



Impulsive Functional Differential Equations with Variable Times

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Abstract—In this note, a Schaefer fixed-point theorem is used to investigate the existence of solutions for first-order impulsive functional differential equations with variable times. © 2004 Elsevier Ltd. All rights reserved.

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1. INTRODUCTION

This note is concerned with the existence of solutions, for the initial value problems (IVP for short), for first-order functional differential equations with impulsive effects

$$y'(t) = f(t, y_t), \quad \text{a.e. } t \in J = [0, T], \quad t \neq \tau_k(y(t)), \quad k = 1, \dots, m, \quad (1)$$

$$y(t^+) = I_k(y(t)), \quad t = \tau_k(y(t)), \quad k = 1, \dots, m, \quad (2)$$

$$y(t) = \phi(t), \quad t \in [-r, 0], \quad (3)$$

where $f : J \times D \rightarrow \mathbb{R}^n$ is a given function, $D = \{\psi : [-r, 0] \rightarrow \mathbb{R}^n; \psi \text{ is continuous everywhere except for a finite number of points } \bar{t} \text{ at which } \psi(\bar{t}) \text{ and } \psi(\bar{t}^+) \text{ exist and } \psi(\bar{t}^-) = \psi(\bar{t})\}$, $\phi \in D$,

$0 < r < \infty$, $\tau_k : \mathbb{R}^n \rightarrow \mathbb{R}$, $I_k : \mathbb{R}^n \rightarrow \mathbb{R}^n$, $k = 1, 2, \dots, m$ are given functions satisfying some assumptions that will be specified later.

For any function y defined on $[-r, T]$ and any $t \in J$, we denote by y_t the element of D defined by

$$y_t(\theta) = y(t + \theta), \quad \theta \in [-r, 0].$$

Here $y_t(\cdot)$ represents the history of the state from time $t - r$, up to the present time t .

Impulsive differential equations have become more important in recent years in some mathematical models of real processes and phenomena studied in physics, chemical technology, population dynamics, biotechnology, and economics. There has been a significant development in impulse theory, in recent years, especially in the area of impulsive differential equations with fixed moments; see the monographs of Bainov and Simeonov [1], Lakshmikantham *et al.* [2], and Samoilenko and Perestyuk [3], and the references therein. The theory of impulsive differential equations with variable time is relatively less developed due to the difficulties created by the state-dependent impulses. Recently, some interesting extensions to impulsive differential equations with variable times have been done by Bajo and Liz [4], Frigon and O'Regan [5–7], Kaul *et al.* [8], Kaul and Liu [9,10], Lakshmikantham *et al.* [11,12], and Liu and Ballinger [13].

The main theorem of this note extends problem (1)–(3) considered by Benchohra *et al.* [14] when the impulse times are constant. Our approach is based on the Schaefer's fixed-point theorem (see [15, p. 29]).

2. PRELIMINARIES

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper.

By $C(J, \mathbb{R}^n)$, we denote the Banach space of all continuous functions from J into \mathbb{R}^n with the norm

$$\|y\|_\infty := \sup\{|y(t)| : t \in J\}.$$

Also, D is endowed with norm $\|\cdot\|$ defined by

$$\|\phi\| := \sup\{|\phi(\theta)| : -r \leq \theta \leq 0\}.$$

In order to define the solutions of (1)–(3), we shall consider the space

$$\begin{aligned} \Omega = \{y : [-r, T] \longrightarrow \mathbb{R}^n : \text{there exist } 0 = t_0 < t_1 < \dots < t_m < t_{m+1} = T, \text{ such that,} \\ t_k = \tau_k(y(t_k)), y(t_k^-) \text{ and } y(t_k^+) \text{ exist, with } y(t_k^-) = y(t_k), \\ k = 1, \dots, m, \text{ and } y \in C([t_k, t_{k+1}], \mathbb{R}^n), k = 0, \dots, m\}. \end{aligned}$$

DEFINITION 2.1. A map $f : J \times D \longrightarrow \mathbb{R}^n$ is said to be L^1 -Carathéodory if

- (i) $t \longmapsto f(t, u)$ is measurable for each $u \in D$;
- (ii) $u \longmapsto f(t, u)$ is continuous for almost all $t \in J$;
- (iii) for each $q > 0$, there exists $h_q \in L^1(J, \mathbb{R}_+)$, such that

$$|f(t, u)| \leq h_q(t), \quad \text{for all } \|u\| \leq q \text{ and for almost all } t \in J.$$

In what follows, we will assume that f is an L^1 -Carathéodory function.

The consideration of this paper is based on the following Schaefer's fixed-point theorem (cf. [15]).

THEOREM 2.2. Let X be a Banach space and $N : X \rightarrow X$ be a completely continuous map. If the set

$$\mathcal{E}(N) = \{y \in X : y = \lambda N(y), \text{ for some } 0 < \lambda < 1\}$$

is bounded, then N has a fixed point.

3. MAIN RESULT

Let us start by defining what we mean by a solution of problem (1)–(3).

DEFINITION 3.1. A function $y \in \Omega$, is said to be a solution of (1)–(3) if y satisfies the equation $y'(t) = f(t, y_t)$, a.e. on J , $t \neq \tau_k(y(t))$, $k = 1, \dots, m$, and the conditions $y(t^+) = I_k(y(t))$, $t = \tau_k(y(t))$ $k = 1, \dots, m$ and $y(t) = \phi(t)$ on $[-r, 0]$.

We are now in a position to state and prove our existence result for problem (1)–(3). For the study of this problem, we first list the following hypotheses.

(H1) The functions $\tau_k \in C^1(\mathbb{R}^n, \mathbb{R})$, for $k = 1, \dots, m$. Moreover,

$$0 < \tau_1(x) < \dots < \tau_m(x) < T, \quad \text{for all } x \in \mathbb{R}^n.$$

(H2) There exist constants c_k , such that $|I_k(x)| \leq c_k$, $k = 1, \dots, m$ for each $x \in \mathbb{R}^n$.

(H3) There exists a continuous nondecreasing function $\psi : [0, \infty) \rightarrow (0, \infty)$ and $p \in L^1(J, \mathbb{R}_+)$, such that

$$|f(t, u)| \leq p(t)\psi(\|u\|),$$

for a.e. $t \in J$ and each $u \in D$ with

$$\int_1^\infty \frac{ds}{\psi(s)} = \infty.$$

(H4) For all $(t, x) \in [0, T] \times \mathbb{R}^n$ and for all $y_t \in D$, we have

$$\langle \tau'_k(x), f(t, y_t) \rangle \neq 1, \quad \text{for } k = 1, \dots, m,$$

where $\langle \cdot, \cdot \rangle$ denotes the scalar product in \mathbb{R}^n .

(H5) For all $x \in \mathbb{R}^n$,

$$\tau_k(I_k(x)) \leq \tau_k(x) < \tau_{k+1}(I_k(x)), \quad \text{for } k = 1, \dots, m.$$

THEOREM 3.2. Assume that Hypotheses (H1)–(H5) hold. Then, the IVP (1)–(3) has at least one solution on $[-r, T]$.

PROOF. The proof will be given in several steps.

STEP 1. Consider the following problem

$$y'(t) = f(t, y_t), \quad \text{a.e. } t \in [0, T], \tag{4}$$

$$y(t) = \phi(t), \quad t \in [-r, 0]. \tag{5}$$

Transform problem (4),(5) into a fixed-point problem. Consider the operator $N : C([-r, T], \mathbb{R}^n) \rightarrow C([-r, T], \mathbb{R}^n)$ defined by:

$$N(y)(t) = \begin{cases} \phi(t), & \text{if } t \in [-r, 0]; \\ \phi(0) + \int_0^t f(s, y_s) ds, & \text{if } t \in [0, T]. \end{cases}$$

We shall show that the operator N is completely continuous.

CLAIM 1. N is continuous.

Let $\{y_n\}$ be a sequence, such that $y_n \rightarrow y$ in $C([-r, T], \mathbb{R}^n)$.

Then,

$$\begin{aligned} |N(y_n)(t) - N(y)(t)| &\leq \int_0^t |f(s, y_{n_s}) - f(s, y_s)| ds \\ &\leq \int_0^T |f(s, y_{n_s}) - f(s, y_s)| ds. \end{aligned}$$

Since f is an L^1 -Carathéodory function, we have by the Lebesgue dominated convergence theorem

$$\|N(y_n) - N(y)\|_\infty \leq \|f(\cdot, y_{n(\cdot)}) - f(\cdot, y(\cdot))\|_{L^1} \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

CLAIM 2. N maps bounded sets into bounded sets in $C([-r, T], \mathbb{R}^n)$.

Indeed, it is enough to show that for any $q > 0$, there exists a positive constant ℓ , such that for each $y \in B_q = \{y \in C([-r, T], \mathbb{R}^n) : \|y\|_\infty \leq q\}$, we have $\|N(y)\|_\infty \leq \ell$. By Definition 3.1 (iii), we have for each $t \in [0, T]$,

$$\begin{aligned} |N(y)(t)| &\leq |\phi(0)| + \int_0^t |f(s, y_s)| \, ds \\ &\leq \|\phi\| + \|h_q\|_{L^1}. \end{aligned}$$

Thus,

$$\|N(y)\|_\infty \leq \|\phi\| + \|h_q\|_{L^1} := \ell.$$

CLAIM 3. N maps bounded sets into equicontinuous sets of $C([-r, T], \mathbb{R}^n)$.

Let $l_1, l_2 \in [0, T]$, $l_1 < l_2$, B_q be a bounded set of $C([-r, T], \mathbb{R}^n)$ as in Claim 2, and let $y \in B_q$. Then,

$$|N(y)(l_2) - N(y)(l_1)| \leq \int_{l_1}^{l_2} h_q(s) \, ds.$$

As $l_2 \rightarrow l_1$, the right-hand side of the above inequality tends to zero. The equicontinuity for the cases $l_1 < l_2 \leq 0$ and $l_1 \leq 0 \leq l_2$ is obvious.

As a consequence of Claims 1–3, together with the Arzela-Ascoli theorem, we can conclude that $N : C([-r, T], \mathbb{R}^n) \rightarrow C([-r, T], \mathbb{R}^n)$ is completely continuous.

CLAIM 4. Now it remains to show that the set

$$\mathcal{E}(N) := \{y \in C([-r, T], \mathbb{R}^n) : y = \lambda N(y), \text{ for some } 0 < \lambda < 1\}$$

is bounded.

Let $y \in \mathcal{E}(N)$. Then, $y = \lambda N(y)$ for some $0 < \lambda < 1$. Thus, for each $t \in [0, T]$,

$$y(t) = \lambda \left(\phi(0) + \int_0^t f(s, y_s) \, ds \right).$$

This implies by (H2),(H3) that for each $t \in J$, we have

$$|y(t)| \leq \|\phi\| + \int_0^t p(s)\psi(\|y_s\|) \, ds.$$

We consider the function μ defined by

$$\mu(t) = \sup \{|y(s)| : -r \leq s \leq t\}, \quad 0 \leq t \leq T.$$

Let $t^* \in [-r, t]$ be such that $\mu(t) = |y(t^*)|$. If $t^* \in [0, T]$, by the previous inequality, we have for $t \in [0, T]$

$$\mu(t) \leq \|\phi\| + \int_0^t p(s)\psi(\mu(s)) \, ds.$$

If $t^* \in [-r, 0]$, then $\mu(t) = \|\phi\|$ and the previous inequality holds. Let us take the right-hand side of the above inequality as $v(t)$. Then, we have

$$c = v(0) = \|\phi\|, \quad \mu(t) \leq v(t), \quad t \in [0, T],$$

and

$$v'(t) = p(t)\psi(\mu(t)), \quad \text{a.e. } t \in [0, T].$$

Using the nondecreasing character of ψ , we get

$$v'(t) \leq p(t)\psi(v(t)), \quad \text{a.e. } t \in [0, T].$$

This implies that for each $t \in [0, T]$

$$\int_{v(0)}^{v(t)} \frac{ds}{\psi(s)} \leq \int_0^T p(s) ds < \int_{v(0)}^{\infty} \frac{ds}{\psi(s)}.$$

Thus, there exists a constant K , such that $v(t) \leq K, t \in [0, T]$, and hence, $\mu(t) \leq K, t \in [0, T]$. Since for every $t \in [0, T], \|y_t\| \leq \mu(t)$, we have

$$\|y\|_{\infty} \leq K' = \max \{ \|\phi\|, K \},$$

where K' depends on T and on the functions p and ψ . This shows that $\mathcal{E}(N)$ is bounded.

Set $X := C([-r, T], \mathbb{R}^n)$. As a consequence of Schaefer's theorem (see [15, p. 29]), we deduce that N has a fixed-point y which is a solution to problem (4),(5). Denote this solution by y_1 .

Define the function

$$r_{k,1}(t) = \tau_k(y_1(t)) - t, \quad \text{for } t \geq 0.$$

(H1) implies that

$$r_{k,1}(0) \neq 0, \quad \text{for } k = 1, \dots, m.$$

If

$$r_{k,1}(t) \neq 0, \quad \text{on } [0, T] \text{ for } k = 1, \dots, m,$$

i.e.,

$$t \neq \tau_k(y_1(t)), \quad \text{on } [0, T] \text{ and for } k = 1, \dots, m,$$

then y_1 is a solution of problem (1)–(3).

It remains to consider the case when

$$r_{1,1}(t) = 0, \quad \text{for some } t \in [0, T].$$

Now, since

$$r_{1,1}(0) \neq 0$$

and $r_{1,1}$ is continuous, there exists $t_1 > 0$, such that

$$r_{1,1}(t_1) = 0 \quad \text{and} \quad r_{1,1}(t) \neq 0, \quad \text{for all } t \in [0, t_1].$$

Thus, by (H1), we have

$$r_{k,1}(t) \neq 0, \quad \text{for all } t \in [0, t_1] \quad \text{and} \quad k = 1, \dots, m.$$

STEP 2. Consider now the following problem

$$y(t) = y_1(t), \quad t \in [t_1 - r, t_1], \tag{6}$$

$$y'(t) = f(t, y_t), \quad \text{a.e. } t \in [t_1, T],$$

$$y(t_1^+) = I_1(y_1(t_1)). \tag{7}$$

Transform problem (6),(7) into a fixed-point problem. Consider the operator $N_1 : C([t_1 - r, T], \mathbb{R}^n) \rightarrow C([t_1 - r, T], \mathbb{R}^n)$ defined by:

$$y(t) = \begin{cases} y_1(t), & \text{if } [t_1 - r, t_1], \\ I_1(y_1(t_1)) + \int_{t_1}^t f(s, y_s) ds, & \text{if } t \in (t_1, T]. \end{cases}$$

As in Step 1, we can show that N_1 is completely continuous, and the set

$$\mathcal{E}(N_1) := \{y \in C([t_1 - r, T], \mathbb{R}^n) : y = \lambda N_1(y), \text{ for some } 0 < \lambda < 1\}$$

is bounded.

Set $X := C([t_1 - r, T], \mathbb{R}^n)$. As a consequence of Schaefer's theorem, we deduce that N_1 has a fixed-point y which is a solution to problem (6),(7). Denote this solution by y_2 . Define

$$r_{k,2}(t) = \tau_k(y_2(t)) - t, \quad \text{for } t \geq t_1.$$

If

$$r_{k,2}(t) \neq 0, \quad \text{on } (t_1, T] \text{ and for all } k = 1, \dots, m,$$

then

$$y(t) = \begin{cases} y_1(t), & \text{if } t \in [0, t_1], \\ y_2(t), & \text{if } t \in (t_1, T], \end{cases}$$

is a solution of problem (1)–(3).

It remains to consider the case when

$$r_{2,2}(t) = 0, \quad \text{for some } t \in (t_1, T].$$

By (H5), we have

$$\begin{aligned} r_{2,2}(t_1^+) &= \tau_2(y_2(t_1^+)) - t_1 \\ &= \tau_2(I_1(y_1(t_1))) - t_1 \\ &> \tau_1(y_1(t_1)) - t_1 \\ &= r_{1,1}(t_1) = 0. \end{aligned}$$

Since $r_{2,2}$ is continuous, there exists $t_2 > t_1$, such that

$$r_{2,2}(t_2) = 0$$

and

$$r_{2,2}(t) \neq 0, \quad \text{for all } t \in (t_1, t_2).$$

It is clear by (H1) that

$$r_{k,2}(t) \neq 0, \quad \text{for all } t \in (t_1, t_2), \quad k = 2, \dots, m.$$

Suppose now, that there is $\bar{s} \in (t_1, t_2]$, such that

$$r_{1,2}(\bar{s}) = 0.$$

From (H5), it follows that

$$\begin{aligned} r_{1,2}(t_1^+) &= \tau_1(y_2(t_1^+)) - t_1 \\ &= \tau_1(I_1(y_1(t_1))) - t_1 \\ &\leq \tau_1(y_1(t_1)) - t_1 \\ &= r_{1,1}(t_1) = 0. \end{aligned}$$

Thus, the function $r_{1,2}$ attains a nonnegative maximum at some point $s_1 \in (t_1, T]$. Since

$$y_2'(t) = f(t, (y_2)_t),$$

then,

$$r'_{1,2}(s_1) = \tau'_1(y_2(s_1))y'_2(s_1) - 1 = 0.$$

Therefore,

$$\langle \tau'_1(y_2(s_1)), f(s_1, (y_2)_{s_1}) \rangle = 1,$$

which is a contradiction by (H4).

STEP 3. We continue this process and taking into account that $y_{m+1} := y|_{[t_m, T]}$ is a solution to the problem

$$\begin{aligned} y(t) &= y_m(t), & t \in [t_m - r, t_m], \\ y'(t) &= f(t, y_t), & \text{a.e. } t \in (t_m, T), \end{aligned} \quad (8)$$

$$y(t_m^+) = I_m(y_{m-1}(t_m)). \quad (9)$$

The solution y of problem (1)–(3) is then defined by

$$y(t) = \begin{cases} y_1(t), & \text{if } t \in [-r, t_1], \\ y_2(t), & \text{if } t \in (t_1, t_2], \\ \vdots \\ y_{m+1}(t), & \text{if } t \in (t_m, T]. \end{cases}$$

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