ON A PROBLEM OF J. ZAKS CONCERNING 5-VALENT 3-CONNECTED PLANAR GRAPHS

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Recently J. Zaks formulated the following Eberhard-type problem:
Let \((p_5, p_6, \ldots)\) be a finite sequence of nonnegative integers; does there exist a 5-valent 3-connected planar graph \(G\) such that it has exactly \(p_k\) \(k\)-gons for all \(k \geq 5\), \(m_i\) of its vertices meet exactly \(i\) triangles, \(4 \leq i \leq 5\), and
\[
m_4 + 2m_5 = 24 + 3 \sum_{k=5}^{\infty} (k-4)p_k.
\]

This paper brings a solution to the problem, and similar problems are considered as well.

1. Introduction

Let \(\mathcal{G}(s, g)\) be the set of \(s\)-connected graphs embedded in an orientable surface \(S_g\) of genus \(g\) which contain no loops or multiple edges, \(s \geq 2\), \(g \geq 0\). For \(G \in \mathcal{G}(s, g)\) let \(p_k(G)\) or \(v_k(G)\) denote the number of \(k\)-gonal faces or \(k\)-valent vertices of \(G\) respectively, where \(k \geq 3\). Further let \(M_i(G)\) denote the set of those vertices of \(G\) which are incident with exactly \(i\) triangles, \(i \geq 0\); let \(m_i(G) = |M_i(G)|\). Let \(G(r, s, g)\) be the subset of \(\mathcal{G}(s, g)\) considering of regular graphs of degree \(r\). Evidently by Euler's relation,
\[
\sum_{k=3}^{\infty} (2r + 2k - rk)p_k = 4r(1 - g).
\]
(Kotzig [12] showed that \(M_4(G) \cup M_5(G) \neq \emptyset\) for every graph \(G\) from \(\mathcal{G}(5, 3, 0)\). Zaks [17] proved the following extension of Kotzig's theorem:

**Theorem 1.** If \(G \in \mathcal{G}(5, 3, g)\), \(g \geq 0\), then
\[
m_4 + 2m_5 \geq 24(1 - g) + 3 \sum_{k=4}^{\infty} (k - 4)p_k;
\]
equality holds if and only if \(m_0 = m_1 = m_2 = 0\).

In the same paper the following Eberhard-type problem is formulated:
Let \((p_5, p_6, \ldots)\) be a finite sequence of nonnegative integers; does there exist a graph \(G \in \mathcal{G}(5, 3, 0)\) such that \(p_k(G) = p_k\) for all \(k \geq 5\) and
\[
m_4(G) + 2m_5(G) = 24 + 3 \sum_{k=5}^{\infty} (k - 4)p_k.
\]
In the present paper a solution of Zaks' problem is given and also an interesting relationship among certain subsets of $\mathcal{H}(5, 3, g)$ and subsets of $\mathcal{H}(2, g)$, $\mathcal{H}(3, 2, g)$ and $\mathcal{H}(4, 3, g)$.

2. Transformations

Two transformations play a central role in our considerations. The first is the $\mu$-transformation which maps each graph $G \in \mathcal{H}(2, g)$ into its medial graph $\mu(G)$ (cf. [14, p. 47] or [11]). The graph $\mu(G)$ is obtained from $G$ by mapping each edge $e$ of $G$ into a vertex $\mu(e)$ of $\mu(G)$ and making two vertices $\mu(e_1)$ and $\mu(e_2)$ adjacent if $e_1$ and $e_2$ are adjacent edges from the boundary of a face $F$ of $G$—see Fig. 1. $\mu(G)$ is an element of $\mathcal{H}(4, 2, g)$ and contains faces of two types: those corresponding to faces of $G$ and those corresponding to its vertices. The number of edges bounding a face $\mu(X)$ of $\mu(G)$ is equal to the number of edges bounding the face $X$ of $G$ or incident with the vertex $X$ of $G$ respectively. This means that for every $k \geq 3$,

$$p_k(\mu(G)) = p_k(G) + v_k(G).$$

(2)

It is evident from Fig. 1 that if two faces $F_1$, $F_2$ or two vertices $V_1$, $V_2$ are adjacent in $G$ then the corresponding faces of $\mu(G)$ will have a common vertex. It is further evident that if $G$ is 2-edge-connected, then $\mu(G)$ is 3-edge-connected and if $G$ is 3-edge-connected, then $\mu(G) \in \mathcal{H}(4, 3, g)$.

The second transformation which will be necessary in the sequel is the $\tau$-transformation. This transformation maps each graph $H$ from $\mathcal{H}(4, s, g)$, $s \geq 2$, $g \geq 0$ into a graph $\tau(H)$ with the following properties:

$$\tau(H) \in \mathcal{H}(5, s, g),$$

(3)

$$M_i(\tau(H)) = \emptyset \quad \text{for all } i \in \{0, 1, 2\},$$

(4)

$$p_k(\tau(H)) = p_k(H) \quad \text{for every } k \geq 4.$$  

(5)

Fig. 1.
A problem concerning 5-valent 3-connected planar graphs

The $\tau$-transformation works as follows: first the graph $H$ is mapped into $\mu(H)$; then every quadrangle of $\mu(H)$ which corresponds to a vertex of $H$ is divided into two triangles by a diagonal (cf. Fig. 2; the dashed line of Fig. 2(a) is the graph $H$, the unbroken line is $\mu(H)$ and finally $\tau(H)$ is shown in Fig. 2(b)).

Since every vertex of $\mu(H)$ is incident with two such quadrangles and the surface $S_g$ in which $\tau(H)$ is embedded is orientable it is possible to make every vertex of $\tau(H)$ 5-valent and incident with at least three triangles.

3. Results

The problem (cf. [4, p. 271] and [11]) whether for every finite sequence of nonnegative integers $(p_4, p_5, \ldots)$ there exists a graph $G$ from $\mathcal{G}(5, 3, g)$ such that $p_k(G) = p_k$ for every $k \geq 4$ remains unsolved ($p_3(G)$ is determined from (E₃)). The best known result so far is that presented by Trenkler [15] (cf. also [3]):

**Theorem 2.** If $(p_4, p_5, \ldots)$ is finite sequence of nonnegative integers such that $p_4 \geq 4$ or $p_4 \geq 2$ and $p_5 \geq 2$ then there exists a graph $G \in \mathcal{G}(5, 3, 0)$ such that $p_k(G) = p_k$ for all $k \geq 4$.

Our result is stated as

**Theorem 3.** Let $g \geq 0$ and $(p_4, p_5, \ldots)$ be a finite sequence of nonnegative integers satisfying the following conditions:

(a) $\sum_{k>5} (k-4)p_k \geq 8(g-1)$.
(b) If $g = 1$ then $(p_4, p_5, \ldots) \neq (p_4, 1)$.
(c) $p_4 \geq c$ where $c$ is a constant dependent on the sequence $(p_4, p_5, \ldots)$.

Then there exists a graph $G \in \mathcal{G}(5, 3, g)$ such that $p_k(G) = p_k$ for all $k \geq 4$ and that $M_i(G)$ is empty for every $i \in \{1, 2, 3\}$.

**Proof.** By [10] for every sequence $(p_4, p_5, \ldots)$ satisfying the conditions of Theorem 3 there exists a graph $H \in \mathcal{G}(4, 3, g)$ such that $p_k(H) = p_k$ for all $k \geq 4$. By (3), (4) and (5), $G = \tau(H)$ has the required properties. □
Remark 1. Evidently condition (a) is always satisfied for $g = 0, 1$.

Remark 2. Using the result of Enns [1] (cf. also [9]) it is possible to show that for $g = 0$

$$c = 2 \sum_{k \geq 5} p_k + \max\{ k \mid p_k \neq 0 \}.$$  

According to Trenkler [16], $c = \max_{k \geq 5} \{ k \mid p_k = 1 \pmod{2} \} - 2$.

Remark 3. Results analogous to those of Theorem 3 can be obtained by using the properties of $\tau$-transformation and results concerning graphs from $\mathcal{G}(4, 3, g)$ obtained e.g. by Enns [1] or Jucovič [9, 10, 11].

The following theorem gives the solution of Zaks' problem.

**Theorem 4.** For every finite sequence of nonnegative integers $(p_5, p_6, \ldots)$ there exists a graph $G \in \mathcal{G}(5, 3, 0)$ such that $p_k(G) = p_k$ for all $k \geq 5$ and

$$m_4(G) + 2m_5(G) = 24 + 3 \sum_{k \geq 5} (k - 4)p_k.$$  

**Proof.** It is sufficient to add a 'large enough' $p_4$ to the sequence $(p_5, p_6, \ldots)$ and then use Theorem 3 with $g = 0$. □

Interesting results concerning graphs from $\mathcal{G}(5, 3, g)$ may be obtained using known properties of graphs from $\mathcal{G}(3, 2, g)$. The following lemma plays a key role.

**Lemma 1.** If $H \in \mathcal{G}(3, 2, g)$ then there exists $G \in \mathcal{G}(5, 3, g)$ such that

(a) $p_k(G) = p_k(H)$ for all $k \geq 4$.

(b) $M_i(G)$ is empty for every $i \in \{0, 1, 2, 3\}$.

If in addition $p_3(H) = 0$ then $M_5(G)$ is also empty.

**Proof.** $G = \tau(\mu(H))$ has the required properties—cf. Fig. 3. (Fig. 3(a) shows $H$ as dashed lines and $\mu(H)$ as unbroken lines; Fig. 3(b) shows $\tau(\mu(H))$.)

**Theorem 5.** Let $(p_4, p_5, \ldots)$ be a finite sequence of nonnegative integers satisfying the condition

(a) $\sum_{k \geq 4} kp_4 = 0 \pmod{3}$, \quad $\sum_{k \neq 0 \pmod{3}} p_k \geq 3$, \quad $p_6 = c + t$, \quad $t = 0, 1, 2, \ldots$,  

or the condition

(b) $\sum_{k \geq 4} kp_k = 0 \pmod{3}$, \quad $\sum_{k \neq 0 \pmod{3}} p_k < 3$, \quad $p_6 = c + 2t$, \quad $t = 0, 1, 2, \ldots$,  

where $c$ is a constant dependent on the sequence $(p_4, p_5, \ldots)$.
Then there exists a graph \( G \in \mathcal{G}(5, 3, 0) \) such that \( p_k(G) = p_k \) for all \( k \geq 4 \) and \( M_i(G) \) is empty for all \( i \in \{0, 1, 2, 3\} \).

**Proof.** By [6, 7] there exists for every sequence of nonnegative integers \((p_3, p_4, p_5, \ldots)\) such that
\[
p_3 = 4 + \frac{1}{3} \sum_{k=4}^\infty (k - 6)p_k,
\]
whose remaining elements satisfy conditions (a) or (b), a graph \( H \in \mathcal{G}(3, 3, 0) \) such that
\[
p_k(H) = p_k \quad \text{for all} \quad k \geq 3.
\]
The rest follows from Lemma 1. \( \square \)

**Remark 4.** The constant \( c \) of Theorem 5 can be estimated using results of Fisher [2], Grünbaum [5], Kraeft [13] and Jucovič [11].

Using Lemma 1 and known properties of graphs from \( \mathcal{G}(3, 2, g) \) (see e.g. [6–8]), various theorems analogous to Theorem 3 may be formulated. An example of such a result is

**Theorem 6.** Let \((p_4, p_5, \ldots)\) be a finite sequence of nonnegative integers such that
\[
p_4 = 0, \quad p_5 = 12 + \sum_{k=7}^\infty (k - 6)p_k, \quad p_6 \geq 8.
\]
Then there exists a graph \( G \in \mathcal{G}(5, 3, 0) \) such that \( p_k(G) = p_k \) for all \( k \geq 4 \) and \( M_i(G) \) is empty for all \( i \in \{0, 1, 2, 3, 5\} \).

**Proof.** By [5] there exists for every sequence of nonnegative integers satisfying
the conditions of this theorem a graph \( H \in \mathcal{G}(3, 3, 0) \) such that
\[
p_3(H) = p_4(H) = 0, \quad p_k(H) = p_k \quad \text{for all } k \geq 5.
\]
To this graph Lemma 1 may be applied.

Let \( \mathcal{G}^*(5, 3, g) = \{ G \mid G \in \mathcal{G}(5, 3, g), M_i(G) = 0 \text{ for } i \in \{0, 1, 2, 3, 5\} \} \) and also
\( \mathcal{K}(3, 2, g) = \{ H \mid H \in \mathcal{G}(3, 2, g), p_3(H) = 0 \} \).

**Theorem 7.** The sets of graphs \( \mathcal{G}^*(5, 3, g) \) and \( \mathcal{K}(3, 2, g) \) are equivalent.

**Proof.** In one direction this is a direct consequence of Lemma 1. As for the opposite direction, let \( G \in \mathcal{G}^*(5, 3, g) \); suppose that there is an arbitrary \( n \)-gon of \( G \) with vertices \( A_1, A_2, \ldots, A_n \). Let \( B_i \) be a vertex of \( G \) adjacent to \( A_i \) and \( A_{i+1} \), \( C_i \) a vertex of \( G \) adjacent to \( B_{i-1}, A_i, B_i \) for \( i = 1, 2, \ldots, n \) (in these definitions put \( i = 1 \) instead of \( i = n + 1 \) and \( i = n \) instead of \( i = 0 \)).

A face of \( G \) which is not incident with \( A_i \) but is incident with \( C_i \) and \( B_{i-1} \) (\( B_i \), neither of the two) will be called \( \alpha_i, \beta_i, \) or \( \gamma_i \) respectively—see Fig. 4. Evidently exactly one of the faces \( \alpha_i, \beta_i, \gamma_i \), \( i = 1, 2, \ldots, n \) is not a triangle. We will show that \( \gamma_i \) is a triangle.

If \( \gamma_i \) is not a triangle, then \( \alpha_i \) and \( \beta_i \) must be triangles. Let \( D_i \) be a vertex incident with \( \beta_i \) and \( \gamma_i \) and distinct from \( C_i \). Since \( D_i \in M_4(G) \) there must exist, in addition to the edge \( D_iB_i \), also the edge \( D_iC_{i+1} \). Thus \( B_i \in M_5(G) \) which is a contradiction since \( M_5(G) \) is empty by hypothesis.

If \( \alpha_i \) is not a triangle for some \( i \), then this must be true for all \( i = 1, 2, \ldots, n \).

Since \( \omega \) was any triangular face, the overall structure of \( G \) must be that shown in Fig. 3(b).

We shall further show that there is an inverse transformation to both \( \tau \)- and \( \mu \)-transformations which map any graph \( G \in \mathcal{G}^*(5, 3, g) \) into a graph \( H \in \mathcal{K}(3, 2, g) \). Note that the set of all triangles of \( G \) may be divided into two subsets. Subset 1 consists of those triangles which have exclusively triangular neighbors (e.g., all triangles \( A_iB_iC_i \)). Subset 2 consists of triangles with one non-triangular neighbor, e.g., triangles \( A_{i-1}A_iB_{i-1}, A_iB_{i-2}C_i, \ldots \). Evidently there are no other
triangles in G. Each triangle from Subset 2 has one triangular neighbor from Subset 1 and one from Subset 2. Since there is an edge common to two triangles from Subset 2 incident with every vertex, we may consider these edges as added in the second step of the \( \tau \)-transformation. By deleting all these edges, we change G into \( G' \), all of whose vertices are 4-valent. After the deletion the two triangles from Subset 2 form one quadrangle which can be considered as corresponding to a suitable 4-valent vertex of \( \mu(H) \) in the first step of the \( \tau \)-transformation. We continue these considerations until we get a graph \( H \in \kappa(3, 2, g) \). This completes the proof. □

**Remark 5.** Let \( \mathcal{G}(5, 3, g) = \{ G \mid G \in \mathcal{G}(5, 3, g), M_i(G) = 0 \text{ for } i \in \{0, 1, 2, 3\} \} \). Then \( \mathcal{G}(5, 3, g) \) and \( \mathcal{G}(3, 2, g) \) are probably also equivalent.

**Remark 6.** Note that if \( H \in \mathcal{G}(3, 3, g) \) then \( G = \mu(H) \) is from \( \mathcal{G}(4, 3, g) \) and moreover, \( M_i(G) \) is empty for \( i = 0, 1 \). This fact is useful in applying analogous proof techniques to proving theorems analogous to Theorem 3. For example,

**Theorem 8.** If a finite sequence of nonnegative integers satisfies the conditions of Theorem 5 (Theorem 6) then there exists a graph \( G \in \mathcal{G}(4, 3, 0) \) such that \( p_k(G) = p_k \) for all \( k \geq 4 \) and \( M_i(G) \) is empty for all \( i \) from \( \{0, 1\} \) (or \( \{0, 1, 3\} \) respectively).

**References**