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Exact formulas for traces of singular moduli of higher level modular functions

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Abstract

Zagier proved that the traces of singular values of the classical j-invariant are the Fourier coefficients of a weight $3/2$ modular form and Duke provided a new proof of the result by establishing an exact formula for the traces using Niebur's work on a certain class of non-holomorphic modular forms. In this short note, by utilizing Niebur's work again, we generalize Duke's result to exact formulas for traces of singular moduli of higher level modular functions.

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1. Introduction and statement of result

The classical j-invariant is defined for z in the complex upper half plane $\mathbb{H}$ by

$$j(z) = q^{-1} + 744 + 196884q + \cdots,$$

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where $q = e(z) = e^{2\pi i z}$ and $J(z) = j(z) - 744$ is the normalized Hauptmodul for the group $\Gamma(1) = PSL_2(\mathbb{Z})$. All the modular groups discussed in this paper are subgroups of $\Gamma(1)$. For a positive integer D congruent to 0 or 3 modulo 4, we denote by $\mathcal{Q}_D$ the set of positive definite integral binary quadratic forms

$$Q(x, y) = [a, b, c] = ax^2 + bxy + cy^2$$

with discriminant $-D = b^2 - 4ac$. The group $\Gamma(1)$ acts on $\mathcal{Q}_D$ by $Q \circ \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) = Q(\alpha x + \beta y, \gamma x + \delta y)$. For each $Q \in \mathcal{Q}_D$, we let

$$z_Q = \frac{-b + i\sqrt{D}}{2a},$$

the corresponding CM point in $\mathbb{H}$ and we write $\Gamma(1)_Q$ for the stabilizer of Q in $\Gamma(1)$. The trace of a singular modulus of discriminant $-D$ is defined as

$$\mathbf{t}_J(D) = \sum_{Q \in \mathcal{Q}_D/\Gamma(1)} \frac{1}{|\Gamma(1)_Q|} J(z_Q).$$

In [9, Theorem 1], Zagier proved that the generating series for the traces of singular moduli

$$g(z) := q^{-1} - 2 - \sum_{\substack{D>0 \\ D \equiv 0,3 \ (\mathrm{mod}\ 4)}} \mathbf{t}_J(D) q^D = q^{-1} - 2 + 248q^3 - 492q^4 + \cdots$$

is a weakly holomorphic modular form (that is, meromorphic with poles only at the cusps) of weight $3/2$ on $\Gamma_0(4)$. Recently, Bruinier, Jenkins, and Ono [2] obtained an explicit formula for the Fourier coefficients of $g(z)$ in terms of Kloosterman sums and Duke [4] derived an exact formula for $\mathbf{t}_J(D)$ as follows:

$$\mathbf{t}_J(D) = -24H(D) + \sum_{\substack{c>0 \\ c \equiv 0 \ (\mathrm{mod}\ 4)}} S_D(c) \sinh\left(\frac{4\pi\sqrt{D}}{c}\right), \tag{1}$$

where

$$S_D(c) = \sum_{x^2 \equiv -D \ (\mathrm{mod}\ c)} e(2x/c) \quad \text{and} \quad H(D) = \sum_{Q \in \mathcal{Q}_D/\Gamma(1)} \frac{1}{|\Gamma(1)_Q|}$$

is the Hurwitz class number. Using these two results together, Duke [4] reestablished Zagier's trace formula [9, Theorem 1].

The purpose of this paper is to give a generalization of (1) to traces of singular values of modular functions of any prime level p. For prime p, let $\Gamma_0^*(p)$ be the group generated by $\Gamma_0(p)$ and the Fricke involution $W_p = \frac{1}{\sqrt{p}}\left(\begin{smallmatrix} 0 & -1 \\ p & 0 \end{smallmatrix}\right)$. Let $\mathcal{Q}_{D,p}$ denote the set of quadratic forms $Q \in \mathcal{Q}_D$ such that $a \equiv 0 \pmod{p}$. The group $\Gamma_0^*(p)$ acts on $\mathcal{Q}_{D,p}$, where the action for elements of $\Gamma_0(p)$ is defined as above and $Q \circ W_p = [pc, -b, a/p]$. Note that the discriminant $-D$ is congruent to a square modulo $4p$. We choose an integer $\beta \pmod{2p}$ with $\beta^2 \equiv -D \pmod{4p}$ and consider the set $\mathcal{Q}_{D,p,\beta} = \{[a,b,c] \in \mathcal{Q}_{D,p} \mid b \equiv \beta \pmod{2p}\}$ on which $\Gamma_0(p)$ acts. For a modular function f for $\Gamma_0^*(p)$, we define the class number $H_p(D)$ (resp. $H_p^*(D)$) and the trace $\mathbf{t}_f(D)$ (resp. $\mathbf{t}_f^*(D)$) by

$$H_p(D) = \sum_{Q \in \mathcal{Q}_{D,p,\beta}/\Gamma_0(p)} \frac{1}{|\Gamma_0(p)_Q|}; \qquad \mathbf{t}_f(D) = \sum_{Q \in \mathcal{Q}_{D,p,\beta}/\Gamma_0(p)} \frac{1}{|\Gamma_0(p)_Q|} f(z_Q),$$

$$H_p^*(D) = \sum_{Q \in \mathcal{Q}_{D,p}/\Gamma_0^*(p)} \frac{1}{|\Gamma_0^*(p)_Q|}; \qquad \mathbf{t}_f^*(D) = \sum_{Q \in \mathcal{Q}_{D,p}/\Gamma_0^*(p)} \frac{1}{|\Gamma_0^*(p)_Q|} f(z_Q).$$

Here $\Gamma_0(p)_Q$ and $\Gamma_0^*(p)_Q$ are the stabilizers of Q in $\Gamma_0(p)$ and $\Gamma_0^*(p)$, respectively. It is easy to see that

$$H_p^*(D) = \begin{cases} \frac{1}{2} H_p(D), & \text{if } \beta \equiv 0 \text{ or } p \pmod{2p}, \\ H_p(D), & \text{otherwise;} \end{cases} \tag{2}$$

and

$$\mathbf{t}_f^*(D) = \begin{cases} \frac{1}{2} \mathbf{t}_f(D), & \text{if } \beta \equiv 0 \text{ or } p \pmod{2p}, \\ \mathbf{t}_f(D), & \text{otherwise.} \end{cases} \tag{3}$$

The modularity for the traces $\mathbf{t}_f(D)$ was established by one of the authors in [6,7] in the case when $\Gamma_0^*(p)$ is of genus zero. If f is the Hauptmodul for such $\Gamma_0^*(p)$ and if we define $\mathbf{t}_f(-1) = -1$, $\mathbf{t}_f(0) = 2$ and $\mathbf{t}_f(D) = 0$ for $D < -1$, then the series $\sum_{n,r} \mathbf{t}_f(4pn - r^2) q^n \zeta^r$, where $\zeta = e(w)$ for a complex number w, is a weak Jacobi form of weight 2 and index p. Meanwhile, using the theta correspondence, Bruinier and Funke [1] generalized Zagier's trace formula to traces of CM values of modular functions of arbitrary level. In particular, they showed that if p is an odd prime and $f = \sum a(n) q^n$ is a modular function for $\Gamma_0^*(p)$ with $a(0) = 0$, then

$$\sum_{\substack{D>0 \\ -D \equiv \square \pmod{4p}}} \mathbf{t}_f^*(D) q^D + \sum_{n \geqslant 1} \big(\sigma(n) + p\sigma(n/p)\big) a(-n) - \sum_{m \geqslant 1} \sum_{n \geqslant 1} m a(-mn) q^{-m^2} \tag{4}$$

is a weakly holomorphic modular form of weight $3/2$ and level $4p$.

Remark. In the forthcoming paper [3], the authors generalize this result on modularity of traces by Bruinier and Funke to any weakly holomorphic modular functions with arbitrary level, including a composite level. We establish that the generating function for traces of singular moduli of a weakly holomorphic modular function, whether its constant term is zero or not, plus certain linear combination of class numbers is a weakly holomorphic modular form of weight $3/2$.

We will obtain in the next section, the following exact formula for $\mathbf{t}_f^*(D)$ which is a generalization of (1).

Theorem 1. *Suppose f is a modular function for $\Gamma_0^*(p)$ with principal part $\sum_{m=1}^{N} a_m e(-mz)$ at $i\infty$, and define for any positive integers m and c,*

$$S_D(m,c) = \sum_{x^2 \equiv -D \pmod{c}} e(2mx/c).$$

Then

$$\mathbf{t}_f^*(D) = \sum_{m=1}^{N} a_m \left[c_m H_p^*(D) + \sum_{\substack{c>0 \\ c\equiv 0 \pmod{4p}}} S_D(m,c) \sinh\left(\frac{4\pi m\sqrt{D}}{c}\right)\right],$$

where

$$c_m = -24\left(\frac{-p^{\alpha+1}}{p+1}\sigma\left(m/p^{\alpha}\right) + \sigma(m)\right) \quad \text{with } p^{\alpha} \| m.$$

As an example, consider

$$f = \left(\frac{\eta(z)}{\eta(37z)}\right)^2 - 2 + 37\left(\frac{\eta(37z)}{\eta(z)}\right)^2,$$

where $\eta(z)$ is the Dedekind eta function defined by $\eta(z) = q^{\frac{1}{24}} \prod_{n=1}^{\infty}(1-q^n)$. Then f is a modular function for $\Gamma_0^*(37)$ which is of genus 1 and has a Fourier expansion of the form $q^{-3} - 2q^{-2} - q^{-1} + 0 + O(q)$. Since the representatives for $\mathcal{Q}_{148,37,0}/\Gamma_0(37)$ are given by $[37,0,1]$ and $[74,-74,19]$, we find from equations (2), (3), and Theorem 1 that

$$24\cdot\frac{3}{38} + \sum_{\substack{c>0 \\ c\equiv 0 \pmod{148}}} \left[S_D(3,c)\sinh\left(\frac{12\pi\sqrt{D}}{c}\right) - 2S_D(2,c)\sinh\left(\frac{8\pi\sqrt{D}}{c}\right) - S_D(1,c)\sinh\left(\frac{4\pi\sqrt{D}}{c}\right)\right] = \frac{1}{2}\left(f\left(\frac{\sqrt{37}i}{37}\right) + f\left(\frac{37+\sqrt{37}i}{74}\right)\right),$$

where the latter is known to be -2.

2. Proof of Theorem 1

Throughout this section, Γ denotes $\Gamma_0^*(p)$. For a positive integer m we consider Niebur's Poincaré series [8]

$$\mathcal{F}_m(z,s) = \sum_{M\in\Gamma_\infty\backslash\Gamma} e(-m \operatorname{Re} Mz)(\operatorname{Im} Mz)^{1/2} I_{s-1/2}(2\pi m \operatorname{Im} Mz), \tag{5}$$

where $I_{s-1/2}$ is the modified Bessel function of the first kind. Then $\mathcal{F}_m(z,s)$ converges absolutely for $\operatorname{Re} s > 1$ and satisfies

$$\mathcal{F}_m(Mz,s) = \mathcal{F}_m(z,s) \quad \text{for } M\in\Gamma \quad \text{and} \quad \triangle\mathcal{F}_m(z,s) = s(1-s)\mathcal{F}_m(z,s), \tag{6}$$

where $\triangle$ is the hyperbolic Laplacian $\triangle = -y^2(\partial_x^2 + \partial_y^2)$ for $z = x + iy$. Niebur showed that $\mathcal{F}_m(z,s)$ has an analytic continuation to $s = 1$ [8, Theorem 5] and that $\mathcal{F}_m(z,s)$ has the following Fourier expansion [8, Theorem 1]; for $\operatorname{Re} s > 1$,

$$\mathcal{F}_m(z,s) = e(-mx)y^{1/2}I_{s-1/2}(2\pi my) + \sum_{n=-\infty}^{\infty} b_n(y,s;-m)e(nx), \tag{7}$$

where $b_n(y,s;-m) \to 0$ $(n \neq 0)$ exponentially as $y \to i\infty$. Hence the pole of $\mathcal{F}_m(z,1)$ at $i\infty$ may occur only in $e(-mx)y^{1/2}I_{1/2}(2\pi my)$, which is equal to

$$\frac{1}{\pi y^{1/2}m^{1/2}} \sinh(2\pi my)y^{1/2}e(-mx) = \frac{1}{2\pi m^{1/2}}\big(e(-mz) - e(-m\bar{z})\big). \tag{8}$$

We normalize $\mathcal{F}_m(z,1)$ by multiplying with $2\pi m^{1/2}$, so that the coefficient of $e(-mz)$ is normalized. Now we need to compute the constant term in $(2\pi m^{1/2})\mathcal{F}_m(z,1)$.

Lemma 2. *Let $\mathcal{F}_m(z,s)$ be the Poincaré series defined in* (5). *Then the constant term in $(2\pi m^{1/2})\mathcal{F}_m(z,1)$ is given by*

$$24\left(\frac{-p^{\alpha+1}}{1+p}\sigma\big(m/p^{\alpha}\big) + \sigma(m)\right) =: -c_m. \tag{9}$$

Proof. It follows from [8, Theorem 1] that $b_0(y,s,-m) = a_m(s)y^{1-s}/(2s-1)$. Here

$$a_m(s) = 2\pi^s m^{s-1/2}\phi_m(s)/\Gamma(s) \quad \text{and} \quad \phi_m(s) = \sum_{c>0} S(m,0;c)c^{-2s}, \tag{10}$$

where $S(m,n;c)$ is the general Kloosterman sum $\sum_{0\leqslant d<|c|} e((ma+nd)/c)$ for $\left(\begin{smallmatrix} a & * \\ c & d\end{smallmatrix}\right) \in \Gamma$. Note that if $M = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in \Gamma = \Gamma_0^*(p)$, then $M \in \Gamma_0(p)$ or M is of the form $\left(\begin{smallmatrix} \sqrt{p}x & y/\sqrt{p} \\ \sqrt{p}z & \sqrt{p}w\end{smallmatrix}\right)$ with $x, y, z, w \in \mathbb{Z}$. In the former case, c is a multiple of p and in the latter case, $c = \sqrt{p}z$ with $p \nmid z$. For $n \in \mathbb{Z}^+$, let $u_m(n)$ denote the sum of mth powers of primitive nth roots of unity. We observe that

$$S(m,0;c) = \begin{cases} u_m(c), & \text{if } p \mid c, \\ u_m(z), & \text{if } c = \sqrt{p}z \text{ with } p \nmid z. \end{cases}$$

If we define

$$u_m^*(n) = \begin{cases} u_m(n), & \text{if } p \mid n, \\ p^{-s}u_m(n), & \text{if } p \nmid n, \end{cases}$$

then

$$\begin{aligned} p^s\phi_m(s)\zeta(2s) &= p^s \sum_{c>0} S(m,0;c)c^{-2s} \sum_{c' \in \mathbb{Z}^+} c'^{-2s} \\ &= \sum_{c\in\mathbb{Z}^+} \big(p^s u_m^*(c)\big)c^{-2s} \sum_{c'\in\mathbb{Z}^+} c'^{-2s} = \sum_{k\in\mathbb{Z}^+}\left(\sum_{c\mid k} p^s u_m^*(c)\right)k^{-2s}. \end{aligned} \tag{11}$$

Note that if $p \nmid k$, then

$$\sum_{c|k} p^s u_m^*(c) = \sum_{c|k} u_m(c) = \begin{cases} k, & \text{if } k \mid m, \\ 0, & \text{if } k \nmid m \end{cases} \tag{12}$$

and if $k = p^l k'$ with $l \geqslant 1$ and $p \nmid k'$, then

$$\sum_{c|k} p^s u_m^*(c) = \sum_{d|k'} p^s u_m^*(d) + \sum_{\substack{c|k \\ p|c}} p^s u_m^*(c) = \sum_{d|k'} u_m(d) + \sum_{\substack{c|k \\ p|c}} p^s u_m(c). \tag{13}$$

By adding $(p^s - 1)\sum_{d|k'} u_m(d)$ on both sides of (13), we obtain

$$\left(p^s - 1\right)\sum_{d|k'} u_m(d) + \sum_{c|k} p^s u_m^*(c) = \sum_{c|k} p^s u_m(c).$$

Since

$$\sum_{d|k'} u_m(d) = \begin{cases} k', & \text{if } k' \mid m, \\ 0, & \text{if } k' \nmid m \end{cases} \quad \text{and} \quad \sum_{c|k} p^s u_m(c) = \begin{cases} p^s k, & \text{if } k \mid m, \\ 0, & \text{if } k \nmid m, \end{cases}$$

we find that

$$\sum_{c|k} p^s u_m^*(c) = \begin{cases} p^s k + (1 - p^s)k', & \text{if } k \mid m, \\ (1 - p^s)k', & \text{if } k \nmid m \text{ and } k' \mid m, \\ 0, & \text{if } k \nmid m \text{ and } k' \nmid m. \end{cases} \tag{14}$$

Writing $m = p^\alpha m'$ with $p \nmid m'$, we can deduce from (12) and (14) that

$$\begin{aligned} \sum_{k \in \mathbb{Z}_+} \left(\sum_{c|k} p^s u_m^*(c)\right) k^{-2s} &= \sum_{k'|m'} k' k'^{-2s} + \sum_{l=1}^{\infty} \sum_{k'|m'} \left(1 - p^s\right) k' \left(p^l k'\right)^{-2s} \\ &\quad + \sum_{l=1}^{\alpha} \sum_{k'|m'} p^s \left(p^l k'\right)\left(p^l k'\right)^{-2s} \\ &= \sigma_{1-2s}(m') + \left(1 - p^s\right)\sigma_{1-2s}(m') \sum_{l=1}^{\infty} \left(p^{-2s}\right)^l + p^s \sum_{1 \leqslant l \leqslant \alpha} \sum_{k'|m'} \left(p^l k'\right)^{1-2s} \\ &= \sigma_{1-2s}(m') \left[1 + \left(1 - p^s\right) \frac{p^{-2s}}{1 - p^{-2s}}\right] + p^s \left(\sigma_{1-2s}(m) - \sigma_{1-2s}(m')\right) \\ &= \frac{-p^{2s}}{1 + p^s} \sigma_{1-2s}\left(m/p^\alpha\right) + p^s \sigma_{1-2s}(m). \end{aligned} \tag{15}$$

Recall that the constant term in $(2\pi m^{1/2})\mathcal{F}_m(z, 1)$ is

$$\lim_{s \to 1} 2\pi m^{1/2} b_0(y, s, -m) = \lim_{s \to 1} 2\pi m^{1/2} a_m(s) y^{1-s}/(2s - 1).$$

By the definition of $a_m(s)$ in (10), it is equal to

$$\lim_{s\to 1} 2\pi m^{1/2}\big(2\pi^s m^{s-1/2}\phi_m(s)/\Gamma(s)\big)y^{1-s}/(2s-1).$$

It follows from (11) and (15) that this limit goes to

$$\frac{4\pi^2 m}{p\zeta(2)}\left(\frac{-p^2}{1+p}\sigma_{-1}\big(m/p^\alpha\big)+p\sigma_{-1}(m)\right).$$

Thus simple calculations lead us to have the constant term of $(2\pi m^{1/2})\mathcal{F}_m(z,1)$ in (9). □

Now we define

$$\mathcal{F}_m^*(z,s)=\big(2\pi m^{1/2}\big)\mathcal{F}_m(z,s)+c_m.$$

Then by (6), (7), (8), and Lemma 2, $\mathcal{F}_m^*(z,1)$ is a Γ-invariant harmonic function and $\mathcal{F}_m^*(z,1)-e(-mz)$ has a zero at $i\infty$. Hence it follows from [8, Theorem 6] that

$$f(z)=\sum_{m=1}^N a_m\mathcal{F}_m^*(z,1)$$

for any modular function f for $\Gamma_0^*(p)$ with principal part $\sum_{m=1}^N a_m e(-mz)$ at $i\infty$. Hence

$$\mathbf{t}_f^*(D)=\sum_{m=1}^N a_m\left(\sum_{Q\in\mathcal{Q}_{D,p}/\Gamma}\frac{1}{|\Gamma_Q|}\mathcal{F}_m^*(z_Q,1)\right). \tag{16}$$

In order to complete the proof of Theorem 1, it suffices to determine the value $\sum_{Q\in\mathcal{Q}_{D,p}/\Gamma}\frac{1}{|\Gamma_Q|}\times\mathcal{F}_m^*(z_Q,1)$.

Lemma 3. *Let $\mathcal{F}_m^*(z,s)=(2\pi m^{1/2})\mathcal{F}_m(z,s)+c_m$, where $\mathcal{F}_m(z,s)$ and c_m are defined in* (5) *and* (9)*, respectively. Then the trace of CM values of $\mathcal{F}_m^*$ is given by*

$$\sum_{Q\in\mathcal{Q}_{D,p}/\Gamma}\frac{1}{|\Gamma_Q|}\mathcal{F}_m^*(z_Q,1)=c_m H_p^*(D)+\sum_{\substack{c>0\\ c\equiv 0 \pmod{4p}}} S_D(m,c)\sinh\left(\frac{4\pi m\sqrt{D}}{c}\right).$$

Proof. We first compute for $\operatorname{Re} s>1$,

$$\sum_{Q\in\mathcal{Q}_{D,p}/\Gamma}\frac{1}{|\Gamma_Q|}\mathcal{F}_m^*(z_Q,s)=c_m H_p^*(D)+2\pi\sqrt{m}\sum_{Q\in\mathcal{Q}_{D,p}/\Gamma}\frac{1}{|\Gamma_Q|}\mathcal{F}_m(z_Q,s). \tag{17}$$

By the Poincaré series expansion of $\mathcal{F}_m(z_Q,s)$ in (5),

$$\sum_{Q\in\mathcal{Q}_{D,p}/\Gamma}\frac{\mathcal{F}_m(z_Q,s)}{|\Gamma_Q|}=\sum_{Q\in\mathcal{Q}_{D,p}/\Gamma_\infty}e(-m\operatorname{Re} z_Q)(\operatorname{Im} z_Q)^{1/2}I_{s-1/2}(2\pi m\operatorname{Im} z_Q). \tag{18}$$

The series on the right-hand side of (18) is equal to

$$\sum_{[ap,b,c]\in\mathcal{Q}_{D,p}/\Gamma_\infty} e\left(\frac{2mb}{4pa}\right)\left(\frac{2\sqrt{D}}{4pa}\right)^{1/2} I_{s-1/2}\left(2\pi m\frac{2\sqrt{D}}{4pa}\right)$$

$$=\sum_{a=1}^{\infty}\sum_{\substack{x \pmod{2ap}\\ x^2\equiv -D \pmod{4ap}}} e\left(\frac{2mx}{4pa}\right)\left(\frac{2\sqrt{D}}{4pa}\right)^{1/2} I_{s-1/2}\left(2\pi m\frac{2\sqrt{D}}{4pa}\right)$$

$$=\sum_{\substack{c>0\\ c\equiv 0 \pmod{4p}}} \frac{1}{2}S_D(m,c)\left(\frac{2\sqrt{D}}{c}\right)^{1/2} I_{s-1/2}\left(2\pi m\frac{2\sqrt{D}}{c}\right),$$

which converges uniformly for $s\in[1,2]$ as explained in [5]. This combined with (17) completes the proof of Lemma 3. □

Theorem 1 now follows from (16) and Lemma 3.

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