Note

Competition polysemy

Miranka Fischermann\textsuperscript{a}, Werner Knoben\textsuperscript{b}, Dirk Kremer\textsuperscript{a}, Dieter Rautenbach\textsuperscript{c}

\textsuperscript{a}Lehrstuhl II für Mathematik, RWTH-Aachen, 52062 Aachen, Germany
\textsuperscript{b}Am Schürkamp 21, 46509 Xanten, Germany
\textsuperscript{c}Forschungsinstitut für Diskrete Mathematik, Lennéstrasse 2, 53113 Bonn, Germany

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Abstract

Following a suggestion of Tanenbaum (Electron. J. Combin. 7 (2000) R43) we introduce the notion of competition polysemic pairs of graphs. A pair of (simple) graphs \((G_1, G_2)\) on the same set of vertices \(V\) is called competition polysemic, if there exists a digraph \(D = (V, A)\) such that for all \(u, v \in V\) with \(u \neq v\), \(uw\) is an edge of \(G_1\) if and only if there is some \(w \in V\) such that \(uw \in A\) and \(vw \in A\) and \(uw\) is an edge of \(G_2\) if and only if there is some \(w \in V\) such that \(uw \in A\) and \(v_2 \in A\). Our main results are a characterization of competition polysemic pairs \((G_1, G_2)\) in terms of edge clique covers of \(G_1\) and \(G_2\) and a characterization of the connected graphs \(G\) for which there exists a tree \(T\) such that \((G, T)\) is competition polysemic.

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1. Introduction

We consider finite simple graphs \(G = (V, E)\) with vertex set \(V\) and edge set \(E\). A clique of \(G\) is the vertex set of a (not necessarily maximal) complete subgraph of \(G\). An edge clique cover of \(G\) is a collection \(\mathcal{C}\) of cliques such that for every edge \(uw \in E\) some clique in \(\mathcal{C}\) contains both vertices \(u\) and \(v\). A block of \(G = (V, E)\) is a maximal 2-connected subgraph of \(G\) and a vertex \(u \in V\) for which \(G - \{u\} = G[V\setminus\{u\}]\) has more components than \(G\) is a cutvertex.

We also consider finite digraphs \(D = (V, A)\) with vertex set \(V\) and arc set \(A\) which may contain loops but no multiple arcs. An arc in \(D\) from \(u\) to \(v\) will be denoted by \(uv\) and the positive (negative) neighbourhood of a vertex \(v \in V\) is \(N^+_v(u) = \{v \in V \mid uv \in A\}\) (\(N^-_v(u) = \{v \in V \mid vu \in A\}\)). For further definitions we refer to [3].

In [11] Tanenbaum introduced the notion of bound polysemy. He called a pair \((G_1, G_2)\) of graphs \(G_1 = (V, E_1)\) and \(G_2 = (V, E_2)\) on a common set of vertices \(V\) bound polysemic, if there exists a reflexive poset \(P = (V, \leq)\) on the set \(V\) such that for all \(u, v \in V\) with \(u \neq v\), \(uw \in E_1\) if and only if there is some \(w \in V\) such that \(u \leq w\) and \(v \leq w\) and \(uv \in E_2\) if and only if there is some \(w \in V\) such that \(w \leq u\) and \(w \leq v\).

In this situation the graphs \(G_1\) and \(G_2\) are called the upper bound graph and the lower bound graph of \(P\), respectively. Upper bound graphs were introduced by McMorris and Zaslavsky in [7] (cf. also the survey [6]).

At the end of [11] Tanenbaum poses the problem of generalizing bound polysemy to competition polysemy using digraphs instead of posets. We will do so in the present paper. Consequently, we call a pair \((G_1, G_2)\) of graphs \(G_1 = (V, E_1)\) and \(G_2 = (V, E_2)\) on a common set of vertices \(V\) competition polysemic, if there exists a digraph \(D = (V, A)\) on the same...
set of vertices such that for all \( u, v \in V \) with \( u \neq v \), \( uv \in E_1 \) if and only if \( N^+_D(u) \cap N^-_D(v) \neq \emptyset \) and \( uv \in E_2 \) if and only if \( N^+_D(u) \cap N^-_D(v) \neq \emptyset \).

In this situation \( D \) is called a realization of \((G_1, G_2)\). Furthermore, the graphs \( G_1 \) and \( G_2 \) are called the competition graph and common enemy graph of \( D \), respectively. Competition graphs were introduced by Cohen [1] to study food web models in ecology and have been studied by various authors (cf. eg. [2,4,8–10]).

Since every poset \( P = (V, \leq) \) corresponds to a digraph \( D = (V, A) \) such that \( u \leq v \) for \( u, v \in V \) if and only if \( uv \in A \), a pair of graphs is bound polysemic only if it is competition polysemic. In this sense competition polysemity generalizes bound polysemity. An unlabeled version of competition polysemity was studied in [5] (see also the corresponding comments in [11]).

In the next section we prove a characterization of competition polysemic pairs. In Section three we consider special cases of competition polysemity and prove a characterization of the connected graphs \( G \) for which there exists a tree \( T \) such that \((G, T)\) is competition polysemic.

2. A characterization of competition polysemity

Tanenbaum derived his characterization of bound polysemic pairs of graphs (Theorem 15 in [11]) from the characterization of upper bound graphs due to McMorris and Zaslavsky [7]. We adopt the same approach and start with the following characterization of competition graphs due to Dutton and Brigham [2] (cf. also [4,9]).

**Theorem 2.1** (cf. Dutton and Brigham [2]). A graph \( G = (V, E) \) is the competition graph of some digraph if and only if there exists an edge clique cover \( C = \{C_1, C_2, \ldots, C_p\} \) of \( G \) with \( p \leq |V| \).

If \( C = \{C_1, C_2, \ldots, C_p\} \) is an edge clique cover of \( G \) with \( p \leq |V| \), then we can choose a set of \( p \) different vertices \( R = \{v_1, v_2, \ldots, v_p\} \subseteq V \). We call \( R \) a set of distinct representatives of the cliques in \( C \). (Note that—par abus de langage—we do not require \( v_i \in C_i \) for \( 1 \leq i \leq p \).

We proceed to our main result in this section.

**Theorem 2.2.** A pair \((G_1, G_2)\) of graphs with \( G_1 = (V, E_1) \) and \( G_2 = (V, E_2) \) is competition polysemic if and only if there exist edge clique covers \( C_1 = \{C_{1,1}, C_{1,2}, \ldots, C_{1,p}\} \) of \( G_1 \) and \( C_2 = \{C_{2,1}, C_{2,2}, \ldots, C_{2,q}\} \) of \( G_2 \) for which there exist sets of distinct representatives \( R_1 = \{v_{1,1}, v_{1,2}, \ldots, v_{1,p}\} \) and \( R_2 = \{v_{2,1}, v_{2,2}, \ldots, v_{2,q}\} \), i.e. \( |R_1| = p, |R_2| = q \leq |V| \), such that

(i) \( v_{2,i} \in C_{1,i} \) if and only if \( v_{1,i} \in C_{2,i} \),

(ii) if \( C_{1,i} \cap C_{2,j} \neq \emptyset \), then there is some \( 1 \leq l \leq q \) such that \( v_{1,i}, v_{1,j} \in C_{2,l} \) and

(iii) if \( C_{2,i} \cap C_{2,j} \neq \emptyset \), then there is some \( 1 \leq l \leq p \) such that \( v_{2,i}, v_{2,j} \in C_{1,l} \).

**Proof.** First, we assume that \((G_1, G_2)\) with \( G_1 = (V, E_1) \) and \( G_2 = (V, E_2) \) is competition polysemic with realization \( D = (V, A) \) and prove the existence of \( C_1, C_2, R_1 \) and \( R_2 \) as in the statement of the theorem.

Let \( V = \{v_1, v_2, \ldots, v_p\} \) and for \( 1 \leq i \leq n \) let \( v_{1,i} = v_{2,i} = v_i, C_{1,i} = N^+_D(v_{1,i}) \) and \( C_{2,i} = N^-_D(v_{2,i}) \). Clearly, \( v_i, v_j \in C_{1,i} \) holds for \( u, v \in V \) with \( u \neq v \) and \( 1 \leq i \leq n \) if and only if \( v_i, v_j \in N^+_D(v) \cap N^-_D(v) \) or equivalently \( uv \in E_1 \). This implies that \( C_1 = \{C_{1,1}, C_{1,2}, \ldots, C_{1,n}\} \) is an edge clique cover of \( G_1 \). By symmetry, \( C_2 = \{C_{2,1}, C_{2,2}, \ldots, C_{2,n}\} \) is an edge clique cover of \( G_2 \). Furthermore, \( v_{2,i} \in C_{2,i} = N^-_D(v_{1,i}) \) holds if and only if \( v_{1,i} \in C_{2,i} = N^-_D(v_{2,i}) \) which implies (i). Finally, if \( C_{1,i} \cap C_{2,j} \neq \emptyset \), then there is some \( 1 \leq l \leq n \) such that \( v_{2,i} \in C_{1,l} \) and \( v_{1,j} \in C_{2,l} \) which implies (ii) and, by symmetry, also (iii). This completes the first part of the proof.

Now, let \((G_1, G_2)\) be a pair of graphs with \( G_1 = (V, E_1) \) and \( G_2 = (V, E_2) \) and let \( C_1, C_2, R_1 \) and \( R_2 \) be as in the statement of the theorem.

Let the digraph \( D \) have vertex set \( V \) and arc set \( A = A_1 \cup A_2 \) where

\[
A_1 = \{u v_i | u \in C_{1,i}, 1 \leq i \leq p\} \quad \text{and} \quad A_2 = \{v_{2,j} u | u \in C_{2,j}, 1 \leq j \leq q\}.
\]

We prove that \((G_1, G_2)\) is competition polysemic with realization \( D \).

Let \( u, v \in E_1 \) with \( u \neq v \). Since \( C_1 \) is an edge clique cover of \( G_1 \), there is some \( 1 \leq i \leq p \) such that \( u, v \in C_{1,i} \). This implies that \( u v_i, v_i u_1 \in A_1 \) and \( v_i u_1 \in N^+_D(u) \cap N^-_D(v) \neq \emptyset \).

Now, let \( x \in N^+_D(u) \cap N^-_D(v) \neq \emptyset \) for \( u, v \in V \) with \( u \neq v \). We have that \( u v_i, v_i u \in A_1 \cup A_2 \).

If \( u v_i, v_i u \in A_1 \), then \( x = v_i u_1 \) and \( u, v \in C_{1,i} \) for some \( 1 \leq i \leq p \). This implies that \( u \in E_1 \). If \( u v_i \in A_2 \) and \( v_i u \in A_2 \), then \( x = v_i u_1 \) and \( u \in C_{1,i} \) for some \( 1 \leq i \leq p \) and \( v = v_i u_2 \) and \( x = v_i u_2 \in C_{2,j} \) for some \( 1 \leq i \leq q \). Condition (i) implies that \( v = v_{2,j} \in C_{1,i} \). Thus, \( u, v \in C_{1,i} \) which implies that \( u \in E_1 \). Similarly, if \( u v_i \in A_2 \) and \( v_i u \in A_2 \) we obtain \( u \in E_1 \). Finally, if
and only if for some tree (iii) implies that there exists some 1 \leq i, j \leq q with i \neq j. Since x \in C_{2,i} \cap C_{2,j} \neq \emptyset, Condition (iii) implies that there exists some 1 \leq l \leq p such that v_{2,i}, v_{2,j} \in C_{1,l}. Thus, v_{2,i}v_{2,j} = uv \in E_1. Hence, in all cases we have uv \in E_1.

We obtain that uv \in E_1 for u, v \in V with u \neq v if and only if N_{G}(u) \cap N_{G}(v) \neq \emptyset which means that G_1 is the competition graph of D. By symmetry, G_2 is the common enemy graph of D and hence (G_1, G_2) is competition polysemic with realization D. This completes the proof. 

We want to point out that it is straightforward but tedious to derive Tanenbaum’s characterization of bound polysemic pairs of graphs (Theorem 15 in [11]) from Theorem 2.2.

3. Special cases of competition polysemic

In Section 4 of [11] Tanenbaum investigates graphs G such that (G, H) is bound polysemic, and H = G or H is the complement \( \bar{G} \) of G. The analogous problems for competition polysemic and most complicated. For example, Tanenbaum shows that (G, G) is bound polysemic if and only if the vertex set of G is the disjoint union of cliques (cf. Theorem 8 in [11]). The following lemma shows that the graphs G such that (G, G) is competition polysemic cannot be characterized by forbidden induced subgraphs.

**Lemma 3.1.** Let G = (V_G, E_G) be a graph. There exists a graph H = (V_H, E_H) of order at most |E_G| such that (G \cup H, G \cup H) is competition polysemic where G \cup H = (V_G \cup V_H, E_G \cup E_H) and V_G \cap V_H = \emptyset.

**Proof.** Let \( \mathcal{C} = \{C_1, C_2, \ldots, C_p\} \) be an edge clique cover of G = (V_G, E_G) such that p is minimum. Since \( \{u, v \mid uv \in E_G\} \) is an edge clique cover of G, we obtain that p \leq |E_G|. Let D = (V_D, A_D) be the digraph with vertex set V_D = V_G \cup \{v_1, v_2, \ldots, v_p\}, where V_G \cap \{v_1, v_2, \ldots, v_p\} = \emptyset, and arc set

\[
A_D = \bigcup_{i=1}^{p} \{v_i v_j, v_j v_i \mid v_j \in C_i\}.
\]

Let G_1 = (V_D, E_1) and G_2 = (V_D, E_2) be the competition graph and common enemy graph of D, respectively. Since N_{G}(v) \cap N_{G}(v) for every vertex v \in V_D, we have G_1 = G_2.

For u, v \in V_G with u \neq v we obtain that uv \in E_G if and only if u, v \in C_i for some 1 \leq i \leq p if and only if v \in N_{G}(u) \cap N_{G}(v) if and only if uv \in E_1.

For u \in V_G and v \in \{v_1, v_2, \ldots, v_p\} we obtain that N_{G}(u) \cup N_{G}(v) = \emptyset and hence uv \notin E_1. Let H = (V_G, V_H) \backslash E_G). Then, H has p \leq |E_G| vertices and G_1 = G_2 = G \cup H. This completes the proof. 

Another result of Tanenbaum is that (G, \( \bar{G} \)) is bound polysemic if and only if G has just one vertex (cf. Theorem 10 in [11]). We will now present graphs G of any order n \geq 2 such that (G, \( \bar{G} \)) is competition polysemic.

**Lemma 3.2.** For n \geq 2 the pairs (K_{1,n−1}, \( \bar{K}_{1,n−1} \)) and (\( \bar{K}_n, K_n \)) are competition polysemic where K_{1,n−1} and \( \bar{K}_n \) denote the star and the edgeless graph of order n, respectively.

**Proof.** Let V = \{v_1, v_2, \ldots, v_n\} and E = \{v_i v_j \mid 2 \leq i \leq n\} and let G = (V, E) and H = (V, \emptyset). Clearly, G \cong K_{1,n−1} and H \cong \( \bar{K}_n \).

Furthermore, let A_G = \{v_i v_j, v_j v_i \mid 2 \leq i \leq n\} and A_H = \{v_1 v_2 \mid 2 \leq i \leq n\}. It is straightforward to verify that the pair (G, \( \bar{G} \)) is competition polysemic with realization D_G = (V, A_G) and that the pair (H, \( \bar{H} \)) is competition polysemic with realization D_H = (V, A_H).

Tanenbaum shows that for any graph G of order n the pair (G, K_n) is bound polysemic if and only if G is an upper bound graph that contains a vertex of degree n − 1 (cf. Theorem 11 in [11]). We have just seen in Lemma 3.2 that (\( \bar{K}_n, K_n \)) is competition polysemic, which shows that the existence of a vertex of degree n − 1 is not necessary for competition polysemic with K_n.

Our main result of this section generalizes Tanenbaum’s characterization of graphs G such that (G, T) is bound polysemic for some tree T in the case of connected graphs. Tanenbaum showed that (G, T) is bound polysemic for some tree T if and only if G is complete and T is a star (cf. Theorem 12 in [11]).
Theorem 3.3. Let $G=(V,E_G)$ be a connected graph. There is a tree $T=(V,E_T)$ such that $(G,T)$ is competition polysemic if and only if

(i) at most one block of $G$ is not complete,
(ii) every cutvertex of $G$ lies in exactly two blocks of $G$ and
(iii) if some block of $G$ is not complete, then the vertex set of this block is the union of two cliques of $G$ that have exactly two common vertices and these vertices lie in no other block of $G$.

Proof. First, we assume that $(G,T)$ is competition polysemic with realization $D$ where $G=(V,E_G)$ is a connected graph and $T=(V,E_T)$ is a tree.

Let $V=\{v_1,v_2,\ldots,v_n\}$ and for $1 \leq i \leq n$ let $v_{1,i}=v_{2,i}=v_i$, $C_{1,i}=N^+_D(v_i)$ and $C_{2,i}=N^-_D(v_i)$. Let $\mathcal{C}_i=\{C_{1,1},C_{1,2},\ldots,C_{1,n}\}$ and $\mathcal{C}_2=\{C_{2,1},C_{2,2},\ldots,C_{2,n}\}$. As in the proof of Theorem 2.2 is follows that $\mathcal{C}_1$, $\mathcal{C}_2$, $R_1$ and $R_2$ are as in the statement of Theorem 2.2. (Note that we use double indices '1,i' or '2,j' for vertices just in order to emphasize that a vertex corresponds to a certain clique in $\mathcal{C}_1$ or $\mathcal{C}_2$, respectively.)

Since $T$ is a tree, $\mathcal{C}_2$ contains exactly $n-1$ different cliques of cardinality 2 and one clique that is a subset of one of the others. Without loss of generality let $C_{2,1} \subseteq C_{2,2}$.

If $v_{2,i} \in C_{1,j} \cap (C_{1,k} \cap C_{1,l})$ for some $1 \leq i \leq n$ and $1 \leq j < k < l \leq n$, then $v_{1,j},v_{1,k},v_{1,l} \in C_{1,1}$, which implies a contradiction to $|C_{2,1}| \leq 2$. Hence, every vertex of $G$ lies in at most two cliques of $\mathcal{C}_1$. We denote this property of $G$ by (*).

If $v_{2,i},v_{2,j} \in C_{1,i} \cap C_{1,j}$ for some $1 \leq i < j \leq n$ and $1 \leq t \leq n$, then $v_{1,i},v_{1,j} \in C_{2,s} \cap C_{2,t}$, which implies that $\{v_{1,i},v_{1,j}\} = C_{2,s} = C_{2,t}$ and hence $\{s,t\} = \{1,2\}$. Thus, for $1 \leq i < j \leq n$ we obtain

$$|C_{1,i} \cap C_{1,j}| = 1, \quad \text{if} \ C_{2,1} \neq \{v_{1,i},v_{1,j}\},$$

(1)

$$|C_{1,i} \cap C_{1,j}| = 2, \quad \text{if} \ C_{2,1} = \{v_{1,i},v_{1,j}\}.$$ 

(2)

If $G$ contains a cycle that is not covered by a single clique in $\mathcal{C}_1$, then there are $t \geq 2$ cliques

$$C_{1,j_1},C_{1,j_2},\ldots,C_{1,j_t}, \in \mathcal{C}_1$$

such that $C_{1,j_i} \neq C_{1,j_{i+1}}$ for every $1 \leq i \leq t-1$ and $C_{1,j_t} \neq C_{1,j_1}$ and $t$ vertices

$$v_{f_1},v_{f_2},\ldots,v_{f_t}$$

such that $v_{f_i} \in C_{1,j_i} \cap C_{1,j_{i+1}}$ for every $1 \leq i \leq t-1$ and $v_{f_t} \in C_{1,j_t} \cap C_{1,j_1}$ with $f_1 \neq f_i$ for $i \neq j$.

We obtain, $v_{1,j_i},v_{1,j_{i+1}} \in C_{2,j_i}$ for every $1 \leq i \leq t-1$ and $v_{1,j_t},v_{1,j_1} \in C_{2,j_t}$. Therefore $v_{1,j_i},v_{1,j_{i+1}} \in E_T$ for every $1 \leq i \leq t-1$ and $v_{1,j_t},v_{1,j_1} \in E_T$. Since $T$ is a tree, we have $t=2$, $C_{1,j_1} = C_{1,j_2} = \{v_{1,j_1},v_{1,j_2}\}$ and $\{f_1,f_2\} = \{1,2\}$.

Hence, every cycle in $G$ that is not covered by a single clique in $\mathcal{C}_1$ is covered by the unique two cliques $C_{1,j_1},C_{1,j_2}$ with $C_{2,1} = C_{2,2} = \{v_{1,j_1},v_{1,j_2}\}$.

This implies that every clique $C_{1,j}$ with $v_{1,j} \notin C_{2,1}$ is the vertex set of a complete block in $G$. Furthermore, if some block $B$ of $G$ is not complete, then $C_{2,1} = C_{2,2}$ and $V(B) \subseteq C_{1,j_1} \cup C_{1,j_2}$ with $C_{2,1} = \{v_{1,j_1},v_{1,j_2}\}$. Since every block of $G$ which contains two vertices of a clique contains the whole clique, we obtain that $V(B) = C_{1,j_1} \cup C_{1,j_2}$.

Thus, at most one block of $G$ is not complete and Condition (i) holds.

Since every cutvertex of $G$ lies in at least two blocks of $G$, we get, by (*), that every cutvertex of $G$ lies in exactly two blocks of $G$ and Condition (ii) holds.

Now, let $G$ contain a block $B$ that is not complete. Then, $V(B) = C_{1,j_1} \cup C_{1,j_2}$ and $C_{2,1} = \{v_{1,j_1},v_{1,j_2}\}$. By (2), we obtain that $|C_{1,j_1} \cap C_{1,j_2}| = 2$. By (*), the two vertices in $C_{1,j_1} \cap C_{1,j_2}$ lie in no clique $C_{1,i}$ with $i \neq j_1, j_2$ and in no block of $G$ besides $B$. Hence Condition (iii) holds. This completes the first part of the proof.

Now, let $G=(V,E_G)$ be a connected graph such that the Conditions (i)–(iii) hold. Let $S$ be the set of cutvertices of $G$.

If one block of $G$ is not complete, then let this block be $B_0$, let $C_0$ and $C_1$ be two cliques of $G$ such that $V(B_0) = C_0 \cup C_1$ and $|C_0 \cap C_1| = 2$. Let $\{x_0,x_1\} = C_0 \cap C_1$ and define $N_i = C_i$ for $i=0,1$.

If all blocks of $G$ are complete, then let $x_0$ be an arbitrary vertex in $V \setminus S$, let $B_0$ be the unique block of $G$ that contains $x_0$, let $x_1 = x_0$ and $N_i = V(B_0)$ for $i=0,1$.

It is straightforward to see that for $1 \leq i \leq |S|$ we can (recursively) choose vertices $x_{i+1} \in S \setminus \{x_j \mid 2 \leq j \leq i\}$ and define sets

$$N_{i+1} = \{x_{i+1}\} \cup \{u \in V \mid ux_{i+1} \in E_G\} \bigcup_{j=0}^{i} N_j.$$
such that every set \( N_i \) for \( 0 \leq i \leq |S| + 1 \) is a clique of \( G \) and if \( i \geq 2 \), then \( N_i \) is the vertex set of a block in \( G \). Furthermore, for \( i \geq 2 \) every cutvertex \( x_i \) of \( G \) lies in \( N_i \) and \( N_j \) for some unique \( j < i \). (See the left part of Fig. 1 for illustration.)

Now, we define the digraph \( D = (V, A) \) with vertex set \( V \) and arc set
\[
A = \{ \overrightarrow{xy} \mid y \in N_j, 0 \leq j \leq |S| + 1 \} \cup \{ \overrightarrow{uv} \in V \}.
\]
(See the right part of Fig. 1 for illustration.)

Let \( E_1 \) and \( E_2 \) be the edge sets of the competition graph and the common enemy graph of \( D \), respectively. Note, that \( N_2^D(x_0) = N_2^D(x_1) = \{x_0, x_1\} \) and for every \( x \in V \backslash \{x_0, x_1\} \) we have \( x \in N_i \{x_i\} \) and \( N_0^D(x) = \{x, x_i\} \) for some \( 0 \leq i \leq |S| + 1 \). Thus, for \( u, v \in V \) with \( u \neq v \) we obtain that \( uv \in E_2 \) if and only if \( \{u, v\} = N_0^D(x) \) for some \( x \in V \) if and only if \( \{u, v\} = \{x, x_i\} \) and \( x \in N_i \{x_i\} \) for some \( 0 \leq i \leq |S| + 1 \). Hence, we obtain that \( G_2 = (V, E_2) \) is a tree, since for every block \( B \) of \( G \) the subgraph \( G_2[V(B)] \) induced by \( V(B) \) in \( G_2 \) is a star, if \( B \) is complete and a double star (=a tree of diameter 3), if \( B = B_0 \) and \( B_0 \) is not complete.

Now, it remains to prove that \( G_1 = (V, E_1) = (V, E_G) = G \). Note that \( N_1^G(x) = N_i \) if \( x = x_i \) for \( 0 \leq i \leq |S| + 1 \) and \( N_0^G(x) = \{x\} \) if \( x \in V \backslash \{x_0, x_1, \ldots, x_{|S|+1}\} \). Let \( uv \) be an edge of \( G \). If \( uv \in E(B_0) \), then \( u, v \in N_i \) for some \( i \in \{0, 1\} \) which implies that \( u, v \in N_2^D(x_i) \) for some \( i \in \{0, 1\} \) and thus \( uv \in E_1 \). If \( uv \in E(B) \) for some block \( B \neq B_0 \), then \( B \) is complete and contains at least one cutvertex. If \( i = \min \{2 \leq j \leq |S| \mid x_j \in V(B)\} \), then \( u, v \in N_i = V(B) \) and \( u, v \in N_2^D(x_i) \) which implies that \( uv \in E_1 \). This yields that \( E_G \subseteq E_1 \).

Conversely, let \( uv \in E_1 \). We have \( u, v \in N_2^D(x) \) for some vertex \( x \in V \) with \( |N_2^D(x)| \geq 2 \). This implies that \( x = x_j \) and \( u, v \in N_j \) for some \( 0 \leq j \leq |S| + 1 \). Since \( N_j \) is a clique in \( G \), we obtain that \( uv \in E_G \). Hence \( E_G = E_1 \) and the proof is complete. □

References