vector of the quotient $\mathcal{F}(0)$. Thus Y_0 works. For future reference, note that $Y_0 + cQ$ would also work for any scalar c. $\square$

We now have sufficient information to compute the $\mathfrak{a}$-decompositions of $I^2(\gamma)$ and $I^3(\gamma)$, our goal in this section.

Proposition 2.5. *For all γ in $\mathbb{C}$, $I(\gamma)$ contains every $\mathfrak{a}$-submodule of $\mathfrak{U}(\mathcal{W})$ equivalent to $\mathcal{F}(k)$ for any $k \geqslant 1$. The ideals $I(0)$ and $I(1)$ are equal and are contained in $\mathfrak{U}^+(\mathcal{W})$. Conversely, if $\gamma \neq 0$ or 1 then $I(\gamma)$ is not equal to $I(\gamma')$ for any $\gamma' \neq \gamma$, and $I(\gamma)$ is not contained in $\mathfrak{U}^+(\mathcal{W})$.*

For $\gamma \neq 0$ or 1, the $\mathfrak{a}$-decomposition of $I^2(\gamma)$ is $\mathbb{C} \oplus (\bigoplus_{i=1}^{\infty} \mathcal{F}(2i))$. The $\mathfrak{a}$-decomposition of $I^2(0)$ is $\bigoplus_{i=0}^{\infty} \mathcal{F}(2i)$. For all γ, the inclusion $I^2(\gamma) \subset I^3(\gamma)$ is $\mathfrak{a}$-split and the $\mathfrak{a}$-decomposition of the quotient $I^3(\gamma)/I^2(\gamma)$ is that of $\mathcal{S}^3\mathcal{F}(-1)$ with the $\mathcal{F}(-3)$ is deleted. Using the notation

$$Q^{e_2} := \mathrm{ad}(e_2)Q, \qquad Z_0(\gamma) := Q - \gamma^2 + \gamma, \qquad Y_0(\gamma) := Y_0 - (\gamma - 1/2)Q,$$

the first few $\mathfrak{a}$-summands and their lowest weight vectors are

$$\begin{array}{llllllll}
I^2(\gamma) \overset{\mathfrak{a}}{\cong} & \underset{Z_0(\gamma)}{\mathbb{C}} & \oplus & \underset{Q^{e_2}}{\mathcal{F}(2)} & \oplus & \mathcal{F}(4) & \oplus \ \cdots & \textit{for } \gamma \neq 0,\ 1, \\
I^2(0) \overset{\mathfrak{a}}{\cong} & \underset{Q}{\mathcal{F}(0)} & \oplus & \underset{Q^{e_2}}{\mathcal{F}(2)} & \oplus & \mathcal{F}(4) & \oplus \ \cdots, & \\
I^3(\gamma) \overset{\mathfrak{a}}{\cong} & I^2(\gamma) & \oplus & \underset{Z_0(\gamma)e_{-1}}{\mathcal{F}(-1)} & \oplus & \big(\underset{Y_0(\gamma)}{\mathcal{F}(0)} \oplus_{\mathfrak{b}} \underset{Q^{e_2}e_{-1}}{\mathcal{F}(1)}\big) & \oplus \ \cdots & \textit{for all } \gamma.
\end{array}$$

Proof. Let V be an $\mathfrak{a}$-copy of $\mathcal{F}(k)$ in $\mathfrak{U}(\mathcal{W})$ for some $k \geqslant 1$ and let Ω_k be its lowest weight vector. The $\mathcal{F}(k)$ with $k \geqslant 1$ are irreducible under $\mathfrak{a}$, so to prove $V \subset I(\gamma)$ it suffices to prove $\pi_\gamma(\Omega_k) = 0$. By Lemma 2.1, if $\pi_\gamma(\Omega_k)$ is non-zero then it is a lowest weight vector of $\mathrm{Diff}(\gamma)$ of weight k. But by Lemma 2.2 there are no such vectors.

The fact that $\mathcal{F}(0)$ has a trivial submodule $\mathbb{C}$ immediately yields $I(0) \subset \mathrm{Ann}_{\mathcal{W}}\,\mathbb{C} = \mathfrak{U}^+(\mathcal{W})$, and the fact that $\mathcal{F}(0)/\mathbb{C}$ is $\mathcal{W}$-equivalent to $\mathcal{F}(1)$ gives $I(0) \subseteq I(1)$. To prove $I(1) = I(0)$, fix $\Omega \in I(1)$. We may assume that Ω is homogeneous, say of weight n. Since Ω annihilates $\mathcal{F}(1)$ it annihilates $\mathcal{F}(0)/\mathbb{C}$, so $\pi_0(\Omega)$ maps $\mathcal{F}(0)$ to $\mathbb{C}$. Therefore $\pi_0(\Omega)$ must kill all weight spaces $\mathcal{F}(0)_m$ except possibly $\mathcal{F}(0)_{-n}$. But $\pi_0(\Omega)$ is a differential operator, and by an easy Zariski density argument any differential operator which kills infinitely many weight spaces must be zero. Thus $I(0) = I(1)$.

Fix $\gamma \neq 0$ or 1. Then $I(\gamma)$ is not in $\mathfrak{U}^+(\mathcal{W})$ because it contains $Q - \gamma^2 + \gamma$. If $I(\gamma) = I(\gamma')$, the actions of Q and Y_0 under π_γ and $\pi_{\gamma'}$ force $\gamma^2 - \gamma = \gamma'^2 - \gamma'$ and $(\gamma - 1/2)(\gamma^2 - \gamma) = (\gamma' - 1/2)(\gamma'^2 - \gamma')$, whence $\gamma = \gamma'$.

Now we come to the $\mathfrak{a}$-decompositions. Corollary 2.4 gives the $\mathfrak{a}$-decompositions of $\mathfrak{U}^2(\mathcal{W})$ and $\mathfrak{U}^3(\mathcal{W})$ and some of their lowest weight vectors. First consider $I^2(\gamma)$ with $\gamma \neq 0$ or 1. By Lemma 2.1, π_γ is a surjective $\mathfrak{a}$-map from $\mathfrak{U}^2(\mathcal{W})$ to $\mathrm{Diff}^2(\gamma)$, and by Lemma 2.2, $\mathrm{Diff}^2(\gamma)$ is $\mathfrak{a}$-equivalent to $\mathcal{F}(0) \oplus \mathcal{F}(-1) \oplus \mathcal{F}(-2)$. We saw above that π_γ kills the $\mathcal{F}(2) \oplus \mathcal{F}(4) \oplus \cdots$ in $\mathfrak{U}^2(\mathcal{W})$, so it restricts to an $\mathfrak{a}$-surjection from the $\mathbb{C} \oplus \mathcal{F}(0) \oplus \mathcal{F}(-1) \oplus \mathcal{F}(-2)$ in $\mathfrak{U}^2(\mathcal{W})$ to $\mathrm{Diff}^2(\gamma)$. Since π_γ preserves weights, weight space dimensions show that the kernel of this restriction must be a trivial module under $\mathfrak{a}$. This proves that $I^2(\gamma)$ has the stated $\mathfrak{a}$-decomposition. We already know that $Z_0(\gamma)$ and Q^{e_2} are lowest weight vectors of the correct weights in $I^2(\gamma)$, so they are correct.

Next consider $I^2(0)$. Note that $\mathfrak{a}$-maps preserve Q-eigenspaces, and recall that Q acts on $\mathcal{F}(\gamma)$ by the scalar $\gamma^2 - \gamma$. We know $I^2(0)$ lies in $\mathfrak{U}^+(\mathcal{W})$, so by Lemma 2.1 it is the kernel of the $\mathfrak{a}$-surjection π_0 from $\mathfrak{U}^{+,2}(\mathcal{W})$ to $\mathrm{Diff}^1(0) \circ D$. By Lemma 2.2, $\mathrm{Diff}^1(0) \circ D$ is an $\mathfrak{a}$-copy of $\mathcal{F}(-1) \oplus \mathcal{F}(-2)$, and by Corollary 2.4, $\mathfrak{U}^{+,2}(\mathcal{W})$ is an $\mathfrak{a}$-copy of $\mathcal{F}(-2) \oplus \mathcal{F}(-1) \oplus \mathcal{F}(0) \oplus \mathcal{F}(2) \oplus \cdots$. Since π_0 kills the $\mathcal{F}(2) \oplus \cdots$ and preserves Q-eigenvalues, it must kill the $\mathcal{F}(0)$ and be bijective on the $\mathcal{F}(-1) \oplus \mathcal{F}(-2)$. This gives the $\mathfrak{a}$-decomposition of $I^2(0)$, and the lowest weight vectors are given by Corollary 2.4.

Finally we come to $I^3(\gamma)$. For all γ, π_γ defines an $\mathfrak{a}$-surjection from $\mathfrak{U}^3(\mathcal{W})/\mathfrak{U}^2(\mathcal{W})$ to $\mathrm{Diff}^3(\gamma)/\mathrm{Diff}^2(\gamma)$. Up to equivalence, this is an $\mathfrak{a}$-surjection from $\mathcal{S}^3\mathcal{F}(-1)$ to $\mathcal{F}(-3)$. We have already seen that it kills all $\mathfrak{a}$-copies of $\mathcal{F}(k)$ with $k \geqslant 1$. From Appendix A we know that the generalized eigenvalue of Q on $\mathcal{F}(0) \oplus_{\mathfrak{b}} \mathcal{F}(1) \cong \mathcal{G}(1)$ is zero. Since π_0 preserves generalized Q-eigenspaces, Lemma 2.3 shows that it is bijective on the $\mathfrak{a}$-copy of $\mathcal{F}(-3)$ and zero on the other summands. This proves that $I^3(\gamma)/I^2(\gamma)$ has the stated $\mathfrak{a}$-decomposition.

It remains to prove that $I^2(\gamma) \subset I^3(\gamma)$ is $\mathfrak{a}$-split and that $I^3(\gamma)$ has the stated lowest weight vectors. It will do to find an $\mathfrak{a}$-submodule of $I^3(\gamma)$ with the same $\mathfrak{a}$-decomposition as $I^3(\gamma)/I^2(\gamma)$ which does not intersect $I^2(\gamma)$. We will check that the given lowest weight vectors are correct as we construct this submodule.

For the $\mathcal{F}(-1)$ we may take $Z_0(\gamma)\mathcal{W}$, as $Z_0(\gamma)$ is an $\mathfrak{a}$-invariant and $\mathcal{W}$ is a copy of $\mathcal{F}(-1)$. For the $\mathcal{G}(1)$, first check that $Y_0(\gamma)$ is in $I^3(\gamma)$. By the remark at the end of the proof of Corollary 2.4, $\mathfrak{U}^3(\mathcal{W})$ contains a $\mathcal{G}(1)$ with lowest weight vectors $Y_0(\gamma)$ and $Q^{e_2}e_1$ for all γ. In fact, a consideration of the generalized Q-eigenspace of eigenvalue 0 in $\mathfrak{U}^{+,3}(\mathcal{W})$ shows that for each γ there is a unique such $\mathcal{G}(1)$, which lies in $I^3(\gamma)$ but does not intersect $\mathfrak{U}^2(\mathcal{W})$.

For the $\mathcal{F}(k)$ with $k \geqslant 2$ we may simply take the images of all such $\mathcal{F}(k)$ in $\mathcal{S}^3\mathcal{F}(-1)$ under the symmetrizer map, as by the first sentence of this proposition they are all in $I^3(\gamma)$ for every γ. $\square$

Let us now give explicit formulas for the elements of the generalized Q-eigenspace of eigenvalue 0 in $\mathfrak{U}^{+,3}(\mathcal{W})$. They are not all necessary for the proofs of the main theorems, but they clarify the role of $\mathcal{G}(1)$ above. We define them and give a lemma stating their properties.

Definition. Define $Z_0 := Q$, and recall from Section 1 our definitions

$$Z_1 := (e_1e_0 - e_2e_{-1} - e_1)/2, \qquad Y_0 := Z_0(e_0 - 1/2) - Z_1e_{-1}.$$

Using them, define

$$Y_1 := Z_0 e_1 - Z_1(e_0 + 1/2), \qquad X_1 := \mathrm{ad}(e_1) Y_0/2.$$

For $n > 1$, define recursively $Z_n := \mathrm{ad}(e_1) Z_{n-1}/(n-1)$,

$$X_n := \mathrm{ad}(e_1) X_{n-1}/n, \qquad Y_n := \bigl(\mathrm{ad}(e_1) Y_{n-1} - 2X_n\bigr)/(n-1)$$

(their subscripts are their weights). Finally, define polynomials

$$P_2(\gamma) := \gamma(\gamma - 1), \qquad P_3(\gamma) := \gamma(\gamma - 1/2)(\gamma - 1).$$

Lemma 2.6. *The images of Z_n, Y_n, and X_n under π_γ are*

$$\pi_\gamma(Z_n) = P_2(\gamma) x^n, \qquad \pi_\gamma(Y_n) = P_3(\gamma) x^n, \qquad \pi_\gamma(X_n) = 0.$$

The action of $\mathfrak{a}$ on these elements is as follows: for $e_m \in \mathfrak{a}$,

$$\begin{aligned} &\mathrm{ad}(e_m) Z_n = n Z_{n+m}, && \mathrm{ad}(Q) Z_n = 0,\\ &\mathrm{ad}(e_m) Y_n = n Y_{n+m} + \bigl(m^2 + m\bigr) X_{n+m}, && \mathrm{ad}(Q) Y_n = -2n X_n,\\ &\mathrm{ad}(e_m) X_n = (n+m) X_{n+m}, && \mathrm{ad}(Q) X_n = 0. \end{aligned}$$

For $n > 1$, we have the closed formulae $Z_n = \mathrm{ad}(e_1^{n-1}) Z_1/(n-1)!$,

$$X_n = \mathrm{ad}\bigl(e_1^{n-1}\bigr) X_1/n!, \qquad Y_n = \mathrm{ad}\bigl(e_1^{n-1}\bigr) Y_1/(n-1)! - 2(n-1) X_n.$$

Finally, X_1 is equal to $(\mathrm{ad}(e_2) Q) e_{-1}/4$.

Proof. For the first sentence, compute the images of Z_1 and Y_1 under π_γ directly. Then check that $\mathrm{ad}(e_{-1})$ maps Z_1 to Z_0 and Y_1 to Y_0, and use the fact that $\pi_\gamma : \mathfrak{U}(\mathcal{W}) \to \mathrm{Diff}(\gamma)$ intertwines ad with $\sigma_{\gamma,0}$ to compute the images under π_γ of Z_0, Y_0, and X_1. For example, $\pi_\gamma(Y_0) = \pi_\gamma(\mathrm{ad}(e_{-1}) Y_1) = \sigma_{\gamma,0}(e_{-1}) \pi_\gamma(Y_1) = P_3(\gamma) \sigma_{\gamma,0}(e_{-1}) x = P_3(\gamma)$. Similarly, use the recursive definitions of Z_n, Y_n, and X_n together with the intertwining property of π_γ to deduce their π_γ-images.

The formulae for the actions of $\mathrm{ad}(e_0)$ and $\mathrm{ad}(e_1)$ are immediate from the recursive definitions. To verify that $\mathrm{ad}(Q)$ kills Z_n and X_n, check that it kills Z_1 and Y_0 and use induction coupled with the fact that Q commutes with e_1. Then use induction to prove the formulae for $\mathrm{ad}(e_{-1})$ applied to Z_n and X_n. It helps to note that $Q = e_0^2 + e_0 - e_{-1} e_1$, so for example, $\mathrm{ad}(e_{-1}) Z_n = \mathrm{ad}(e_{-1} e_1) Z_{n-1}/(n-1)$ is the same as $\mathrm{ad}(e_0^2 + e_0 - Q) Z_{n-1}/(n-1)$. Armed with the formulae already established, one can prove the formulae for $\mathrm{ad}(Q)$ and $\mathrm{ad}(e_{-1})$ applied to Y_n similarly.

The first two closed formulae are clear, and the last may be proven by induction. The final sentence can be checked directly but it admits a more elegant proof: by Corollary 2.4, $\mathfrak{U}_1^3(\mathcal{W})^{e_{-1}}$ is 1-dimensional and contains both $(\mathrm{ad}(e_2) Q) e_{-1}$ and X_1. A symbol calculation yields the constant of proportionality. $\square$

The following corollary of Lemma 2.6 gives the $\mathfrak{a}$-decomposition of the generalized Q-eigenspace of eigenvalue 0 in $\mathfrak{U}^3(\mathcal{W})$ explicitly. The proof is a simple application of Corollary 2.4

and will be omitted. Henceforth we will use the following extensions of the definitions of Proposition 2.5:

$$Z_0(\gamma) := Z_0 - P_2(\gamma), \qquad Y_n(\gamma) := Y_n - (\gamma - 1/2)Z_n. \tag{4}$$

Corollary 2.7. *The generalized Q-eigenspace of eigenvalue* 0 *in* $\mathfrak{U}^3(\mathcal{W})$ *has a basis consisting of* 1 *and all of the* Z_n, Y_n, *and* X_n. *The* Z_n *span the* $\mathfrak{a}$-*copy of* $\mathcal{F}(0)$ *in* $\mathfrak{U}^{+,2}(\mathcal{W})$. *The* Y_n *and* X_n *span the* $\mathfrak{a}$-*copy* $\mathcal{G}(1)$ *in* $\mathfrak{U}^{+,3}(\mathcal{W})$, *and the* X_n *span its* $\mathfrak{a}$-*submodule* $\mathcal{F}(1)$. *The* $\mathcal{F}(1)$ *in* $\mathfrak{U}^3(\mathcal{W})$ *is unique but the* $\mathcal{G}(1)$ *is not: for each* γ *in* $\mathbb{C}$, *the* $Y_n(\gamma)$ *and* X_n *span a* $\mathcal{G}(1)$ *which lies in* $I^3(\gamma)$.

The transpose involution of $\mathfrak{U}(\mathcal{W})$. We conclude this section with a discussion of the transpose involution. For any Lie algebra $\mathfrak{g}$, the *transpose* $\Omega \mapsto \Omega^T$ is the algebra anti-involution of $\mathfrak{U}(\mathfrak{g})$ that is -1 on $\mathfrak{g}$. Note that it is ad-covariant. It is easy to check that for any $\mathfrak{g}$-module M, the annihilator of its dual M^* is the transpose of the annihilator of M, i.e., $\mathrm{Ann}_{\mathfrak{g}}(M^*) = (\mathrm{Ann}_{\mathfrak{g}} M)^T$.

Here we will prove that Theorems 1.1 and 1.2 imply $I(\gamma)^T = I(1-\gamma)$, suggesting a link between $\mathcal{F}(1-\gamma)$ and $\mathcal{F}(\gamma)^*$. This will be made precise in Section 5: the two are equal for tensor density modules over the circle.

Proposition 2.8. *Under* T, Z_n *has even parity and* Y_n *and* X_n *have odd parity. In particular,* Q *and* Y_0 *are the unique* (*up to a scalar*) *elements of* $\mathfrak{U}_0^{+,3}(\mathcal{W})^{e_{-1}}$ *of* T-*parities* 1 *and* -1, *respectively.*

Thus $Z_0(\gamma)^T = Z_0(\gamma) = Z_0(1-\gamma)$ *and* $Y_n(\gamma)^T = -Y_n(1-\gamma)$, *so*

$$I(\gamma)^T = I(1-\gamma).$$

Proof. It is enough to prove the T-parity statements for Z_1 and Y_1, as they generate the other elements under the adjoint action of $\mathfrak{a}$. This can be done directly. The final equation is then immediate from Theorems 1.1 and 1.2. □

3. Proof of Theorem 1.3

It is necessary to prove Theorem 1.3 before Theorems 1.1 and 1.2. Throughout the next two sections we will use the abbreviations

$$I := \bigcap_{\gamma\in\mathbb{C}} I(\gamma), \qquad Q^{e_2} := \mathrm{ad}(e_2)Q, \qquad \mathfrak{U} := \mathfrak{U}(\mathcal{W}), \qquad \mathfrak{U}^+ := \mathfrak{U}^+(\mathcal{W})$$

whenever it is convenient. The proof of $I = \langle Q^{e_2}\rangle_{\mathcal{W}}$ is by far the hardest part of the theorem, so we will begin by proving the other statements.

By Proposition 2.5, Q^{e_2} is the lowest weight vector of an $\mathfrak{a}$-copy of $\mathcal{F}(2)$ in I^2. Therefore Q^{e_2} is in $\langle I_n^2\rangle_{\mathcal{W}}$ for any $n \geqslant 2$, as $\mathcal{F}(2)$ is irreducible under $\mathfrak{a}$. This reduces proving $I = \langle I_n^2\rangle_{\mathcal{W}}$ to proving $I = \langle Q^{e_2}\rangle_{\mathcal{W}}$. The second sentence of the theorem was already proven in Proposition 2.5.

For the second paragraph we will use the *Gel'fand–Fuks module* GF, an indecomposable module of $\mathcal{W}$ of length 2 with submodule $\mathcal{F}(2)$ and quotient $\mathbb{C}$. As a space it is $\mathbb{C} \oplus \mathcal{F}(2)$, with basis $\{1, dx^2 x^n\colon n \geqslant 0\}$. The action on $dx^2 x^n$ is as in $\mathcal{F}(2)$ and the action on 1 is

$$e_n(1) := \left(n^3 - n\right) dx^2 x^{n-2}, \tag{5}$$

the famous Gel'fand–Fuks cocycle (see for example [FF80]). Note that GF is split under $\mathfrak{a}$: Q acts on $\mathbb{C}$ by 0 and on $\mathcal{F}(2)$ by 2.

The key point is that I is not in $\mathrm{Ann}_{\mathcal{W}}\,\mathrm{GF}$, as $Q^{e_2}(1) = -Qe_2(1) = -12\,dx^2$. Note that any Ω in I kills the submodule $\mathcal{F}(2)$ as well as the trivial quotient, so $\Omega(\mathrm{GF}) = \mathbb{C}\Omega(1)$ and $\Omega(1)$ is in $\mathcal{F}(2)$. If Ω is a lowest weight vector of weight k, then $\Omega(1)$ is in $\mathcal{F}(2)_k^{e_{-1}}$ because 1 is also a lowest weight vector. For $k \neq 2$ this space is 0, so Ω is in $\mathrm{Ann}_{\mathcal{W}}\,\mathrm{GF}$. If Ω is any vector of weight $k \leqslant 1$ then $\Omega(1)$ is in $\mathcal{F}(2)_k$, which is zero, so again Ω is in $\mathrm{Ann}_{\mathcal{W}}\,\mathrm{GF}$. This proves the first sentence of the second paragraph.

For the second sentence we use the easy fact that $\mathfrak{U}_0(\mathfrak{a})$ is $\mathbb{C}[Q, e_0]$. Since Q and e_0 act on $\mathcal{F}(\gamma)_n$ by $P_2(\gamma)$ and (n), respectively, no element of $\mathfrak{U}_0(\mathfrak{a})$ can kill $\mathcal{F}(\gamma)_n$ for all γ and n. But if $I \cap \mathfrak{U}(\mathfrak{a})$ were non-zero it would be a sum of finite-dimensional $\mathfrak{a}$-submodules of I, so $I \cap \mathfrak{U}_0(\mathfrak{a})$ would be non-zero.

It remains to prove $I = \langle Q^{e_2}\rangle_{\mathcal{W}}$. The proof is long and has three parts. Define a subspace J of $\mathfrak{U}$ by

$$J := \mathrm{Span}\big\{e_0^i e_{-1}^j,\, e_n e_0^i e_{-1}^j\colon\ i \geqslant 0,\ j \geqslant 0,\ n \geqslant 1\big\}.$$

First we will prove that J is complementary to I. Second we will prove that $\langle I^2\rangle_{\mathcal{W}} + J$ is all of $\mathfrak{U}$, whence $I = \langle I^2\rangle_{\mathcal{W}}$. Finally we will prove that $\langle Q^{e_2}\rangle_{\mathcal{W}}$ contains I^2.

Note that I is the annihilator of the module $\bigoplus_\gamma \mathcal{F}(\gamma)$. In order to prove that $I \oplus J = \mathfrak{U}$, we define a useful variant of this module. Let

$$\mathcal{F}(\Gamma) := \mathbb{C}[\Gamma, x],$$

where x and Γ are indeterminates. Define an action π of $\mathcal{W}$ on $\mathcal{F}(\Gamma)$ by

$$\pi(fD) := fD + \Gamma f', \quad \text{i.e.,} \quad \pi(fD)\big(\Gamma^j g(x)\big) := \Gamma^j(fg' + \Gamma f'g). \tag{6}$$

It is easy to verify that π is a representation; it amounts to the fact that $fD \mapsto f'$ is a 1-cocycle of $\mathcal{W}$.

For all $\gamma \in \mathbb{C}$, it is clear that the evaluation map

$$\mathrm{eval}_\gamma : \mathcal{F}(\Gamma) \to \mathcal{F}(\gamma), \qquad \mathrm{eval}_\gamma\big(g(\Gamma, x)\big) := dx^\gamma g(\gamma, x) \tag{7}$$

is a $\mathcal{W}$-covariant surjection. Moreover, an element g of $\mathcal{F}(\Gamma)$ is 0 if and only if $\mathrm{eval}_\gamma g = 0$ for all γ in a Zariski-dense subset of $\mathbb{C}$. It follows that $\mathrm{Ann}_{\mathcal{W}}(\Gamma) = I$. (As we alluded to at the end of the introduction, I is actually the annihilator of the universal Verma module of $\mathfrak{a}$, extended to a $\mathcal{W}$-module. This can be seen by checking that $\mathcal{F}(\Gamma)$ is the dual of the universal Verma module and noting that $I = I^T$ by Proposition 2.8.)

Since $I = \mathrm{kernel}(\pi|_{\mathfrak{U}})$, to prove $I \oplus J = \mathfrak{U}$ it suffices to prove that π maps J bijectively to the image $\pi(\mathfrak{U})$. In order to describe this image, we define the algebra of differential operators on $\mathcal{F}(\Gamma)$ to be

$$\mathrm{Diff}(\Gamma) := \mathbb{C}[\Gamma, x, D],$$

where Γ, x, and D are indeterminates such that Γ is central and $[D, x] = 1$. In other words, $\mathrm{Diff}(\Gamma)$ is the Weyl algebra over $\mathbb{C}[\Gamma]$. Note that eval_γ extends to a map from $\mathrm{Diff}(\Gamma)$ to $\mathrm{Diff}(\gamma)$ by $h(\Gamma, x, D) \mapsto h(\gamma, x, D)$. One checks that $\mathrm{eval}_\gamma \circ \pi = \pi_\gamma$.

Let $\mathrm{Diff}^r(\Gamma)$ be the filtration of $\mathrm{Diff}(\Gamma)$ by *total* (Γ, D)*-degree,* and given any subspace V of $\mathrm{Diff}(\Gamma)$, write V^r for $V \cap \mathrm{Diff}^r(\Gamma)$:

$$\mathrm{Diff}^r(\Gamma) := \mathrm{Span}\{\Gamma^j x^n D^k \colon n \geqslant 0,\ j+k \leqslant r\}, \qquad V^r := V \cap \mathrm{Diff}^r(\Gamma).$$

Since $\pi(\mathcal{W}) \subseteq \mathrm{Diff}^1(\Gamma)$, we have $\pi(\mathfrak{U}^r) \subseteq \mathrm{Diff}^r(\Gamma)$, i.e., $\pi(\mathfrak{U}^r) \subseteq \pi(\mathfrak{U})^r$ (in fact, we will see in Lemma 3.2 that the two are equal). The next two lemmas give the image of π and prove $I \oplus J = \mathfrak{U}$.

Lemma 3.1. *For all* $r \geqslant 1$,

$$\pi(\mathfrak{U}^{+,r}) = P_2(\Gamma)\,\mathrm{Diff}^{r-2}(\Gamma) \oplus \mathrm{Span}\{x^n D^k + n\Gamma x^{n-1}D^{k-1} \colon n \geqslant 0,\ 1 \leqslant k \leqslant r\}.$$

Proof. During this proof we write $\langle P_2\rangle^r$ for $P_2(\Gamma)\,\mathrm{Diff}^{r-2}(\Gamma)$ and Δ^r for the span of the $x^n D^k + n\Gamma x^{n-1}D^{k-1}$ with $n \geqslant 0$ and $1 \leqslant k \leqslant r$.

First we prove that the right-hand side (RHS) is in the left-hand side (LHS). We have $\pi(e_{n-1}e_{-1}^{k-1}) = x^n D^k + n\Gamma x^{n-1}D^{k-1}$, so Δ^r is in the LHS. It follows from Lemma 2.6 that $\pi(Z_n) = P_2(\Gamma)x^n$ and $\pi(Y_n) = P_3(\Gamma)x^n$. Taking products of these two and $\pi(e_{-1}) = D$, we see that $\langle P_2\rangle^r$ is in the LHS.

We shall prove that the LHS is in the RHS by induction on r. At $r = 1$ it reduces to $\pi(\mathcal{W}) = \Delta^1$, which is clear. Assume that it holds at $r-1$. We must prove that $\pi(e_{i_1}\cdots e_{i_r})$ is in $\langle P_2\rangle^r \oplus \Delta^r$ for arbitrary $i_1, \ldots, i_r$.

Let us work in $\mathrm{Diff}^r(\Gamma)$ modulo $\langle P_2\rangle^r \oplus \pi(\mathfrak{U}^{+,r-1})$: given two elements H_1 and H_2 of $\mathrm{Diff}^r(\Gamma, 0)$, write $H_1 \equiv H_2$ whenever $H_1 - H_2$ is in $\langle P_2\rangle^r \oplus \pi(\mathfrak{U}^{+,r-1})$. A calculation yields

$$\pi(e_n e_m - e_{n+m+1}e_{-1} - (m+1)e_{n+m}) = (n+1)(m+1)P_2(\Gamma)x^n,$$

which leads to $\pi(e_{i_1}\cdots e_{i_r}) \equiv \pi(e_{i_1}\cdots e_{i_{r-2}}e_{i_{r-1}+i_r+1}e_{-1})$. Applying this repeatedly gives

$$\pi(e_{i_1}\cdots e_{i_r}) \equiv \pi(e_{i+r-1}e_{-1}^{r-1}) = x^{i+r}D^r + (i+r)\Gamma x^{i+r-1}D^{r-1} \in \Delta^r,$$

where $i = i_1 + \cdots + i_r$. This concludes the proof. □

Lemma 3.2. $\mathfrak{U} = I \oplus J$, *and* $\pi : J^r \to \pi(\mathfrak{U})^r$ *is bijective for all* r.

Proof. As we mentioned, the second fact implies the first. For the second, by induction on degree it is enough to prove that π induces a bijection from J^r/J^{r-1} to $\pi(\mathfrak{U})^r/\pi(\mathfrak{U})^{r-1}$. This is true for $r = 0$ and 1 by Lemma 3.1, so the induction begins.

We may restrict to a fixed weight n, as J is $\mathrm{ad}(e_0)$-invariant and π preserves weights. Thus we must prove that π induces a bijection from J_n^r/J_n^{r-1} to $\pi(\mathfrak{U})_n^r/\pi(\mathfrak{U})_n^{r-1}$ for all n. First consider the case $n \geqslant 0$. Here Lemma 3.1 shows that $\pi(\mathfrak{U})_n^r/\pi(\mathfrak{U})_n^{r-1}$ is r-dimensional, with a basis given by the cosets of $x^{n+r}D^r + (n+r)x^{n+r-1}D^{r-1}$ and $P_2(\Gamma)\Gamma^{i-2}x^{n+r-i}D^{r-i}$ for $2 \leqslant i \leqslant r$.

On the other hand, J_n^r/J_n^{r-1} is also r-dimensional: the cosets of $e_{n+i}e_0^{r-i-1}e_{-1}^i$ with $0 \leqslant i \leqslant r-1$ are a basis. Since $\pi(e_{n+i}e_0^{r-i-1}e_{-1}^i)$ is of total (Γ, D)-degree r and Γ-degree $r-i$, it is clear that π induces an injection from J_n^r/J_n^{r-1} to $\pi(\mathfrak{U})_n^r/\pi(\mathfrak{U})_n^{r-1}$, and hence a bijection as their dimensions are the same.

The cases with $n<0$ are similar. We leave them to the reader, mentioning only that J_n^r/J_n^{r-1} and $\pi(\mathfrak{U})_n^r/\pi(\mathfrak{U})_n^{r-1}$ are both $(r+n)$-dimensional for $-r<n<0$, 1-dimensional for $n=-r$, and 0-dimensional otherwise. □

We come now to the second part of the proof of $I=\langle Q^{e_2}\rangle_{\mathcal{W}}$, in which we prove that $I=\langle I^2\rangle_{\mathcal{W}}$. Towards this aim, define

$$\kappa:\mathfrak{U}\to\mathfrak{U},\quad \kappa:=\text{projection to } I \text{ along } J.$$

Lemma 3.3. *κ commutes with the left, right, and adjoint actions of $\mathfrak{b}$ on $\mathfrak{U}$. It also preserves degrees*: $\kappa(\mathfrak{U}^r)=I^r$.

Proof. For the first sentence, check that I and J are both invariant under the right and adjoint actions of $\mathfrak{b}$. For the second, recall from Lemma 3.2 that given any Ω in $\mathfrak{U}^r$, there exists Ω_J in J^r such that $\pi(\Omega_J)=\pi(\Omega)$. Hence $\kappa(\Omega)=\Omega-\Omega_J$ is in I^r. □

Lemma 3.4. $I=\langle I^2\rangle_{\mathcal{W}}$.

Proof. Since $\langle I^2\rangle_{\mathcal{W}}\subseteq I$ and $\mathfrak{U}=I\oplus J$, we need only prove $\mathfrak{U}=\langle I^2\rangle_{\mathcal{W}}+J$. It will suffice to prove that κ has image $\langle I^2\rangle_{\mathcal{W}}$. The strategy is to prove by induction on degree that $\kappa(\mathfrak{U}^r)\subset\langle I^2\rangle_{\mathcal{W}}$. By Lemma 3.3, this is true for $r=2$. Assume it for some r. To prove it for $r+1$, fix $X\in\mathcal{W}$ and $\Omega\in\mathfrak{U}^r$. We must prove $\kappa(X\Omega)\in\langle I^2\rangle_{\mathcal{W}}$.

Write $\kappa^\perp$ for $1-\kappa$, the projection of $\mathfrak{U}$ to J along I. Multiply $\Omega=\kappa(\Omega)+\kappa^\perp(\Omega)$ by X and apply κ to obtain

$$\kappa(X\Omega)=\kappa\big(X\kappa(\Omega)\big)+\kappa\big(X\kappa^\perp(\Omega)\big).$$

Here $\kappa(X\kappa(\Omega))$ is equal to $X\kappa(\Omega)$ because $X\kappa(\Omega)$ is already in I. Moreover, $\kappa(\Omega)$ is in $\langle I^2\rangle_{\mathcal{W}}$ by induction, so $X\kappa(\Omega)$ is also. Therefore we come down to proving $\kappa(X\kappa^\perp(\Omega))\in\langle I^2\rangle_{\mathcal{W}}$.

Since $\kappa^\perp(\Omega)$ is in J, $X\kappa^\perp(\Omega)$ is a linear combination of terms of the form $Xe_ne_0^ie_{-1}{}^j$ and $Xe_0^ie_{-1}{}^j$. By Lemma 3.3, $\kappa(Xe_ne_0^ie_{-1}{}^j)$ equals $\kappa(Xe_n)e_0^ie_{-1}{}^j$ and $\kappa(Xe_n)$ is in I^2, so $\kappa(Xe_ne_0^ie_{-1}{}^j)$ is in $\langle I^2\rangle$. The terms $Xe_0^ie_{-1}{}^j$ are in J, so κ kills them. Thus the proof is done. □

The final step in the proof of Theorem 1.3 is the following lemma. It and Lemma 3.4 yield $I=\langle I^2\rangle_{\mathcal{W}}=\langle Q^{e_2}\rangle_{\mathcal{W}}$. Recall that a module is said to be *completely indecomposable,* or *uniserial,* if all of its subquotients are indecomposable.

Lemma 3.5. *$\mathcal{S}^2\mathcal{F}(-1)$, $I^2(\gamma)$, and I^2 are all completely indecomposable $\mathcal{W}$-modules. They are generated under the adjoint action of $\mathcal{W}$ by the vectors $e_{-1}{}^2$, $Q-P_2(\gamma)$ for $P_2(\gamma)\neq 0$, Z_1 for $P_2(\gamma)=0$, and Q^{e_2}, respectively. In particular, I^2 is contained in $\langle Q^{e_2}\rangle_{\mathcal{W}}$.*

Proof. We give two proofs of this result. In the first it is obtained as a corollary of some deeper results from earlier papers, while in the second it is proven directly. We will in fact prove that $\mathcal{S}^2\mathcal{F}(\gamma)$ is completely indecomposable for almost all γ.

To obtain the lemma as a corollary we must first use the fact that for any γ, $\mathcal{S}^2\mathcal{F}(\gamma)$ is completely indecomposable over $\mathrm{Vec}(\mathbb{R})$ if and only if the tensor density modules $\mathcal{S}^2\mathcal{F}_{S^1}(a,\gamma)$

of $\mathrm{Vec}(S^1)$ (see Section 5) are completely indecomposable for all a. A proof of this can be deduced from Section 6 of [Co01].

Over S^1 there is a duality between $\mathcal{F}_{S^1}(a,\gamma)$ and $\mathcal{F}_{S^1}(-a,1-\gamma)$ (see Section 5 again). This duality gives rise to a more subtle duality between certain modules of pseudodifferential operators, known as the *Adler trace* or the *non-commutative residue* (see for example [CMZ97] or [CS04]). Taken together, these dualities yield a $\mathrm{Vec}(S^1)$-equivalence from $\bigotimes^2 \mathcal{F}_{S^1}(a,\gamma)$ to the module of pseudodifferential operators of order $\leqslant -1$ from $\mathcal{F}_{S^1}(-a,1-\gamma)$ to $\mathcal{F}_{S^1}(a,\gamma)$. This pseudodifferential operator module is in general $\mathfrak{a}$-equivalent to $\bigoplus_{i=0}^{\infty} \mathcal{F}_{S^1}(2a, 2\gamma+i)$, and the submodule corresponding to the symmetric product $\mathcal{S}^2\mathcal{F}_{S^1}(a,\gamma)$ is $\mathfrak{a}$-equivalent to $\bigoplus_{i=0}^{\infty} \mathcal{F}_{S^1}(2a, 2\gamma+2i)$.

Conditions under which this submodule is completely indecomposable under $\mathrm{Vec}(S^1)$ may be extracted from [CMZ97], and this extraction has been carried out in several papers, see for example [Co05]. The conclusion is that $\mathcal{S}^2\mathcal{F}(\gamma)$ is completely indecomposable under $\mathrm{Vec}(\mathbb{R})$ if and only if the scalars $b'_{2\gamma+2i+2,2\gamma+2i}$ from Eq. (6) of [Co05] are non-zero for all $i \in \mathbb{N}$. This is exactly the same condition we derive below, and it holds for $\gamma = -1$. The rest of the lemma follows easily, as we will explain.

Now let us give an elementary self-contained proof. By Lemma A.6, the $\mathfrak{b}$-decomposition of $\mathcal{S}^2\mathcal{F}(\gamma)$ is $\bigoplus_{i=0}^{\infty} \mathcal{F}(2\gamma+2i)$ for all γ. By Corollary A.3, its $\mathfrak{a}$-decomposition is the same whenever the scalars $P_2(2\gamma+2i)$, the eigenvalues of Q on $\mathcal{F}(2\gamma+2i)$, are distinct for all $i \in \mathbb{N}$. As is quickly checked, this occurs precisely when γ is not in $-1/4 - \mathbb{N}/2$. In this case $\mathcal{S}^2\mathcal{F}(\gamma)$ contains a unique $\mathfrak{a}$-copy of $\mathcal{F}(2\gamma+2i)$ for each $i \in \mathbb{N}$, namely, the $P_2(2\gamma+2i)$-eigenspace of Q. Let us write $V(i)$ for this $\mathfrak{a}$-submodule of $\mathcal{S}^2\mathcal{F}(\gamma)$, and define

$$U(2\gamma+2i) := \bigoplus_{j=i}^{\infty} V(j).$$

First we claim that the $U(i)$ are $\mathcal{W}$-invariant. To prove this, let $\mathcal{W}^+$ be the subalgebra of $\mathcal{W}$ spanned by $e_2, e_3, \ldots$. Then $\mathcal{W} = \mathcal{W}^+ \oplus \mathfrak{a}$, so $\mathfrak{U}(\mathcal{W}) = \mathfrak{U}(\mathcal{W}^+) \otimes \mathfrak{U}(\mathfrak{a})$. Since $U(i)$ is $\mathfrak{a}$-invariant, $\mathfrak{U}(\mathcal{W})U(i)$ is $\mathfrak{U}(W^+)U(i)$, whose set of weights is $2\gamma + 2i + \mathbb{N}$. If $\mathfrak{U}(\mathcal{W})U(i)$ were larger than $U(i)$, it would have to contain a non-trivial submodule of $V(j)$ for some $j < i$. But this would force it to contain some vectors of weight $< 2\gamma + 2i$, a contradiction. Hence $U(i)$ is $\mathcal{W}$-invariant.

Next we find conditions on γ implying that the action of $\mathcal{W}$ on the lowest weight vector of $\mathcal{S}^2\mathcal{F}(\gamma)$ of weight $2\gamma+2i$ generates all of its lowest weight vectors of higher weight, that is to say, conditions under which the lowest weight vector of $V(i)$ generates the lowest weight vectors of $V(j)$ for all $j > i$. We claim that Eq. (8) below gives such conditions. To prove this, consider the element

$$S := 4e_2e_0 - 2e_2 - 3e_1^2 \in \mathfrak{U}(\mathcal{W}).$$

It is easy to check that $e_{-1}S \in \mathfrak{U}(\mathcal{W})e_{-1}$, which implies that S preserves lowest weight vectors, i.e., it maps any lowest weight vector of weight λ either to zero or to a lowest weight vector of weight $\lambda+2$. (Such elements of $\mathfrak{U}(\mathcal{W})$ are discussed in more generality in Section 3 of [Co01]: they make up what is called the *step algebra*.)

Note that $(dx^\gamma)^2$ is the lowest weight vector of $V(0)$. Therefore if $S^i(dx^\gamma)^2$ is not zero, it must be a lowest weight vector of $V(i)$. It follows that conditions of the desired type are given by $S^i(dx^\gamma)^2 \neq 0$ for all i. By keeping track only of the coefficient of $dx^\gamma \cdot x^{2i}\, dx^\gamma$, we find that these conditions are

$$\gamma \notin -1/4 - \mathbb{N}/2 \quad \text{and} \quad 2i^2 + (8\gamma - 1)i + 6\gamma^2 \neq 0 \quad \forall i \in \mathbb{N}. \tag{8}$$

Next we claim that $U(i)/U(i+1)$ is $\mathcal{W}$-equivalent to $\mathcal{F}(2\gamma + 2i)$. In fact, any $\mathcal{W}$-module that is $\mathfrak{a}$-equivalent to some $\mathcal{F}(\lambda)$ must actually be $\mathcal{W}$-equivalent to $\mathcal{F}(\lambda)$; we leave this to the reader. Since the $\mathcal{F}(\lambda)$ are $\mathcal{W}$-irreducible for $\lambda \neq 0$, this is enough to prove that $\mathcal{S}^2(\gamma)$ is completely indecomposable whenever γ satisfies Eq. (8) and in addition $2\gamma + 2i \neq 0$ for any $i \in \mathbb{N}$, i.e., $\gamma \notin -\mathbb{N}$.

When $\gamma \in -\mathbb{N}$, it is still true that $\mathcal{S}^2\mathcal{F}(\gamma)$ is completely indecomposable whenever Eq. (8) holds, but the proof requires a little more effort. For $\gamma = 0$ it follows from the indecomposability of $\mathcal{F}(0)$. For $\gamma \in \mathbb{Z}^-$, we must prove that the lowest weight vector of the $\mathfrak{a}$-copy of $\mathcal{F}(-2)$ in $\mathcal{S}^2\mathcal{F}(\gamma)$ generates the whole $\mathfrak{a}$-copy of $\mathcal{F}(0)$, not merely its lowest weight vector, which spans a trivial $\mathcal{W}$-submodule of $\mathcal{F}(0)$. One way to do this is to appeal to the fact, proven in [FF80], that there is no indecomposable $\mathcal{W}$-module composed of $\mathcal{F}(-2)$ and $\mathbb{C}$. We give a direct proof in the case of interest to us, $\gamma = -1$.

It will do to prove that the action of $\mathcal{W}$ on the lowest weight vector $(dx^{-1})^2$ of $V(0)$ generates the weight 1 subspace $V(1)_1$. Here $\mathcal{S}^2\mathcal{F}(-1)_1$ is only 2-dimensional and is $V(0)_1 \oplus V(1)_1$. The line $V(0)_1$ lies in the finite-dimensional $\mathfrak{a}$-submodule of $V(0)$ and is in the span of the products of dx^{-1}, $x\,dx^{-1}$, and $x^2\,dx^{-1}$, so we need only note that $e_3(dx^{-1})^2$ is not in their span.

This proves that $\mathcal{S}^2\mathcal{F}(-1)$ is completely indecomposable, and therefore all of its submodules are also. In the notation above, its submodules $U(1)$ and $U(2)$ have $\mathfrak{a}$-decompositions $\bigoplus_{i=j}^{\infty} \mathcal{F}(2i)$ for $j = 0$ and 1, respectively. Since $\mathcal{F}(0)$ contains a trivial $\mathcal{W}$-submodule, we have the intermediate $\mathcal{W}$-submodule $V(1)_0 \oplus U(2)$ inside $U(1)$. Using Proposition 2.5, one finds the following $\mathcal{W}$-equivalences: $I^2(\gamma) \cong V(1)_0 \oplus U(2)$ for $\gamma \neq 0$, $I^2(0) \cong U(1)$, and $I^2 \cong U(2)$. This completes the proof of the lemma, and hence also of Theorem 1.3. □

4. Proofs of Theorems 1.1 and 1.2

We continue with the notation of Section 3. Recall that the representation π of $\mathcal{W}$ on $\mathcal{F}(\Gamma)$ maps $\mathfrak{U} = \mathfrak{U}(\mathcal{W})$ into the algebra $\mathrm{Diff}(\Gamma) = \mathbb{C}[\Gamma, x, D]$. Its image is given by Lemma 3.1 and its kernel is I. The strategy in this section is to reduce questions about $I(\gamma)$ to questions about $\pi(I(\gamma))$.

We begin with a lemma which is useful in the proofs of both theorems. Recall the maps $\mathrm{eval}_\gamma : \mathrm{Diff}(\Gamma) \to \mathrm{Diff}(\gamma)$. For any subset G of any associative algebra A, let $\langle G \rangle_A$ denote the two-sided ideal in A generated by G. (Thus for example, if G is in $\mathfrak{U}$ then $\langle G \rangle_{\mathcal{W}}$ and $\langle G \rangle_{\mathfrak{U}}$ have the same meaning.)

Lemma 4.1. *For all $\gamma \in \mathbb{C}$,*

(1) $\mathrm{kernel}(\mathrm{eval}_\gamma : \mathrm{Diff}(\Gamma) \to \mathrm{Diff}(\gamma)) = \langle \Gamma - \gamma \rangle_{\mathrm{Diff}(\Gamma)}$.
(2) $\pi(I(\gamma)) = \mathrm{kernel}(\mathrm{eval}_\gamma|_{\pi(\mathfrak{U})}) = \langle \Gamma - \gamma \rangle_{\mathrm{Diff}(\Gamma)} \cap \pi(\mathfrak{U})$.
(3) *For any subset G of $\mathfrak{U}$, we have $\pi(\langle G \rangle_{\mathcal{W}}) = \langle \pi(G) \rangle_{\pi(\mathfrak{U})}$.*
(4) $\pi(\mathfrak{U}) = \mathbb{C}1 \oplus \langle P_2(\Gamma) \rangle_{\mathrm{Diff}(\Gamma)} \oplus \mathrm{Span}\{x^n D^k + n\Gamma x^{n-1} D^{k-1} \colon n \geqslant 0,\ k \geqslant 1\}$.

(5) *Let H be any two-sided ideal in $\mathfrak{U}$. Then $H = I(\gamma)$ if and only if $I \subset H$ and $\pi(H) = \pi(I(\gamma))$.*
(6) $\mathrm{Diff}(\Gamma)^{e_{-1}} = \mathbb{C}[\Gamma, D]$.

Proof. (1) can be proven using the polynomial division algorithm because Γ is central in $\mathrm{Diff}(\Gamma)$. Since $\pi_\gamma = \mathrm{eval}_\gamma \circ \pi$ and $I(\gamma) = \mathrm{kernel}(\pi_\gamma)$, we have $\pi(I(\gamma)) = \mathrm{kernel}(\mathrm{eval}_\gamma|_{\pi(\mathfrak{U})})$. Hence (1) implies (2). (3) is clear, (4) follows from Lemma 3.1, and (5) is immediate from $I = \mathrm{kernel}(\pi)$.

For (6), we can write any element T of $\mathrm{Diff}(\Gamma)$ uniquely as $\sum_{i,j,k} a_{ijk}\Gamma^i x^j D^k$. The statement then follows from $\pi(e_{-1})T = \sum_{i,j,k} j a_{ijk}\Gamma^i x^{j-1} D^k$. □

Proof of Theorem 1.1. The distinctness of the $I(\gamma)$ was proven in Proposition 2.5. By Corollary 2.4, $\mathfrak{U}_0^3(\mathcal{W})^{e_{-1}}$ has basis $\{1, Q, Y_0\}$, and by Proposition 2.5 its intersection with $I(\gamma)$ has basis $\{Z_0(\gamma), Y_0(\gamma)\}$, which clearly has the same span as the generating set given in the theorem.

Now fix some $\gamma \neq 0$ or 1. To prove that $I(\gamma) = \langle Z_0(\gamma), Y_0(\gamma)\rangle_{\mathcal{W}}$, note that $\mathrm{ad}(e_2)Z_0(\gamma) = Q^{e_2}$. Therefore $I \subset \langle Z_0(\gamma), Y_0(\gamma)\rangle_{\mathcal{W}}$, so by (3) and (5) of Lemma 4.1 we need only prove that

$$\pi\big(I(\gamma)\big) = \big\langle \pi\big(Z_0(\gamma)\big), \pi\big(Y_0(\gamma)\big)\big\rangle_{\pi(\mathfrak{U})}.$$

We know by (2) of Lemma 4.1 that $\pi(I(\gamma)) = \langle \Gamma - \gamma\rangle_{\mathrm{Diff}(\Gamma)} \cap \pi(\mathfrak{U})$, and Lemma 2.6 gives

$$\pi\big(Z_0(\gamma)\big) = P_2(\Gamma) - P_2(\gamma) = (\Gamma - \gamma)(\Gamma + \gamma - 1),$$
$$\pi\big(Y_0(\gamma)\big) = P_3(\Gamma) - (\gamma - 1/2)P_2(\Gamma) = (\Gamma - \gamma)P_2(\Gamma).$$

Thus we must prove

$$\langle \Gamma - \gamma\rangle_{\mathrm{Diff}(\Gamma)} \cap \pi(\mathfrak{U}) = \big\langle (\Gamma - \gamma)(\Gamma + \gamma - 1), (\Gamma - \gamma)P_2(\Gamma)\big\rangle_{\pi(\mathfrak{U})}. \tag{9}$$

Clearly the LHS contains the RHS. Let us make a warning remark: it is not possible to replace the two generators of the RHS by their GCD (greatest common divisor), $\Gamma - \gamma$, because it is not an element of $\pi(\mathfrak{U})$. The next lemma describes the LHS. □

Lemma 4.2. *For $\gamma \neq 0$ or 1,*

$$\begin{aligned}
&\langle \Gamma - \gamma\rangle_{\mathrm{Diff}(\Gamma)} \cap \pi(\mathfrak{U}) \\
&\quad = \big\langle (\Gamma - \gamma)P_2(\Gamma)\big\rangle_{\mathrm{Diff}(\Gamma)} \\
&\qquad \oplus \big(P_2(\Gamma) - P_2(\gamma)\big)\big(\mathbb{C} \oplus \mathrm{Span}\big\{x^n D^k + n\Gamma x^{n-1} D^{k-1}\colon n \geqslant 0,\ k \geqslant 1\big\}\big).
\end{aligned}$$

Proof. By (4) of Lemma 4.1, the LHS contains the RHS. For the other direction note that here $P_2(\gamma) \neq 0$, so eval_γ maps $P_2(\Gamma)\mathbb{C}[x, D]$ bijectively to $\mathrm{Diff}(\gamma)$. Since $\mathrm{Diff}(\Gamma) = \mathbb{C}[x, D] \oplus \langle \Gamma - \gamma\rangle_{\mathrm{Diff}(\Gamma)}$, (4) of Lemma 4.1 shows that $\pi(\mathfrak{U})$ is the direct sum of $P_2(\Gamma)\mathbb{C}[x, D]$ and the RHS. Thus the lemma follows from (2) of Lemma 4.1. □

By this lemma, the proof of Eq. (9) comes down to proving that the RHS of Eq. (9) contains the RHS of the equation in the lemma. Since $(\Gamma-\gamma)(\Gamma+\gamma-1)=P_2(\Gamma)-P_2(\gamma)$, the expression for $\pi(\mathfrak{U})$ in (4) of Lemma 4.1 shows that the RHS of Eq. (9) contains the second term of the RHS in the lemma, as well as $(P_2(\Gamma)-P_2(\gamma))P_2(\Gamma)\operatorname{Diff}(\Gamma)$ and $(\Gamma-\gamma)P_2^2(\Gamma)\operatorname{Diff}(\Gamma)$. In light of the fact that the GCD of $(P_2(\Gamma)-P_2(\gamma))P_2(\Gamma)$ and $(\Gamma-\gamma)P_2^2(\Gamma)$ is $(\Gamma-\gamma)P_2(\Gamma)$ for $\gamma\neq 0$ or 1, Eq. (9) is proven.

Thus far we have proven the statements of Theorem 1.1 up through its main point, that $Z_0(\gamma)$ and $Y_0(\gamma)$ generate $I(\gamma)$. Now recall $Y_1(\gamma)$ from Eq. (4). By Lemma 2.6, $\operatorname{ad}(e_{-1})Y_1(\gamma)=Y_0(\gamma)$. Thus it follows from $\operatorname{ad}(e_0)(Z_0(\gamma)+Y_1(\gamma))=Y_1(\gamma)$ that $Z_0(\gamma)+Y_1(\gamma)$ is a single inhomogeneous cubic generator of $I(\gamma)$.

For the other two types of single generators, define

$$\Omega_{-1}:=Y_0(\gamma)e_{-1},\qquad \Omega_0:=Y_0(\gamma)e_0-X_1e_{-1}.$$

Both Ω_{-1} and Ω_0 are in $I(\gamma)$, as X_1 is in I by the last sentence of Lemma 2.6. One checks that Ω_{-1} is a lowest weight vector, that $\operatorname{ad}(e_{-1})\Omega_0=\Omega_{-1}$, and that both Ω_{-1} and Ω_0 are eigenvectors of $\operatorname{ad}(Q)$ of eigenvalue 2.

We claim that $Z_0(\gamma)+\Omega_0$ is a single homogeneous quartic generator. To prove this, apply $\operatorname{ad}(Q)$ to it to show that it generates both $Z_0(\gamma)$ and Ω_0. From Ω_0 we get Ω_{-1}, and from $Z_0(\gamma)$ we get Q^{e_2} and hence I. We have $\pi(Z_0(\gamma))=P_2(\Gamma)-P_2(\gamma)$ and $\pi(\Omega_{-1})=(\Gamma-\gamma)P_2(\Gamma)D$, so by (3) and (5) of Lemma 4.1 we need only prove that

$$\bigl\langle P_2(\Gamma)-P_2(\gamma),(\Gamma-\gamma)P_2(\Gamma)D\bigr\rangle_{\pi(\mathfrak{U})}=\pi\bigl(I(\gamma)\bigr).$$

Conjugating $(\Gamma-\gamma)P_2(\Gamma)D$ by $P_2(\Gamma)x$ shows that $(\Gamma-\gamma)P_2^2(\Gamma)$ is in the LHS of this equation. The GCD of $(P_2(\Gamma)-P_2(\gamma))P_2(\Gamma)$ and $(\Gamma-\gamma)P_2^3(\Gamma)$ is $(\Gamma-\gamma)P_2(\Gamma)$ for $\gamma\neq 0$ or 1, so the proof concludes just as did the proof of Eq. (9).

The same argument shows that $Z_0(\gamma)+\Omega_{-1}$ is a single inhomogeneous quartic lowest weight generator. To prove the last statement of the first paragraph of Theorem 1.1, suppose that Ω is a single homogeneous lowest weight generator. By Proposition 2.5, $I(\gamma)\not\subset\mathfrak{U}^+$, so $\Omega\notin\mathfrak{U}^+$. Therefore Ω must be of weight 0, so $\pi(\Omega)$ is a lowest weight vector of weight 0 such that $\langle\pi(\Omega)\rangle_{\pi(\mathfrak{U})}=\pi(I(\gamma))$.

By (2) and (6) of Lemma 4.1, $\pi(\Omega)$ is in $(\Gamma-\gamma)\mathbb{C}[\Gamma]\cap\pi(\mathfrak{U})$. Thus by Lemma 4.2, $\pi(\Omega)$ is at least quadratic in Γ. It follows that $\langle\pi(\Omega)\rangle_{\pi(\mathfrak{U})}$ cannot contain two elements of $\mathbb{C}[\Gamma]$ whose GCD is $\Gamma-\gamma$. But $\pi(I(\gamma))$ does contain two such elements, namely $\pi(Z_0(\gamma))$ and $\pi(Y_0(\gamma))$. This is a contradiction.

Finally we come to the second paragraph of the theorem. We just saw that $Z_0(\gamma)$ cannot generate $I(\gamma)$, as it is a homogeneous lowest weight vector. The rest follows from Duflo's theorem and Proposition 2.5.

Proof of Theorem 1.2. We saw $I(0)=I(1)$ in Proposition 2.5, and the second sentence of the theorem was proven in Lemma 2.6. (Indeed, by Corollaries 2.4 and 2.7, Z_1 is the weight 1 element of the copy of $\mathcal{F}(0)$ in $\mathfrak{U}^2$. The point of the second sentence of the theorem is that the lowest weight vector of the quotient of this copy of $\mathcal{F}(0)$ by its trivial submodule is the image of Z_1.)

Clearly Z_1 generates Z_0, hence Q^{e_2} and I. Therefore by (3) and (5) of Lemma 4.1, $I(0)=\langle Z_1\rangle_{\mathcal{W}}$ will follow if we prove $\pi(I(0))=\langle\pi(Z_1)\rangle_{\pi(\mathfrak{U})}$. It is easy to see from (2) and (4) of

Lemma 4.1 that $\pi(I(0)) = P_2(\Gamma)\,\mathrm{Diff}(\Gamma)$, and Lemma 2.6 gives $\pi(Z_1) = P_2(\Gamma)x$. Thus we must prove

$$\big\langle P_2(\Gamma)x\big\rangle_{\pi(\mathfrak{U})} = P_2(\Gamma)\,\mathrm{Diff}(\Gamma).$$

Obviously the LHS is in the RHS. For the converse, conjugate $P_2(\Gamma)x$ by the $\pi(e_n)$ to see that it generates $P_2(\Gamma)x^n$ for all n. Since both D and $P_2(\Gamma)\,\mathrm{Diff}(\Gamma)$ are in $\pi(\mathfrak{U})$, the LHS contains $P_2(\Gamma)\mathbb{C}[x, D]$ and $P_2^2(\Gamma)\,\mathrm{Diff}(\Gamma)$. It remains to prove that it contains $\Gamma P_2(\Gamma)\mathbb{C}[x, D]$. This follows from the fact that it contains $P_2(\Gamma)\mathbb{C}[x, D]$ and $P_2(\Gamma)(x^n D^k + n\Gamma x^{n-1} D^{k-1})$ for all $n \geqslant 0$ and $k \geqslant 1$.

To prove the second paragraph of the theorem we will use an indecomposable module V of $\mathcal{W}$ composed of $\mathbb{C}$ and $\mathcal{F}(1)$, a relative of the Gel'fand–Fuks module used in the proof of Theorem 1.3. The space of V is $\mathbb{C} \oplus \mathcal{F}(1)$ and the action of $\mathcal{W}$ is as follows: on $\mathcal{F}(1)$ it is as usual, and on 1 it is

$$e_n(1) = \big(n^2 + n\big)\,dx\,x^{n-1}. \tag{10}$$

It is easy to check that this action makes V a $\mathcal{W}$-module, see for example [FF80]. (We remark that under $\mathfrak{a}$, V is a submodule of $\mathcal{G}(1)$. In fact, $\mathcal{G}(n)$ extends to a $\mathcal{W}$-module if and only if n is 1, 3/2, 2, or 5/2 [FF80].)

Note that Z_1 maps $1 \in V$ to $-dx$, so $I(0) \not\subset \mathrm{Ann}_{\mathcal{W}} V$. Thus to prove that $I(0)$ is not generated by $I(0)^{e_{-1}}$ it suffices to prove $I(0)^{e_{-1}} \subset \mathrm{Ann}_{\mathcal{W}} V$. Proposition 2.5 gives $I(0) = I(1) \subset \mathfrak{U}^+$, so $I(0)$ annihilates both the submodule $\mathcal{F}(1)$ and the trivial quotient of V. Hence any $\Omega \in I(0)$ maps V to $\mathbb{C}\Omega(1) \subset \mathcal{F}(1)$.

Fix $\Omega \in I(0)^{e_{-1}}$. We may assume that Ω is homogeneous, say of weight k. To prove $\Omega(V) = 0$ we need only prove $\Omega(1) = 0$, and we know that $\Omega(1)$ is a lowest weight vector of weight k in $\mathcal{F}(1)$. If $k \neq 1$, $\mathcal{F}(1)_k^{e_{-1}} = 0$ implies $\Omega(1) = 0$. If $k = 1$, Theorem 1.3 implies $\Omega \in \langle Q^{e_2}\rangle_{\mathcal{W}}$, so $\Omega(1) = 0$ because $Q^{e_2}(1) = 0$. This proves $I(0) \neq \langle I(0)^{e_{-1}}\rangle_{\mathcal{W}}$.

The intersection $I(0) \cap \mathfrak{U}(\mathfrak{a})$ is $\langle Q\rangle_{\mathfrak{a}}$ by Duflo's theorem, so it does not generate $I(0)$ by the last paragraph. To prove that $\bigoplus_{k\leqslant 0} I(0)_k$ does not generate I it is enough to note that $I(0)_k(V) = I(0)_k(1)$ lies in $\mathcal{F}(1)_k$, which is zero for $k \leqslant 0$. Finally, for $k \geqslant 1$ $I(0)_k$ contains Z_k, which generates Z_1 under the action of $\mathrm{ad}(e_{-1})$. □

Remarks. It is not hard to see from our proofs that there is no γ such that $I(\gamma)$ has a single homogeneous generator which is an eigenvector of $\mathrm{ad}(Q)$. On the other hand, for $\gamma \neq 0$ or 1 we do not know if $I(\gamma)$ has a single homogeneous generator of degree 3. It follows from our methods that any such generator would have to be of the form $(a + be_0)(Q - \gamma^2 + \gamma) + c(Y_0 - (\gamma - 1/2)(\gamma^2 - \gamma))$ for a, b, and c all non-zero.

Inside $I(0)$ we have the flag of ideals

$$\big\langle I(0) \cap \mathfrak{U}(\mathfrak{a})\big\rangle_{\mathcal{W}} = \big\langle Q\big\rangle_{\mathcal{W}} \subseteq \big\langle Q, Y_0\big\rangle_{\mathcal{W}} \subseteq \big\langle I(0)_0\big\rangle_{\mathcal{W}} \subseteq I(0).$$

By Theorem 1.2, $\langle I(0)_0\rangle_{\mathcal{W}}$ is proper in $I(0)$, and one can use our methods to prove that $\langle Q\rangle_{\mathcal{W}}$ is proper in $\langle Q, Y_0\rangle_{\mathcal{W}}$. However, we do not know whether or not $\langle Q, Y_0\rangle_{\mathcal{W}}$ is proper $\langle I(0)_0\rangle_{\mathcal{W}}$. One way to approach such questions is to study the annihilators of indecomposable modules composed of tensor density modules, such as the modules GF and V used above.

5. Results over the circle

In this section we state our results over $\mathrm{Vec}(S^1)$; the proofs are given in Section 6. We take $\mathrm{Vec}(S^1)$ to be the complex polynomial vector fields on the circle:

$$\mathrm{Vec}(S^1) := \mathrm{Span}\{e_n = x^{n+1}D\colon n \in \mathbb{Z}\}.$$

Note that $\mathrm{Vec}(S^1)$ contains $\mathrm{Vec}(\mathbb{R})$ as a subalgebra, so we may speak of weight spaces, lowest weight vectors, etc., for modules of $\mathrm{Vec}(S^1)$ just as for modules of $\mathrm{Vec}(\mathbb{R})$.

For any scalars a and γ, we write $\mathcal{F}_{S^1}(a,\gamma)$ for the $\mathrm{Vec}(S^1)$-module of polynomial tensors of degree γ and e_0-spectrum $a+\mathbb{Z}$:

$$\mathcal{F}_{S^1}(a,\gamma) := \mathrm{Span}\{x^{\lambda-\gamma}\,dx^{\gamma}\colon \lambda \in a+\mathbb{Z}\}.$$

It is convenient to treat all of the $\mathcal{F}_{S^1}(a,\gamma)$ with a given γ together. To this end, define

$$\mathcal{F}_{S^1}(\gamma) := \bigoplus_{0\leqslant \mathrm{Re}(a)<1} \mathcal{F}_{S^1}(a,\gamma) = \mathrm{Span}\{x^{\lambda}\,dx^{\gamma}\colon \lambda \in \mathbb{C}\}.$$

The tensor density module $\mathcal{F}(\gamma)$ of $\mathrm{Vec}(\mathbb{R})$ studied in the preceding sections is a $\mathrm{Vec}(\mathbb{R})$-submodule of $\mathcal{F}_{S^1}(\gamma,\gamma)$. The action of $\mathrm{Vec}(S^1)$ on $\mathcal{F}_{S^1}(\gamma)$ is the natural extension of the action π_γ of $\mathrm{Vec}(\mathbb{R})$ on $\mathcal{F}(\gamma)$, and we again denote it by π_γ. Thus

$$\pi_\gamma(e_n)\big(x^{\lambda-\gamma}\,dx^{\gamma}\big) := (\lambda+n\gamma)x^{\lambda+n-\gamma}\,dx^{\gamma}.$$

We now state our main results over S^1; they are completely parallel to the results over $\mathbb{R}$. Recall the projective subalgebra $\mathfrak{a}$ and its Casimir operator Q.

Theorem 5.1. *For all a and γ, $\mathrm{Ann}_{\mathrm{Vec}(S^1)}\mathcal{F}_{S^1}(a,\gamma) = \mathrm{Ann}_{\mathrm{Vec}(S^1)}\mathcal{F}_{S^1}(\gamma)$. The intersection of this ideal with $\mathfrak{U}(\mathrm{Vec}(\mathbb{R}))$ is the $\mathrm{Vec}(\mathbb{R})$-annihilator of both $\mathcal{F}_{S^1}(\gamma)$ and $\mathcal{F}(\gamma)$, i.e., $\mathrm{Ann}_{\mathrm{Vec}(\mathbb{R})}\mathcal{F}_{S^1}(\gamma) = \mathrm{Ann}_{\mathrm{Vec}(\mathbb{R})}\mathcal{F}(\gamma)$. We have*

$$\mathrm{Ann}_{\mathrm{Vec}(S^1)}\mathcal{F}_{S^1}(\gamma) = \big\langle \mathrm{Ann}_{\mathrm{Vec}(\mathbb{R})}\mathcal{F}(\gamma)\big\rangle_{\mathrm{Vec}(S^1)}.$$

In particular, any set of generators of $\mathrm{Ann}_{\mathrm{Vec}(\mathbb{R})}\mathcal{F}(\gamma)$ over $\mathfrak{U}(\mathrm{Vec}(\mathbb{R}))$ is also a set of generators of $\mathrm{Ann}_{\mathrm{Vec}(S^1)}\mathcal{F}_{S^1}(\gamma)$ over $\mathfrak{U}(\mathrm{Vec}(S^1))$.

(1) *For $\gamma \neq 0$ or 1, the ideals $\mathrm{Ann}_{\mathrm{Vec}(S^1)}\mathcal{F}_{S^1}(\gamma)$ are all distinct and are not contained in $\mathfrak{U}^+(\mathrm{Vec}(S^1))$. None of them is generated by any single lowest weight vector, nor by its intersection with either $\mathfrak{U}(\mathfrak{a})$ or $\mathfrak{U}^2(\mathrm{Vec}(S^1))$, both of which generate only $\langle Q-\gamma^2+\gamma\rangle_{\mathrm{Vec}(S^1)}$.*

(2) *$\mathrm{Ann}_{\mathrm{Vec}(S^1)}\mathcal{F}_{S^1}(0)$ is equal to $\mathrm{Ann}_{\mathrm{Vec}(S^1)}\mathcal{F}_{S^1}(1)$ and lies in $\mathfrak{U}^+(\mathrm{Vec}(S^1))$. It is not generated by its lowest weight vectors, nor by its intersection with $\mathfrak{U}(\mathfrak{a})$, which generates only $\langle Q\rangle_{\mathrm{Vec}(S^1)}$.*

Theorem 5.2. *The ideal $\bigcap_{\gamma\in\mathbb{C}}\mathrm{Ann}_{\mathrm{Vec}(S^1)}\mathcal{F}_{S^1}(\gamma)$ is generated by its intersection with $\mathfrak{U}(\mathrm{Vec}(\mathbb{R}))$, and hence by Q^{e_2}. It is not generated by any lowest weight vector of weight $\neq 2$.*

Recall the transpose involution from the end of Section 2, where we observed that $(\mathrm{Ann}_{\mathrm{Vec}(\mathbb{R})} \mathcal{F}(\gamma))^T = \mathrm{Ann}_{\mathrm{Vec}(\mathbb{R})} \mathcal{F}(1-\gamma)$. Let us give a conceptual proof of the analogous fact over S^1. There is a non-degenerate $\mathcal{W}$-invariant form $(\cdot,\cdot)$ pairing $\mathcal{F}(\gamma)$ and $\mathcal{F}(1-\gamma)$, defined by

$$\left(x^n\,dx^\gamma, x^m\,dx^{1-\gamma}\right) := \frac{1}{2\pi i}\oint_{S^1} x^{n+m}\,dx = \delta_{n+m,-1}.$$

This form defines a $\mathrm{Vec}(S^1)$-equivalence between $\mathcal{F}(1-\gamma)$ and the restricted dual of $\mathcal{F}(\gamma)$. As we noted in Section 2, $\mathrm{Ann}_{\mathfrak{g}} M^* = (\mathrm{Ann}_{\mathfrak{g}} M)^T$ for any module M of any Lie algebra $\mathfrak{g}$. After checking that this applies equally to restricted duals, one has the following proposition.

Proposition 5.3. $(\mathrm{Ann}_{\mathrm{Vec}(S^1)} \mathcal{F}_{S^1}(\gamma))^T = \mathrm{Ann}_{\mathrm{Vec}(S^1)} \mathcal{F}_{S^1}(1-\gamma)$.

The *Cartan involution* θ of $\mathrm{Vec}(S^1)$, the orientation-reversing automorphism, will be useful during the proofs. It is defined by

$$\theta(e_n) := -e_{-n}.$$

Given any module (π, M) of $\mathrm{Vec}(S^1)$, we call the action $\pi \circ \theta$ of $\mathrm{Vec}(S^1)$ on M the *opposite action.* We write M^{opp} for the space M with this action. Note that $\mathrm{Ann}_{\mathrm{Vec}(S^1)} M^{\mathrm{opp}} = \theta(\mathrm{Ann}_{\mathrm{Vec}(S^1)} M)$. One can check that $\mathcal{F}_{S^1}(a,\gamma)^{\mathrm{opp}}$ is equivalent to $\mathcal{F}_{S^1}(-a,\gamma)$, so $\mathcal{F}_{S^1}(\gamma)^{\mathrm{opp}}$ is equivalent to itself. This gives the following lemma.

Lemma 5.4. $\mathrm{Ann}_{\mathrm{Vec}(S^1)}(\gamma)$ *is θ-invariant for all γ.*

6. Proofs over the circle

Recall our abbreviations $\mathcal{W} := \mathrm{Vec}(\mathbb{R})$, $I(\gamma) := \mathrm{Ann}_{\mathcal{W}} \mathcal{F}(\gamma)$, and $I := \bigcap_\gamma I(\gamma)$. In this section we will use the abbreviations

$$\mathcal{V} := \mathrm{Vec}\big(S^1\big), \qquad I_{S^1}(\gamma) := \mathrm{Ann}_{\mathcal{V}} \mathcal{F}_{S^1}(\gamma), \qquad I_{S^1} := \bigcap_{\gamma\in\mathbb{C}} I_{S^1}(\gamma).$$

We will also need $\mathrm{Diff}_{S^1}(\gamma)$, the algebra of differential operators of integral weight from $\mathcal{F}_{S^1}(\gamma)$ to itself:

$$\mathrm{Diff}_{S^1}(\gamma) := \mathrm{Span}\big\{x^n D^k\colon\ n\in\mathbb{Z},\ k\in\mathbb{N}\big\}.$$

Write $\mathrm{Diff}^k_{S^1}(\gamma)$ for its order filtration. (In fact, as an algebra $\mathrm{Diff}_{S^1}(\gamma)$ is independent of γ; only its $\mathcal{V}$-module structure is γ-dependent. One can easily verify that $\mathrm{Diff}^k_{S^1}(\gamma)$ is $\mathcal{V}$-invariant and $\mathrm{Diff}^k_{S^1}(\gamma)/\mathrm{Diff}^{k-1}_{S^1}(\gamma)$ is naturally isomorphic to $\mathcal{F}_{S^1}(0,-k)$, but we will not use this fact.)

Following the same strategy we used over the line, we will prove Theorem 5.2 before Theorem 5.1. The proofs go largely as in Section 3, but the differences are sufficient to necessitate rewriting.

Proof of Theorem 5.2. The proof that I_{S^1} is not generated by any lowest weight vector of weight $\neq 2$ goes just as over the line, but with the Gel'fand–Fuks module GF replaced by GF_{S^1}, its analog over the circle. Here GF_{S^1} is the $\mathcal{V}$-module composed of $\mathbb{C}$ and $\mathcal{F}_{S^1}(0,2)$ defined by Eq. (5) (it is the coadjoint module of the Virasoro Lie algebra).

In order to prove $I_{S^1} = \langle Q^{e_2}\rangle_{\mathcal{V}}$, define $J_{S^1} \subset \mathfrak{U}(\mathcal{V})$ by

$$J_{S^1} := \mathrm{Span}\bigl\{1, e_1e_{-2}, e_n e_0^i e_{-1}^j \colon i \geqslant 0,\ j \geqslant 0,\ n \in \mathbb{Z}\bigr\}.$$

The proof has the same three parts it had over the line. First we prove that J_{S^1} is complementary to I_{S^1}. Second we prove that $\langle I^2_{S^1}\rangle_{\mathcal{V}} + J_{S^1}$ is all of $\mathfrak{U}(\mathcal{V})$, so $I_{S^1} = \langle I^2_{S^1}\rangle_{\mathcal{V}}$. Finally we prove that $\langle Q^{e_2}\rangle_{\mathcal{V}}$ contains $I^2_{S^1}$.

As in Section 3, I_{S^1} is the annihilator of the module $\bigoplus_\gamma \mathcal{F}_{S^1}(\gamma)$. Define

$$\mathcal{F}_{S^1}(\Gamma) := \mathrm{Span}\bigl\{\Gamma^j x^n \colon j \in \mathbb{N},\ n \in \mathbb{C}\bigr\},$$

where the $\Gamma^j x^n$ are all linearly independent. Extend the action π of $\mathcal{W}$ on $\mathcal{F}(\Gamma)$ to an action of $\mathcal{V}$ on $\mathcal{F}_{S^1}(\Gamma)$ by Eq. (6), extend the evaluation map eval_γ to a $\mathcal{V}$-covariant surjection from $\mathcal{F}_{S^1}(\Gamma)$ to $\mathcal{F}_{S^1}(\gamma)$ by Eq. (7), and verify $\mathrm{Ann}_{\mathcal{V}}\mathcal{F}_{S^1}(\Gamma) = I_{S^1}$.

Since $I_{S^1} = \mathrm{kernel}(\pi|_{\mathfrak{U}(\mathcal{V})})$, to prove $I_{S^1} \oplus J_{S^1} = \mathfrak{U}(\mathcal{V})$ it suffices to prove that π maps J_{S^1} bijectively to the image $\pi(\mathfrak{U}(\mathcal{V}))$. To describe this image we need the algebra of differential operators of integral weight on $\mathcal{F}_{S^1}(\Gamma)$:

$$\mathrm{Diff}_{S^1}(\Gamma) := \mathbb{C}\bigl[\Gamma, x, x^{-1}, D\bigr],$$

where Γ, x, and D are indeterminates such that Γ is central and $[D, x^n] = nx^{n-1}$. Extend eval_γ to a map from $\mathrm{Diff}_{S^1}(\Gamma)$ to $\mathrm{Diff}_{S^1}(\gamma)$ by $\Gamma \mapsto \gamma$ as before, and verify $\mathrm{eval}_\gamma \circ \pi = \pi_\gamma$.

Let $\mathrm{Diff}^r_{S^1}(\Gamma)$ be the filtration of $\mathrm{Diff}_{S^1}(\Gamma)$ by total (Γ, D)-degree, and given any subspace W of $\mathrm{Diff}_{S^1}(\Gamma)$, write W^r for $W \cap \mathrm{Diff}^r_{S^1}(\Gamma)$. Since $\pi(\mathcal{V}) \subseteq \mathrm{Diff}^1(\Gamma)$, we have $\pi(\mathfrak{U}^r(\mathcal{V})) \subseteq \mathrm{Diff}^r_{S^1}(\Gamma)$, i.e., $\pi(\mathfrak{U}^r(\mathcal{V})) \subseteq \pi(\mathfrak{U}(\mathcal{V}))^r$. The next two lemmas are the analogs of Lemmas 3.1 and 3.2.

Lemma 6.1. *For all $r \geqslant 1$,*

$$\pi\bigl(\mathfrak{U}^{+,r}(\mathcal{V})\bigr) = P_2(\Gamma)\,\mathrm{Diff}^{r-2}_{S^1}(\Gamma) \oplus \mathrm{Span}\bigl\{x^n D^k + n\Gamma x^{n-1} D^{k-1} \colon n \in \mathbb{Z},\ 1 \leqslant k \leqslant r\bigr\}.$$

Proof. The proof is the same as that of Lemma 3.1, except that since Z_n and Y_n are not defined for $n < 0$, we apply $\pi(e_{n-1})$ to $P_2(\Gamma)x$ and $P_3(\Gamma)x$ to prove that $P_2(\Gamma)x^n$ and $P_3(\Gamma)x^n$ are in the LHS for all $n \in \mathbb{Z}$. □

Lemma 6.2. $\mathfrak{U}(\mathcal{V}) = I_{S^1} \oplus J_{S^1}$, *and* $\pi : J^r_{S^1} \to \pi(\mathfrak{U}(\mathcal{V}))^r$ *is bijective for all* r.

Proof. The proof is the same as that of the $n \geqslant 0$ case of Lemma 3.2, except that we must use the new basis element e_1e_{-2} of J_{S^1} together with e_0e_{-1} to prove that J^2_{-1}/J^1_{-1} covers $\pi(\mathfrak{U}(\mathcal{V}))^2_{-1}/\pi(\mathfrak{U}(\mathcal{V}))^1_{-1}$. □

We come now to the second part of the proof, in which we prove that $I_{S^1} = \langle I^2_{S^1}\rangle_{\mathcal{V}}$. Define $\kappa_{S^1} : \mathfrak{U}(\mathcal{V}) \to \mathfrak{U}(\mathcal{V})$ to be projection to I_{S^1} along J_{S^1}.

Lemma 6.3. *κ_{S^1} preserves weights and degrees: it commutes with* $\mathrm{ad}(e_0)$ *and satisfies* $\kappa_{S^1}(\mathfrak{U}^r) = I^r$. *Its commutators with the left, right, and adjoint actions of the elements of* $\mathfrak{b}$ *are zero on* $\mathfrak{U}_n(\mathcal{V})$ *for all* $n \neq -1$. *Its restriction to* $\mathfrak{U}(\mathrm{Vec}(\mathbb{R}))$ *is* κ.

Proof. For the first two sentences, copy the proof of Lemma 3.3, noting that the left, right, and adjoint actions of $\mathfrak{b}$ map $(J_{S^1})_n$ to J_{S^1} for $n \neq -1$. The last sentence follows from $I \subset I_{S^1}$ and $J \subset J_{S^1}$. □

Lemma 6.4. $I_{S^1} = \langle I^2_{S^1} \rangle_{\mathcal{V}}$.

Proof. It follows from Lemma 5.4 that both I_{S^1} and $I^2_{S^1}$ are θ-invariant. Since θ negates weights, it will suffice to prove $(I_{S^1})_n = (\langle I^2_{S^1} \rangle_{\mathcal{V}})_n$ for all weights $n \geqslant 0$. Check as in the proof of Lemma 3.4 that it will do to prove $\kappa_{S^1}(\mathfrak{U}^r_n(\mathcal{V})) \subseteq \langle I^2_{S^1} \rangle_{\mathcal{V}}$ for all $r \geqslant 0$ and $n \geqslant 0$.

We proceed by induction, starting from $r = 2$. By the Poincaré–Birkhoff–Witt theorem, we need only prove that $\kappa_{S^1}(e_{i_1} \cdots e_{i_{r+1}})$ is in $\langle I^2_{S^1} \rangle_{\mathcal{V}}$ for all $i_1 \leqslant \cdots \leqslant i_{r+1}$ such that the weight $\sum_{j=1}^{r+1} i_j$ is non-negative. Moreover, we may assume $i_1 \leqslant -2$, as otherwise the result holds by Lemma 3.4. The result now follows from the argument of Lemma 3.4 with $X = e_{i_1}$ and $\Omega = e_{i_2} \cdots e_{i_{r+1}}$. The point is that since Ω is of positive weight, we can use Lemma 6.3 in place of Lemma 3.3. □

The third and final step in the proof of Theorem 5.2 is the following lemma. It and Lemma 6.4 yield $I_{S^1} = \langle Q^{e_2} \rangle_{\mathcal{V}}$. Its proof is long and completely different in flavor from that of Lemma 3.5.

Lemma 6.5. $I^2_{S^1}$ *is contained in* $\langle Q^{e_2} \rangle_{\mathcal{V}}$.

Proof. During this proof let us write U for $I^2_{S^1}$. We proceed in three steps. First we show that $U_0 \subset \langle Q^{e_2} \rangle_{\mathcal{V}}$. Next we show that $\langle Q^{e_2} \rangle_{\mathcal{V}}$ is θ-invariant, so it contains U_{-n} if and only if it contains U_n. Finally we show that it contains U_n for all $n > 0$.

To begin the first step, for each $n \in \mathbb{Z}^+$ define $\mathfrak{a}_n$ to be the subalgebra $\mathrm{Span}\{e_{-n}, e_0, e_n\}$ of $\mathcal{V}$. It is a copy of $\mathfrak{sl}_2$ with Casimir operator

$$Q_n := e_0^2 - ne_0 - e_n e_{-n} = e_0^2 + ne_0 - e_{-n} e_n.$$

Note that $\mathfrak{a}_1 = \mathfrak{a}$ and $Q_1 = Q$. We will consider Q_n to be defined for all $n \in \mathbb{Z}$ by this formula, and it is often convenient to use the facts that $Q_{-n} = Q_n$ and $Q_0 = 0$. It is not hard to check that $\pi(Q_n) = n^2 P_2(\Gamma)$, so $Q_n/n^2 - Q_m/m^2$ is in U for all $n \neq 0$. Using the fact that $\{1, e_0, e_n e_{-n}\colon n \geqslant 0\}$ is a basis of $\mathfrak{U}^2_0(\mathcal{V})$, one sees that $\{Q_n - n^2 Q_1\colon n \geqslant 2\}$ is a basis of U_0.

We claim that for all n and m,

$$\mathrm{ad}(Q_n)Q_m = (m-n)^2 Q_{m+n} - 2\big(m^2 - 2n^2\big)Q_m + (m+n)^2 Q_{m-n} - 2n^2 Q_n \tag{11}$$

(as we have mentioned, this is not to be confused with $[Q_n, Q_m]$). The proof is an elementary calculation. It can be shortened by noting that $\mathrm{ad}(Q_n)Q_m$ is equal to $m^2\,\mathrm{ad}(Q_n)(Q_m/m^2 - Q_n/n^2)$, so it is in U. This permits one to work at the symbol level.

To prove that $U_0 \subset \langle Q^{e_2}\rangle_{\mathcal{V}}$, we must prove that $Q_n - n^2 Q_1$ is in $\langle Q^{e_2}\rangle_{\mathcal{V}}$ for all $n \geqslant 2$. By Eq. (11),

$$\mathrm{ad}(Q_1)\big(Q_n - n^2 Q_1\big) = (n-1)^2 Q_{n+1} - 2\big(n^2-2\big)Q_n + (n+1)^2 Q_{n-1} - 2Q_1,$$

so it will suffice to prove that $Q_2 - 4Q_1$ is in $\langle Q^{e_2}\rangle_{\mathcal{V}}$.

We claim that $\langle Q^{e_2}\rangle_{\mathcal{V}}$ contains $\mathrm{ad}(Q_n)Q_1$ for all n. This is because $\mathrm{ad}(Q_n)Q_1 = -\mathrm{ad}(e_{-n}e_n)Q_1$, and $\mathrm{ad}(e_1)^{n-2}Q^{e_2}$ is a non-zero multiple of $\mathrm{ad}(e_n)Q_1$ for all $n \geqslant 2$. Now by Eq. (11),

$$\mathrm{ad}(Q_n)Q_1 = (n-1)^2 Q_{n+1} - 2n^2 Q_n + (n+1)^2 Q_{n-1} + 2\big(2n^2-1\big)Q_1.$$

It follows without too much calculation that $\mathrm{ad}(Q_2)Q_1$, $\mathrm{ad}(Q_3)Q_1$, $\mathrm{ad}(Q_4)Q_1$, and $\mathrm{ad}(Q_2)^2 Q_1$ are linearly independent (one can work modulo Q_1), so their span is equal to that of $\{Q_n - n^2 Q_1\colon 2 \leqslant n \leqslant 5\}$. This proves that $U_0 \subset \langle Q^{e_2}\rangle_{\mathcal{V}}$.

The next step in the proof of Lemma 6.5 is to prove that $\langle Q^{e_2}\rangle_{\mathcal{V}}$ is θ-invariant. By the first step, $\langle Q^{e_2}\rangle_{\mathcal{V}}$ contains $Q_2 - 4Q_1$. Applying $\mathrm{ad}(e_{-2})$ to this element shows that $\langle Q^{e_2}\rangle_{\mathcal{V}}$ also contains $\mathrm{ad}(e_{-2})Q$, which is $-\theta(Q^{e_2})$. The second step follows.

The final step is to prove $U_n \subset \langle Q^{e_2}\rangle_{\mathcal{V}}$ for all $n > 0$. We will use the cross section J_{S^1} and the projection $\kappa_{S^1}: \mathfrak{U}(\mathcal{V}) \to I_{S^1}$ along J_{S^1} from earlier in this section. Recall from Lemma 6.3 that κ_{S^1} restricts to κ on $\mathfrak{U}(\mathrm{Vec}(\mathbb{R}))$. It will do to prove $\kappa_{S^1}(e_i e_j) \in \langle Q^{e_2}\rangle_{\mathcal{V}}$ for all $i \leqslant j$ with $i + j > 0$. The result follows from Theorem 1.3 for $i \geqslant -1$, as there $\kappa_{S^1}(e_i e_j) = \kappa(e_i e_j)$ is in $I = \langle Q^{e_2}\rangle_{\mathrm{Vec}(\mathbb{R})}$. Thus we must prove $\kappa_{S^1}(e_{-i}e_{i+k}) \in \langle Q^{e_2}\rangle_{\mathcal{V}}$ for all $i \geqslant 2$ and $k \geqslant 1$.

We proceed by induction on i. The $i = 2$ case is essentially the same as the general case, so assume that the result is proven for $2, 3, \ldots, i-1$. Note that it follows (without calculation) from the form of J_{S^1} that $\kappa_{S^1}(e_{-i}e_i) = i^2 Q_1 - Q_i$. We know from above that this is an element of $\langle Q^{e_2}\rangle_{\mathcal{V}}$, so $\mathrm{ad}(e_k)\kappa_{S^1}(e_{-i}e_i)$ is also in $\langle Q^{e_2}\rangle_{\mathcal{V}}$ for all k. In particular, it is κ_{S^1}-invariant.

Since $\kappa_{S^1}(e_{-i}e_i) - e_{-i}e_i$ is an element of $\mathfrak{U}_0^2(\mathrm{Vec}(\mathbb{R}))$, we find that

$$\mathrm{ad}(e_k)\kappa_{S^1}(e_{-i}e_i) - (i-k)e_{-i}e_{i+k} + (i+k)e_{k-i}e_i$$

is an element of $\mathfrak{U}_k^2(\mathrm{Vec}(\mathbb{R}))$ for all $k > 0$. Applying κ_{S^1} shows that

$$\mathrm{ad}(e_k)\kappa_{S^1}(e_{-i}e_i) - (i-k)\kappa_{S^1}(e_{-i}e_{i+k}) + (i+k)\kappa_{S^1}(e_{k-i}e_i)$$

is an element of $I = \langle Q^{e_2}\rangle_{\mathrm{Vec}(\mathbb{R})}$ for all $k > 0$. Solving for $\kappa_{S^1}(e_{-i}e_{i+k})$ shows that we are done by the induction hypothesis except in the case that $k = i$. This case can be handled by a similar argument involving $\mathrm{ad}(e_1)\kappa_{S^1}(e_{-i}e_{2i-1})$. This completes the proof of Lemma 6.5, and hence also of Theorem 5.2. $\square$

Proof of Theorem 5.1. For the first sentence, suppose that Ω is in $\mathrm{Ann}_{\mathcal{V}}\,\mathcal{F}_{S^1}(a,\gamma)$. Then $\pi_\gamma(\Omega)$ is an element of $\mathrm{Diff}_{S^1}(\gamma)$ which annihilates $\mathcal{F}_{S^1}(a,\gamma)$, so by Zariski density it is zero. Thus $\mathrm{Ann}_{\mathcal{V}}\,\mathcal{F}_{S^1}(a,\gamma)$ is in $I_{S^1}(\gamma)$ for all a. The converse is clear.

The second sentence is proven similarly. Suppose that Ω is in $\mathrm{Ann}_{\mathcal{W}}\,\mathcal{F}(\gamma)$. Then $\pi_\gamma(\Omega)$ is an element of $\mathrm{Diff}_{S^1}(\gamma)$ which annihilates $\mathcal{F}(\gamma)$, the $\mathcal{W}$-submodule of $\mathcal{F}_{S^1}(\gamma,\gamma)$ of vectors of weight in $\gamma + \mathbb{N}$. Hence by Zariski density it annihilates all of $\mathcal{F}_{S^1}(\gamma)$. The converse is again clear.

The proof of the main statement of Theorem 5.1, the displayed equation, follows Section 4 very closely. First use the proof of Theorem 5.2 to check that Lemma 4.1 holds for S^1 with the obvious modifications (for example, in (4) "$n \geqslant 0$" becomes "$n \in \mathbb{Z}$"). Then simply recopy the relevant parts of the proofs of Theorems 1.1 and 1.2.

Most of the rest of the proof also goes as in Section 4. We only mention two points. First, for $\gamma \neq 0$ or 1 the S^1-analog of Lemma 4.2 shows that $\pi(I^2_{S^1}(\gamma))$ is the 1-dimensional space spanned by $P_2(\Gamma) - P_2(\gamma)$, just as for $\mathrm{Vec}(\mathbb{R})$. It follows that $I^2_{S^1}(\gamma)$ does not generate $I_{S^1}(\gamma)$.

Second, to prove that $I_{S^1}(0)$ is not generated by its lowest weight vectors we argue as in the proof of Theorem 1.2, but using the S^1-analog V_{S^1} of the module V used before. Here V_{S^1} is the indecomposable module with quotient $\mathbb{C}$ and submodule $\mathcal{F}_{S^1}(0)$ defined by Eq. (10). □

Remark. A general result describing those subsets S of $\mathfrak{U}(\mathrm{Vec}(\mathbb{R}))$ such that $\langle S\rangle_{\mathrm{Vec}(S^1)} \cap \mathfrak{U}(\mathrm{Vec}(\mathbb{R})) = \langle S\rangle_{\mathrm{Vec}(\mathbb{R})}$ would be interesting. Such a result would streamline the proofs in this section.

7. Questions

In the introduction we pointed out two topics for further study: the annihilators of the tensor density modules of $\mathrm{Vec}(\mathbb{R}^m)$ and the annihilators of the differential operator modules of $\mathrm{Vec}(\mathbb{R})$. In this section we will pose a few specific questions in these areas.

Tensor density modules of Vec($\mathbb{R}^m$). The tensor density modules $\mathcal{F}_{\mathrm{Vec}(\mathbb{R}^m)}(\gamma)$ of $\mathrm{Vec}(\mathbb{R}^m)$ can be defined geometrically for all non-negative real γ, or algebraically for all complex γ [LO99]. As in Section 3, replacing γ by an indeterminate Γ gives a "universal" tensor density module $\mathcal{F}_{\mathrm{Vec}(\mathbb{R}^m)}(\Gamma)$. The projective subalgebra of $\mathrm{Vec}(\mathbb{R}^m)$ is a copy of $\mathfrak{sl}_{m+1}$, and under this $\mathfrak{sl}_{m+1}$ the module $\mathcal{F}_{\mathrm{Vec}(\mathbb{R}^m)}(\Gamma)$ is the dual of what is known as a *universal Verma module* (in this case relative to the subalgebra $\mathfrak{gl}_m$).

Recall Theorem 1.3: the ideal $\mathrm{Ann}_{\mathrm{Vec}(\mathbb{R})} \mathcal{F}_{\mathbb{R}}(\Gamma)$ is generated by $\mathrm{ad}(e_2)Q$, and this is in various senses its "best" generator. Here Q is the Casimir operator, the generator of the center of $\mathfrak{U}(\mathfrak{sl}_2)$ as a polynomial algebra, and the coset of e_2 is the lowest weight vector of the $\mathfrak{sl}_2$-module $\mathrm{Vec}(\mathbb{R})/\mathfrak{sl}_2$.

In the case of $\mathrm{Vec}(\mathbb{R}^m)$, the center of $\mathfrak{U}(\mathfrak{sl}_{m+1})$ is a polynomial algebra generated by m elements $Q_2, Q_3, \ldots, Q_{m+1}$, where Q_i is of degree i. The action of Q_i on $\mathcal{F}_{\mathrm{Vec}(\mathbb{R}^m)}(\Gamma)$ is multiplication by a polynomial of degree i in Γ, the polynomial being given by the value of the Harish-Chandra homomorphism on Q_i. Since multiplication by Γ commutes with the action of $\mathrm{Vec}(\mathbb{R}^m)$, any element of the form $\mathrm{ad}(X)Q_i$ with $X \in \mathrm{Vec}(\mathbb{R}^m)$ and $2 \leqslant i \leqslant m+1$ is in $\mathrm{Ann}_{\mathrm{Vec}(\mathbb{R}^m)} \mathcal{F}_{\mathrm{Vec}(\mathbb{R}^m)}(\Gamma)$.

Note that $\mathrm{ad}(X)Q_i$ depends only on the image of X in $\mathrm{Vec}(\mathbb{R}^m)/\mathfrak{sl}_{m+1}$. It is known that as an $\mathfrak{sl}_{m+1}$-module, this quotient is generated by its lowest weight vectors. Therefore an affirmative answer to the following question would be a natural generalization of Theorem 1.3, as well as a useful tool for the study of the ideals $\mathrm{Ann}_{\mathrm{Vec}(\mathbb{R}^m)} \mathcal{F}_{\mathrm{Vec}(\mathbb{R}^m)}(\gamma)$.

Question 7.1. Is $\mathrm{Ann}_{\mathrm{Vec}(\mathbb{R}^m)} \mathcal{F}_{\mathrm{Vec}(\mathbb{R}^m)}(\Gamma)$ generated as a two-sided ideal by the finite collection of elements $\mathrm{ad}(X)Q_i$, where X runs over the lowest weight vectors of the $\mathfrak{sl}_{m+1}$-module $\mathrm{Vec}(\mathbb{R}^m)/\mathfrak{sl}_{m+1}$ and $2 \leqslant i \leqslant m+1$? If so, is this generating set minimal, or optimal in some other sense?

Annihilators of differential operator modules. Recall from Section 1 the $\mathrm{Vec}(\mathbb{R})$-module $\mathrm{Diff}(\gamma, p)$ of differential operators from $\mathcal{F}(\gamma)$ to $\mathcal{F}(\gamma + p)$. As we have mentioned, its order filtration is invariant and the associated subquotients are tensor density modules: one has the short exact sequence

$$0 \to \mathrm{Diff}^{k-1}(\gamma, p) \to \mathrm{Diff}^{k}(\gamma, p) \to \mathcal{F}(p-k) \to 0.$$

This sequence is rarely split under $\mathrm{Vec}(\mathbb{R})$, and the way in which $\mathrm{Diff}(\gamma, p)$ is built from its Jordan–Hölder composition series $\{\mathcal{F}(p-k)\colon k \in \mathbb{N}\}$ is non-trivial and highly interesting. It has been the subject of numerous articles, for example [FF80,CMZ97,LO99,Ga00,CS04], to name only a few.

To our knowledge, nothing is known about the annihilators of the modules $\mathrm{Diff}(\gamma, p)$. We would be interested in any results bearing on the following question.

Question 7.2. Are the ideals $\mathrm{Ann}_{\mathrm{Vec}(\mathbb{R})} \mathrm{Diff}(\gamma, p)$ non-trivial for any (γ, p)? If so, find optimal generating sets for them.

Let us discuss two possible approaches to this question. For the first, note that $\mathrm{Ann}_{\mathrm{Vec}(\mathbb{R})} \mathrm{Diff}(\gamma, p)$ is equal to the intersection $\bigcap_{k=0}^{\infty} \mathrm{Ann}_{\mathrm{Vec}(\mathbb{R})} \mathrm{Diff}^{k}(\gamma, p)$, so one could try to study it by analyzing the ideals $\mathrm{Ann}_{\mathrm{Vec}(\mathbb{R})} \mathrm{Diff}^{k}(\gamma, p)$. The Jordan–Hölder composition series of the module $\mathrm{Diff}^{k}(\gamma, p)$ is $\{\mathcal{F}(p-i)\colon 0 \leqslant i \leqslant k\}$, so it is in the category of *bounded modules*: modules with a finite composition series of tensor density modules.

Bounded modules have been studied in several papers, for example [FF80,MP92,BO98,Ge01, Co01,Co05]. They always have non-trivial annihilators: if B is a bounded module composed of $\mathcal{F}(\gamma_1), \ldots, \mathcal{F}(\gamma_k)$, it is easy to see that

$$\prod_{1}^{k} \mathrm{Ann}_{\mathrm{Vec}(\mathbb{R})} \mathcal{F}(\gamma_i) \subseteq \mathrm{Ann}_{\mathrm{Vec}(\mathbb{R})} B \subseteq \bigcap_{1}^{k} \mathrm{Ann}_{\mathrm{Vec}(\mathbb{R})} \mathcal{F}(\gamma_i).$$

If B is simply the direct sum $\bigoplus_1^k \mathcal{F}(\gamma_i)$, then its annihilator is all of $\bigcap_1^k \mathrm{Ann}_{\mathrm{Vec}(\mathbb{R})} \mathcal{F}(\gamma_i)$. The extent to which $\mathrm{Ann}_{\mathrm{Vec}(\mathbb{R})} B$ is smaller than this intersection is a measure of its indecomposability: roughly, "the smaller the annihilator, the more indecomposable the module." This suggests the following question, which is independent of Question 7.2.

Question 7.3. Consider the indecomposable bounded modules of $\mathrm{Vec}(\mathbb{R})$ of length 2 discovered in [FF80] and investigated further in [MP92] and [BO98], and more generally, the uniserial (i.e., completely indecomposable) bounded modules of higher length discovered in [Co01] and [Co05]. Find optimal generating sets for their annihilators. How much larger (respectively, smaller) are their annihilators than the product (respectively, intersection) of the annihilators of their composition series?

We remark that these modules are constructed from 1-cochains which are generalizations of the 1-cocycles defined by Eqs. (5) and (10). Therefore an analysis of the annihilators of the modules GF and V defined by those equations would constitute an important first step towards answering Question 7.3. In essence, the question is asking for the annihilators of the Gel'fand–Fuks cocycle and its relatives.

To describe the second approach to Question 7.2, recall the ideals $I(\gamma, p, m)$ in $\mathfrak{U}(\mathrm{Vec}(\mathbb{R}))$ from the introduction. The definition given there is equivalent to

$$I(\gamma, p, m) = \mathrm{Ann}_{\mathrm{Vec}(\mathbb{R})}\left(\bigoplus_{k=0}^{\infty} \mathrm{Diff}^k(\gamma, p)/\mathrm{Diff}^{k-m}(\gamma, p)\right).$$

Clearly $\mathrm{Ann}_{\mathrm{Vec}(\mathbb{R})}\,\mathrm{Diff}(\gamma, p)$ is equal to $\bigcap_{m=1}^{\infty} I(\gamma, p, m)$, so one can try to study it by analyzing the $I(\gamma, p, m)$. A simple Zariski density argument shows that $I(\gamma, p, 1)$ is nothing but the ideal $I = \bigcap_{\gamma} \mathrm{Ann}_{\mathrm{Vec}(\mathbb{R})}\,\mathcal{F}(\gamma)$, which was shown to be $\langle Q^{e_2}\rangle_{\mathrm{Vec}(\mathbb{R})}$ in Theorem 1.3. Combining this theorem with the results of [CMZ97] and [Ga00], one finds that the same is true for $I(\gamma, p, 2)$, but not for any $I(\gamma, p, m)$ with $m \geqslant 3$. Thus we come to the following question, which, in addition to being an important step towards answering Question 7.2, is interesting in its own right.

Question 7.4. Find homogeneous lowest weight generators for the ideals $I(\gamma, p, m)$. How do these ideals vary with γ and p?

We stated in the introduction that $I(\gamma, p, m)$ is non-trivial, as it contains all lowest weight vectors of $\mathfrak{U}(\mathrm{Vec}(\mathbb{R}))$ of weight $\geqslant m$, i.e., all of $\bigoplus_{n \geqslant m} \mathfrak{U}(\mathrm{Vec}(\mathbb{R}))_n^{e_{-1}}$. Let us briefly outline a proof of this fact. First one uses the fact that $\mathrm{Diff}(\gamma, p)$ is composed of $\mathcal{F}(p), \mathcal{F}(p-1), \ldots$ to construct a vector space isomorphism from $\mathrm{Diff}(\gamma, p)$ to $\bigoplus_{k=0}^{\infty} \mathcal{F}(p-k)$. Such an isomorphism is known as a *total symbol* or a *quantization* [CMZ97,LO99,Ga00,CS04]. It is used to carry the natural representation of $\mathrm{Vec}(\mathbb{R})$ on $\mathrm{Diff}(\gamma, p)$ over to an equivalent representation $\pi(\gamma, p)$ on $\bigoplus_{k=0}^{\infty} \mathcal{F}(p-k)$. This representation is then viewed as a block matrix with entries

$$\pi(\gamma, p)_{ij} : \mathrm{Vec}(\mathbb{R}) \to \mathrm{Hom}\big(\mathcal{F}(j), \mathcal{F}(i)\big), \quad i, j \in p - \mathbb{N}.$$

Since the filtration $\mathrm{Diff}^k(\gamma, p)$ is invariant, the quantization may be chosen so that the block matrix is lower triangular with diagonal entries $\pi(\gamma, p)_{ii} = \pi_i$, the usual representation on $\mathcal{F}(i)$. From this point of view, one finds that

$$I(\gamma, p, m) = \big\{\Omega \in \mathfrak{U}\big(\mathrm{Vec}(\mathbb{R})\big) \colon \pi(\gamma, p)_{ij}(\Omega) = 0 \text{ whenever } i - j < m\big\}.$$

Next one proves that the quantization may be chosen so that the entries $\pi(\gamma, p)_{ij}$ take values in $\mathrm{Diff}(j, i-j)$ and are covariant with respect to the affine subalgebra $\mathfrak{b} = \mathrm{Span}\{e_{-1}, e_0\}$. It follows that if Ω is in $\mathfrak{U}(\mathrm{Vec}(\mathbb{R}))_n^{e_{-1}}$, then $\pi(\gamma, p)_{ij}(\Omega)$ is in $\mathrm{Diff}(j, i-j)_n^{e_{-1}}$. But this space is zero whenever $i - j < m \leqslant n$, so in such cases Ω is in $I(\gamma, p, m)$.

Let us remark that in more detailed (but still inconclusive) examinations of the $I(\gamma, p, m)$, we have been led to study the associative algebra

$$\mathfrak{U}\big(\mathrm{Vec}(\mathbb{R})\big)^{e_{-1}}/e_{-1}\mathfrak{U}\big(\mathrm{Vec}(\mathbb{R})\big)^{e_{-1}}.$$

This algebra seems to be quite interesting. We can compute its dimension in each weight and degree (its weights are non-negative), but we need more precise information to analyze the $I(\gamma, p, m)$. For example, generators and relations would be helpful.

In conclusion, note that in some sense "the ideals $I(\gamma, p, m)$ are to I as the ideals $\mathrm{Ann}_{\mathrm{Vec}(\mathbb{R})}\,\mathrm{Diff}^k(\gamma, p)$ are to the annihilators of the tensor density modules." For example, one can make the following statement precise: "$I(\gamma, p, 3)$ consists of those elements of I that are annihilated by the Gel'fand–Fuks cocycle of the universal tensor density module $\mathcal{F}(\Gamma)$." We expect the second approach to Question 7.2 to be more successful than the first, because we feel that I, the annihilator of the universal tensor density module, is a more fundamental object than the annihilator of any specific tensor density module.

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We thank Olivier Mathieu for many helpful remarks, in particular for telling us his extension of Duflo's theorem (see Section 1) and Lemma A.4, and for explaining that generically one should expect $\operatorname{Ann}\mathcal{F}(\gamma)$ to be generated by its 0-weight space. The first author thanks the University of Bourgogne for its hospitality while the work was completed.

Appendix A. Modules of $\mathfrak{sl}_2\mathbb{C}$

In this section we collect some elementary results on $\mathfrak{sl}_2\mathbb{C}$-modules which we need during the body of the paper. We continue to use the realization of $\mathfrak{sl}_2\mathbb{C}$ as the subalgebra $\mathfrak{a} = \operatorname{Span}_{\mathbb{C}}\{e_{-1}, e_0, e_1\}$ of $\operatorname{Vec}(\mathbb{R})$. Here $e_i = x^{i+1}D$, so $[e_0, e_i] = ie_i$ and $[e_{-1}, e_1] = 2e_0$.

We recall some notation for the convenience of the reader. The Borel subalgebra $\mathfrak{b}$ and the Casimir operator Q of $\mathfrak{a}$ are

$$\mathfrak{b} = \operatorname{Span}_{\mathbb{C}}\{e_{-1}, e_0\}, \qquad Q = e_0^2 - e_0 - e_1e_{-1} = e_0^2 + e_0 - e_{-1}e_1.$$

Given any $\mathfrak{a}$-module V, write V_μ for the *μ-weight space* of V, the μ-eigenspace of the action of e_0. Write $V^{e_{-1}}$ and V^{e_1} for the kernels of the actions of e_{-1} and e_1, the *lowest weight vectors* and *highest weight vectors* of V, respectively.

The tensor density modules $\mathcal{F}(\gamma)$ and the actions π_γ of $\mathfrak{a}$ on them are

$$\mathcal{F}(\gamma) = \operatorname{Span}_{\mathbb{C}}\big\{dx^\gamma x^k\colon k \in \mathbb{N}\big\}, \qquad \pi_\gamma(e_i) = x^{i+1}D + \gamma(i+1)x^i$$

(recall that by $\mathbb{N}$ we mean all non-negative integers, including zero). It is easy to check that as $\mathfrak{a}$-modules, they have the following properties:

(1) For all γ, $\mathcal{F}(\gamma)$ has a unique (up to a scalar) lowest weight vector, dx^λ. The action of e_{-1} is surjective and Q acts by the scalar $\gamma^2 - \gamma$.
(2) For γ not in $-\mathbb{N}/2$, $\mathcal{F}(\gamma)$ is irreducible under $\mathfrak{a}$ and has no highest weight vectors.
(3) For γ in $-\mathbb{N}/2$, $\mathcal{F}(\gamma)$ is indecomposable of length 2 under $\mathfrak{a}$. It has a unique (up to a scalar) highest weight vector, $dx^\gamma x^{-2\gamma}$, the highest weight vector of the unique non-trivial proper submodule of $\mathcal{F}(\gamma)$. We will denote this submodule by $L(-\gamma)$; up to equivalence, it is the unique irreducible finite-dimensional module of $\mathfrak{a}$ of highest weight $-\gamma$. It is $(1-2\gamma)$-dimensional, with basis

$$L(-\gamma) = \operatorname{Span}_{\mathbb{C}}\big\{dx^\gamma, dx^\gamma x, \ldots, dx^\gamma x^{-2\gamma}\big\}.$$

The quotient $\mathcal{F}(\gamma)/L(-\gamma)$ is isomorphic to $\mathcal{F}(1-\gamma)$. The differential operator $dx^{1-2\gamma}D^{1-2\gamma}$ is an $\mathfrak{a}$-covariant surjection from $\mathcal{F}(\gamma)$ to $\mathcal{F}(1-\gamma)$ with kernel $L(-\gamma)$ (it is the well-known *Bol operator*; see for example [CMZ97,BO98], or [CS04]).

Our goal in this appendix is to classify $\mathfrak{a}$-modules which are "good" in the following sense. In conjunction with Lemmas A.6 and A.5, this will allow us to completely determine the $\mathfrak{a}$-structure of $\mathfrak{U}(\mathrm{Vec}(\mathbb{R}))$.

Definition. An $\mathfrak{a}$-module is said to be *admissible* if e_0 acts on it semisimply with finite-dimensional weight spaces. We will say that such a module is *good* if, in addition to being admissible, its weights are bounded below and e_{-1} acts on it surjectively.

We have noted that tensor density modules are good. It turns out that good modules have a simple classification: they are all direct sums of tensor density modules and one other class of modules, "extended tensor density modules." We now define these modules and state a lemma giving some of their properties. The lemma is elementary and we leave its proof to the reader.

Definition. For any γ in $1+\mathbb{N}/2$, the *extended tensor density module* $\mathcal{G}(\gamma)$ is the space $\mathcal{F}(1-\gamma)\oplus\mathcal{F}(\gamma)$ together with the linear map $\pi_{\mathcal{G}(\gamma)}$ from $\mathfrak{a}$ to $\mathrm{End}\,\mathcal{G}(\gamma)$ defined as follows. For e_i in $\mathfrak{b}$, i.e., $i=-1$ or 0, $\pi_{\mathcal{G}(\gamma)}(e_i)$ is simply $\pi_{1-\gamma}(e_i)\oplus\pi_\gamma(e_i)$. The endomorphism $\pi_{\mathcal{G}(\gamma)}(e_1)$ acts on the summand $\mathcal{F}(\gamma)$ as $\pi_\gamma(e_1)$ and on the summand $\mathcal{F}(1-\gamma)$ as $\pi_{1-\gamma}(e_1)+dx^{2\gamma-1}D^{2\gamma-2}$.

Lemma A.1. *$\pi_{\mathcal{G}(\gamma)}$ is a representation of $\mathfrak{a}$ on $\mathcal{G}(\gamma)=\mathcal{F}(1-\gamma)\oplus\mathcal{F}(\gamma)$. It is indecomposable: its restriction to the summand $\mathcal{F}(\gamma)$ is π_γ, so $\mathcal{F}(\gamma)$ is a submodule, and the quotient module $\mathcal{G}(\gamma)/\mathcal{F}(\gamma)$ is naturally equivalent to $\pi_{1-\gamma}$.*

$\mathcal{G}(\gamma)$ is good. It is $\mathfrak{b}$-equivalent to $\mathcal{F}(1-\gamma)\oplus\mathcal{F}(\gamma)$. It has a 2-dimensional space of lowest weight vectors: $\mathcal{G}(\gamma)^{e_{-1}}=\mathrm{Span}\{dx^{1-\gamma},dx^\gamma\}$. It has no highest weight vectors and no finite-dimensional submodules.

The Casimir operator acts with generalized eigenvalue $\gamma^2-\gamma$, such that $\pi_{\mathcal{G}(\gamma)}(Q)-\gamma^2+\gamma$ is 2-step nilpotent. More precisely, $\pi_{\mathcal{G}(\gamma)}(Q)$ acts as $\gamma^2-\gamma$ on the submodule $\mathcal{F}(\gamma)$ and as $\gamma^2-\gamma-dx^{2\gamma-1}D^{2\gamma-1}$ on the summand $\mathcal{F}(1-\gamma)$.

Remark. We thank the referee for pointing out that the extended tensor density modules are in fact the duals of indecomposable projective objects of the well-known category $\mathcal{O}(\mathfrak{sl}_2)$, introduced in [BGG76]. More precisely, $\mathcal{G}(\gamma)$ is the unique indecomposable injective object in the dual of $\mathcal{O}(\mathfrak{sl}_2)$ having $\mathcal{F}(\gamma)$ as a submodule.

The next proposition classifies the good modules, and its corollary shows that any good module is determined up to equivalence by its weight space dimensions and its finite-dimensional submodules. The corollary is immediate from the proposition and the fact that $\mathcal{G}(\gamma)$ is $\mathfrak{b}$-equivalent to $\mathcal{F}(1-\gamma)\oplus\mathcal{F}(\gamma)$, so we omit its proof.

Proposition A.2. *Any good $\mathfrak{a}$-module V is equivalent to a direct sum of countably many tensor density modules and extended tensor density modules*:

$$V \overset{\mathfrak{a}}{\cong} \Big(\bigoplus_{\gamma\in\mathbb{C}} m_\gamma\mathcal{F}(\gamma)\Big)\oplus\Big(\bigoplus_{\gamma\in 1+\mathbb{N}/2} n_\gamma\mathcal{G}(\gamma)\Big).$$

The multiplicities m_γ and n_γ are as follows:

(1) *For γ not in either $1+\mathbb{N}/2$ or $-\mathbb{N}/2$, $m_\gamma = \dim V_\gamma^{e_{-1}}$.*
(2) *For γ in $-\mathbb{N}/2$, $m_\gamma = \dim V_{-\gamma}^{e_1}$.*
(3) *For γ in $1+\mathbb{N}/2$, m_γ and n_γ are determined by*

$$m_\gamma + n_\gamma = \dim V_\gamma^{e_{-1}}, \qquad m_{1-\gamma} + n_\gamma = \dim V_{1-\gamma}^{e_{-1}}.$$

Proof. The weight spaces of V are finite-dimensional and Q preserves them, so V decomposes as the direct sum $\bigoplus_{q\in\mathbb{C}} \mathrm{GES}_q V$, where $\mathrm{GES}_q V$ denotes the generalized eigenspace of Q of eigenvalue q, an $\mathfrak{a}$-submodule of V. Recall that Q acts on $V_\gamma^{e_{-1}}$ by $P_2(\gamma) = \gamma^2 - \gamma$. It follows that $(\mathrm{GES}_{P_2(\gamma)} V)^{e_{-1}}$ is $(V_\gamma \oplus V_{1-\gamma})^{e_{-1}}$. Therefore it suffices to prove the proposition on each $\mathrm{GES}_q V$ separately.

Henceforth we assume that $V = \mathrm{GES}_q V$ for some q, and we fix a γ with $\mathrm{Re}(\gamma) \geqslant 1/2$ such that $P_2(\gamma) = q$. Since $e_{-1} : V_\mu \to V_{\mu-1}$ is surjective for all μ and $V^{e_{-1}}$ is in $V_\gamma \oplus V_{1-\gamma}$, the map $e_{-1} : V_\mu \to V_{\mu-1}$ is bijective for $\mu \neq \gamma$ or $1-\gamma$. Conversely, the alternate expression $Q = e_0^2 + e_0 - e_{-1}e_1$ implies that $e_{-1}e_1$ acts on $V_{\mu-1}$ with generalized eigenvalue $P_2(\mu) - P_2(\gamma)$, so $e_1 : V_{\mu-1} \to V_\mu$ is bijective for $\mu \neq \gamma$ or $1-\gamma$.

We claim that $V_\mu = 0$ unless μ is in $\gamma + \mathbb{N}$ or $1-\gamma+\mathbb{N}$. This is because V's weights are bounded below, so if $V_\mu \neq 0$ then some power of e_{-1} maps it to $V^{e_{-1}}$.

The easy case is when either $\gamma+\mathbb{N}$ and $1-\gamma+\mathbb{N}$ have empty intersection or $\gamma = 1/2$. Here $V_\gamma^{e_{-1}}$ is necessarily all of V_γ. Let $m_\gamma = \dim V_\gamma$, let $u_1, \ldots, u_{m_\gamma}$ be a basis of V_γ, and let $U^i = \mathbb{C}[e_1]u_i$. Use the fact that Q acts by $P_2(\gamma)$ on U^i to prove that it is a submodule equivalent to $\mathcal{F}(\gamma)$. Then use the bijectivity properties of e_1 to prove that the U^i are independent and $\bigoplus_i U^i = \bigoplus_{\mathbb{N}} V_{\gamma+n}$. Construct the $\mathcal{F}(1-\gamma)$'s similarly.

The other case is when $\gamma \in 1+\mathbb{N}/2$. Here $V_{1-\gamma}^{e_{-1}}$ is all of $V_{1-\gamma}$, but $V_\gamma^{e_{-1}}$ might not be all of V_γ. We pick out the $\mathcal{F}(1-\gamma)$'s first. Fix $W_\gamma \subseteq V_\gamma$ such that $e_{-1} : W_\gamma \to V_{\gamma-1}^{e_1}$ is bijective. Let $m_{1-\gamma} = \dim W_\gamma$, let $w_1, \ldots, w_{m_{1-\gamma}}$ be a basis of W_γ, and let $W^i = \mathfrak{U}(\mathfrak{a})w_i$. Check that Q acts by $P_2(\gamma)$ on W^i and then that W^i is a submodule equivalent to $\mathcal{F}(1-\gamma)$.

Next we pick the $\mathcal{G}(\gamma)$'s. Fix $U_{\gamma-1} \subseteq V_{\gamma-1}$ complementary to $V_{\gamma-1}^{e_1}$ and let $n_\gamma = \dim U_{\gamma-1}$. Then n_γ is as in the proposition because $\dim V_{\gamma-1} = \dim V_{1-\gamma}$ by the bijectivity properties of $e_{\pm 1}$.

Fix $U_\gamma \subseteq V_\gamma$ such that $e_{-1} : U_\gamma \to U_{\gamma-1}$ is a bijection, let $u_1, \ldots, u_{n_\gamma}$ be a basis of U_γ, and define U^i to be $\mathfrak{U}(\mathfrak{a})u_i$. We claim that U^i is a submodule equivalent to $\mathcal{G}(\gamma)$. To verify this, first note that $e_1 : U_{\gamma-1} \to e_1(V_{\gamma-1})$ is a bijection. The crucial point is that $e_1(V_{\gamma-1})$ is in $V_\gamma^{e_{-1}}$. This is because Q acts as $P_2(\gamma)$ on $V_{1-\gamma}$ and hence on all V_μ with $\mu < \gamma$, so $e_{-1}e_1$ kills $V_{\gamma-1}$. We leave the rest to the reader.

We pick the $\mathcal{F}(\gamma)$'s last. Let $X_\gamma \subseteq V_\gamma^{e_{-1}}$ be any complement of $e_1(V_{\gamma-1})$ and define $m_\gamma = \dim X_\gamma$. Check that m_γ is as in the proposition, let $x_1, \ldots, x_{m_\gamma}$ be a basis of X_γ, and define $X^i = \mathbb{C}[e_1]x_i$. Then X^i is a submodule equivalent to $\mathcal{F}(\gamma)$.

Finally, use $V_\gamma = W_\gamma \oplus U_\gamma \oplus V_\gamma^{e_{-1}}$ and $V_\gamma^{e_{-1}} = X_\gamma \oplus e_1(V_{\gamma-1})$ together with the bijectivity properties of $e_{\pm 1}$ to prove that V is the direct sum of all of the W^i, U^i, and X^i. □

Corollary A.3. *Any good $\mathfrak{a}$-module V is $\mathfrak{b}$-equivalent to a direct sum $\bigoplus_{\gamma\in\mathbb{C}} b_\gamma \mathcal{F}(\gamma)$ of tensor density modules. The multiplicities b_γ are determined by the weight space dimensions of V: $b_\gamma = \dim V_\gamma^{e_{-1}} = \dim V_\gamma - \dim V_{\gamma-1}$.*

By Proposition A.2 *we know that V is $\mathfrak{a}$-equivalent to $\bigoplus_{\gamma\in\mathbb{C}}(m_\gamma\mathcal{F}(\gamma)\oplus n_\gamma\mathcal{G}(\gamma))$ for some multiplicities m_γ and n_γ (where we define n_γ to be zero for γ not in $1+\mathbb{N}/2$). The b_γ are related to these multiplicities by $b_\gamma = m_\gamma + n_\gamma + n_{1-\gamma}$ for all γ.*

Let c_γ be the multiplicity of the finite-dimensional irreducible module $L(\gamma)$ in V. Then $c_\gamma = \dim V_\gamma^{e_1}$, and the b_γ and c_γ together determine the m_γ and n_γ as follows. For γ not in either $1+\mathbb{N}/2$ or $-\mathbb{N}/2$, $m_\gamma = b_\gamma$. For γ in $-\mathbb{N}/2$, $m_\gamma = c_{-\gamma}$. For γ in $1+\mathbb{N}/2$, $n_\gamma = b_{1-\gamma} - c_{\gamma-1}$ and $m_\gamma = b_\gamma - n_\gamma = b_\gamma - b_{1-\gamma} + c_{\gamma-1}$.

We conclude this appendix with Corollary A.5 and Lemma A.6, which, in light of Corollary A.3, completely determine the $\mathfrak{a}$-structure of $\mathcal{S}^k\mathcal{F}(\gamma)$. First we prove Lemma A.4, which we learned from O. Mathieu.

Given any module V of any Lie algebra $\mathfrak{g}$, let V_f be the *locally finite part* of V: the sum of all finite-dimensional submodules of V. Let $\mathfrak{g}_f$ be the locally finite part of $\mathfrak{g}$ under the adjoint action, an ideal in $\mathfrak{g}$.

Lemma A.4. $(V\otimes W)_f = V_f\otimes W_f$ *for any complex modules V and W of any complex Lie algebra $\mathfrak{g}$. In particular, $(\bigotimes V)_f = \bigotimes(V_f)$, $(\mathcal{S}V)_f = \mathcal{S}(V_f)$, $(\Lambda V_f) = \Lambda(V_f)$, and $(\mathfrak{U}\mathfrak{g})_f = \mathfrak{U}(\mathfrak{g}_f)$.*

Proof. It is enough to prove the first sentence, for which it suffices to prove $(V\otimes W)_f \subseteq V\otimes(W_f)$. Write $\mathfrak{U}^n$ for $\mathfrak{U}^n(\mathfrak{g})$. For any $t\in V\otimes W$ not in $V\otimes(W_f)$, we must prove that $\mathfrak{U}^n t\neq \mathfrak{U}^{n-1}t$ for all n. Express t as $\sum_{i=1}^k v_i\otimes w_i$, where the v_i are linearly independent. Let $W' = \mathrm{Span}\{w_1,\dots,w_n\}$. Note that by assumption $W'\not\subseteq W_f$, so $\mathfrak{U}^n W'\neq \mathfrak{U}^{n-1}W'$ for all n.

For all $\Omega\in\mathfrak{U}^n$, we have $\Omega t\equiv \sum_i v_i\otimes \Omega w_i$ modulo $V\otimes \mathfrak{U}^{n-1}W'$, which implies that $\mathfrak{U}^n t\not\subseteq V\otimes\mathfrak{U}^{n-1}W'$. But $\mathfrak{U}^{n-1}t\subseteq V\otimes \mathfrak{U}^{n-1}W'$. □

Corollary A.5. *For $k\geqslant 1$ and γ not in $-\mathbb{N}/2$, $\mathcal{S}^k\mathcal{F}(\gamma)$ has no finite-dimensional $\mathfrak{a}$-submodules. For γ in $-\mathbb{N}/2$, the sum of all of its finite-dimensional $\mathfrak{a}$-submodules is $\mathcal{S}^kL(-\gamma)$.*

Proof. Apply Lemma A.4 with $\mathfrak{g}=\mathfrak{a}$ and $V=\mathcal{F}(\gamma)$. □

Lemma A.6. *For $k\geqslant 1$, the kth symmetric power $\mathcal{S}^k\mathcal{F}(\gamma)$ is a good $\mathfrak{a}$-module. For $k\geqslant 2$, its $\mathfrak{b}$-decomposition is*

$$\mathcal{S}^k\mathcal{F}(\gamma)\overset{\mathfrak{b}}{\cong}\bigoplus_{i_2,i_3,\dots,i_k=0}^{\infty}\mathcal{F}(2i_2+3i_3+\cdots+ki_k+k\gamma).$$

Proof. To prove that $\mathcal{S}^k\mathcal{F}(\gamma)$ is good it will suffice to prove that $\pi_\gamma(e_{-1})=D$ acts on it surjectively. We may assume by induction on k that $\mathcal{S}^{k-1}\mathcal{F}(\gamma)$ is good. By definition, D acts as a derivation of the symmetric algebra structure, so the induction assumption together with $D(dx^\gamma)=0$ shows that D maps the subspace $dx^\gamma\cdot\mathcal{S}^{k-1}\mathcal{F}(\gamma)$ of $\mathcal{S}^k\mathcal{F}(\gamma)$ surjectively to itself. Another easy induction argument, this time on j, shows that D maps $(\bigoplus_{i=0}^j\mathbb{C}\,dx^\gamma x^i)\cdot\mathcal{S}^{k-1}\mathcal{F}(\gamma)$ surjectively to itself for all j. Thus $\mathcal{S}^k\mathcal{F}(\gamma)$ is good.

By Corollary A.3, the $\mathfrak{b}$-equivalence will follow if we prove that the weight space dimensions of the two sides match. For this, note that

$$\bigl\{\bigl(dx^\gamma x^{r_1}\bigr)\cdots\bigl(dx^\gamma x^{r_k}\bigr)\colon r_1\leqslant\cdots\leqslant r_k\bigr\}$$

is a basis of $\mathcal{S}^k\mathcal{F}(\gamma)$ consisting of monomial weight vectors. At $k=2$ it is easy to see that this basis has the same weight multiplicities as $\bigoplus_i \mathcal{F}(2i+2\gamma)$, so we may induct on k.

For arbitrary k, the weight space dimensions of the span of those monomials with $r_1=0$ are the same as the weight space dimensions of $\mathcal{S}^{k-1}\mathcal{F}(\gamma)$, but all shifted by γ because of the leading factor dx^γ. Similarly, the weight space dimensions of the span of those monomials with r_1 fixed at some value r are the same as those of $\mathcal{S}^{k-1}\mathcal{F}(\gamma)$, but all shifted by $\gamma+kr$ because of the leading factor $dx^\gamma x^r$ and the fact that $r_1 \leqslant \cdots \leqslant r_k$. The result for k now follows from the result for $k-1$ by summing over all $r_1 \geqslant 0$. □

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