Homogenization of the Euler system in a 2D porous medium

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Abstract

We study the homogenization of the Euler system in a periodic porous medium (of period $\varepsilon$) by using the notion of two-scale convergence. At the limit, we recover a system which couples a cell problem with the macroscopic one.

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1. Introduction

We define a porous medium as the periodic repetition of an elementary cell of size $\varepsilon$ in a domain $\Omega$. The domain $\Omega$ is a bounded domain of $\mathbb{R}^2$, the periodic torus $\Omega = T$ or the whole space $\Omega = \mathbb{R}^2$. The solid part $\mathcal{Y}_\varepsilon$ of the porous medium is also of size $\varepsilon$. 
Hence, the domain $\Omega_\varepsilon$ is defined as the intersection of $\Omega$ with the fluid part. We consider an incompressible perfect fluid governed by the Euler equation. We consider the following system of equations:

$$\begin{cases}
\partial_t u^\varepsilon + \varepsilon u^\varepsilon \cdot \nabla u^\varepsilon = -\nabla p^\varepsilon + f^\varepsilon(x), \\
\text{div}(u^\varepsilon) = 0, \\
u^\varepsilon \cdot n = 0 \quad \text{on } \partial \Omega_\varepsilon, \\
u^\varepsilon|_{t=0} = u_0^\varepsilon,
\end{cases} \tag{1}$$

where $u^\varepsilon$ is the velocity, $p^\varepsilon$ is the pressure, $f^\varepsilon$ is an exterior force and $n$ is the outward normal vector to $\Omega_\varepsilon$. The derivation of the system (1) will be given later. Arguing as in the book of A. Bensoussan, J.-L. Lions and G. Papanicolaou [3] (see also [4]) and the book of E. Sanchez-Palencia [14], we make an asymptotic development using both a microscopic scale and a macroscopic scale. Hence, we can derive a (formal) limit system. Indeed taking $u^\varepsilon$ of the form,

$$u^\varepsilon = u^0(t, x, x/\varepsilon) + \varepsilon u^1(t, x, x/\varepsilon) + \cdots,$$

we get formally the following system for $v(t, x, y) = u^0(t, x, y),

$$\begin{cases}
\partial_t v + v \cdot \nabla_y v = -\nabla_x p(x, y) - \nabla_x q(x) + f(t, x, y), \\
\text{div}_y(v) = 0, \\
\text{div}_x(\int_{\mathcal{Y}_f} v(x, y) \, dy) = 0, \\
v(x, y) \cdot n = 0 \quad \text{on } \Omega \times \partial \mathcal{Y}_f, \\
(\int_{\mathcal{Y}_f} v(x, y) \, dy) \cdot n = 0 \quad \text{on } \partial \Omega, \\
v|_{t=0} = v_0,
\end{cases} \tag{2}$$

where $\mathcal{Y}_f$ is the fluid part (which will be defined below), $f(t, x, y)$ and $v_0(x, y)$ are the two-scale limits of the sequences $f^\varepsilon$ and $u_0^\varepsilon$ and here $n$ is the inward normal vector to $\mathcal{Y}_f$. The precise notion of two-scale convergence will be recalled in Section 1.3. In the sequel, we denote $\mathfrak{v}(t, x) = \int_{\mathcal{Y}_f} v(t, x, y) \, dy$.

This paper is devoted to the rigorous proof of convergence of the solutions to the initial system (1) toward a solution to the limit system (2). We will also explain the difficulties encountered if we replace the Euler system by the Navier–Stokes one with Dirichlet boundary conditions due to the presence of some boundary layers. In the next subsection, we give a precise formulation of the problem, than we recall the notion of two scale convergence. In the second part, we prove the existence of solutions for the initial system as well as for the limit system. In the third section, we give a proof of the two-scale convergence. Section four is devoted to the analysis of the Navier–Stokes case. We will explain why we cannot prove a similar result in the Navier–Stokes case with Dirichlet boundary condition. This is indeed linked to the problem of uniqueness of weak solutions to the Navier–Stokes system which is still an open problem in the 3D case. Finally, in Appendix A, we give the construction of some special functions which are used to deal with the 2D-Euler system in a domain with holes.
1.1. Formulation of the problem

Let $Y = [0, 1]^2$ be a unit open cube of $\mathbb{R}^2$. Let $Y_s$ (the solid part) be a closed subset of $Y$. We assume that $Y_s = \bigcup_{i=1}^{N} h_i$ where for all $i$, $1 \leq i \leq N$, the interior of $h_i$ is a simply connected regular domain and the $h_i$ are disjoint closed sets of $Y$. In the sequel $N$ will be called the number of holes and $h_1, h_2, \ldots, h_N$ are the $N$ holes. Then, we define the fluid part by $Y_f = Y - Y_s$. By repeating the domain $Y_f$ by $Y$ periodicity we get the fluid domain $E_f$, which can also be defined as

$$E_f = \{(y_1, y_2) \in \mathbb{R}^2 \mid \exists (k_1, k_2) \in \mathbb{Z}^2, \text{ such that } (y_1 - k_1, y_2 - k_2) \in Y_f\}. \quad (3)$$

In the same way, we can define the solid part $E_s = \mathbb{R}^2 - E_f$,

$$E_s = \{(y_1, y_2) \in \mathbb{R}^2 \mid \exists (k_1, k_2) \in \mathbb{Z}^2, \text{ such that } (y_1 - k_1, y_2 - k_2) \in Y_s\}. \quad (4)$$

It is easy to see that $E_f$ is a connected domain, while $E_s$ is formed by separate holes. In the sequel, we denote $Y_k = Y + k$, $Y_{s,k} = Y_s + k$, $h_{i,k} = h_i + k$ and $Y_{f,k} = Y_f + k$ for all $k \in \mathbb{Z}^2$. Hence, for all $\varepsilon$, we can define the domain $\tilde{\Omega}_\varepsilon$ as the intersection of $\Omega$ with the fluid domain rescaled by $\varepsilon$, namely $\tilde{\Omega}_\varepsilon = \Omega \cap \varepsilon E_f$. However, for some technical reasons and to get a Lipschitz connected domain, we have to remove the solid parts which intersect the boundary so instead of working in $\tilde{\Omega}_\varepsilon$, we define

$$\Omega_\varepsilon = \Omega - \bigcup \{\varepsilon Y_{s,k}, \text{ where } k \in \mathbb{Z}^2, \varepsilon Y_{s,k} \subset \Omega\}$$

and we notice that $\tilde{\Omega}_\varepsilon \subset \Omega_\varepsilon$. We denote $K_\varepsilon = \{k \mid \varepsilon Y_{s,k} \subset \Omega\}$.

**Remark 1.1.** We point out that it is also possible to just remove the holes which intersect the boundary and take $\Omega_\varepsilon = \Omega - \bigcup \{\varepsilon h_{i,k}, 1 \leq i \leq N, k \in \mathbb{Z}^2, \varepsilon h_{i,k} \subset \Omega\}$. The results, we are going to prove also apply to this case. As we will see from the proof one of the essential requirements about the domain is that it satisfies the following estimate:

$$\|P u\|_{L^p(\Omega_\varepsilon)} \leq C \|u\|_{L^p(\Omega_\varepsilon)},$$

where $P$ denotes the operator of projection onto divergence-free vectors and $C$ is a constant which is independent of $\varepsilon$ (see Masmoudi [8]).

1.2. Scaling

In this subsection, we give the scaling which yields the system (1). Indeed, let us start from the Navier–Stokes system (or the Euler system $\alpha = 0$),

$$\partial_t v + v \cdot \nabla v - \alpha \nabla v + \nabla q = 0.$$  

Then, taking $v = \varepsilon u$ and $\alpha = \varepsilon^2 v$, we get:

$$\partial_t u + \varepsilon u \cdot \nabla u - \varepsilon^2 v \Delta u + \nabla q = 0 \quad (5)$$

which is Eq. (1) with $v = 0$.  


There are more general scalings which yield Eq. (1). Indeed, taking $v(t, x) = \lambda u(t/\mu, x)$ and $\alpha \mu = \varepsilon^2 \lambda$ with $\mu \lambda = \varepsilon$, we see that $u$ satisfies (5). Notice that the scaling given above corresponds to the case $\lambda = \varepsilon$ and $\mu = 1$.

In Sections 2 and 3, we will only deal with the Euler case, namely $\nu = 0$. In Section 4, we will give some remarks relative to the Navier–Stokes case.

1.3. Two-scale convergence

The notion of two-scale convergence is aimed at a better description of sequences of oscillating functions with a known scale. It was introduced by G. Nguetseng [12,13] and later extended by G. Allaire [1] where one can find the mathematical setting we use here.

Definition 1.2. Let $u^\varepsilon$ be a sequence of functions such that $u^\varepsilon \in L^2(\Omega^\varepsilon)$ and $\|u^\varepsilon\|_{L^2(\Omega^\varepsilon)}$ is bounded uniformly in $\varepsilon$. If $v(x, y) \in L^2(\Omega \times \mathcal{Y}_f)$, then we say that $u^\varepsilon$ two-scale converges to $v$ if and only if $\forall \psi \in C(\Omega \times \mathcal{Y}_f)$, we have

$$\lim_{\varepsilon \to 0} \int_{\Omega^\varepsilon} u^\varepsilon(x) \psi\left(x, \frac{x}{\varepsilon}\right) dx = \int_{\Omega \times \mathcal{Y}_f} v(x, y) \psi(x, y) dx dy.$$  (6)

Moreover, we say that $u^\varepsilon$ two-scale converges strongly to $v$ if and only if $v(x, y) \in L^2(\Omega, C(\mathcal{Y}_f))$ and we have:

$$\lim_{\varepsilon \to 0} \left\| u^\varepsilon(x) - v\left(x, \frac{x}{\varepsilon}\right) \right\|_{L^2(\Omega^\varepsilon)} = 0,$$  (7)

and

$$\lim_{\varepsilon \to 0} \left\| u^\varepsilon(x) \right\|_{L^2(\Omega^\varepsilon - \tilde{\Omega}_\varepsilon)} = 0.$$  (8)

Remark 1.3. (1) We notice that if $x \in \tilde{\Omega}_\varepsilon$ then $x/\varepsilon \in E_f$ and that if we prolong $v(x, y)$ by 0 if $y \in \mathcal{Y}_f$, then the two conditions in the definition of the strong two-scale convergence are equivalent to

$$\lim_{\varepsilon \to 0} \left\| u^\varepsilon(x) - v\left(x, \frac{x}{\varepsilon}\right) \right\|_{L^2(\tilde{\Omega}\varepsilon)} = 0.$$  (9)

(2) We also point out that if $v(x, y) \in L^2(\Omega, C(\mathcal{Y}_f))$ then $u^\varepsilon$ two-scale converges to $v$ and $\|u^\varepsilon(x)\|_{L^2(\tilde{\Omega}\varepsilon)}$ converges to $\|v(x, y)\|_{L^2(\Omega \times \mathcal{Y}_f)}$ if and only if $u^\varepsilon$ two-scale converges strongly to $v$.

(3) The notion of two-scale convergence can be extended to the case $u^\varepsilon$ also depends on time, $u^\varepsilon \in L^2((0, T); \Omega^\varepsilon)$. Then, we say that $u^\varepsilon$ two-scale converges to $v \in L^2((0, T) \times \Omega \times \mathcal{Y}_f)$ if and only if $\forall \psi \in C((0, T) \times \Omega \times \mathcal{Y}_f)$, we have:
\[
\lim_{\varepsilon \to 0} \int_0^T \int_{\Omega_\varepsilon} u^\varepsilon(t, x) \psi \left( t, x, \frac{x}{\varepsilon} \right) \, dx \, dt = \int_0^T \int_{\Omega \times Y_f} v(t, x, y) \psi(t, x, y) \, dx \, dy. \tag{10}
\]

1.4. Statement of the results

We will state two results. The first one concerns the Cauchy problem for the limit system and the second one concerns the convergence of a sequence of the solutions to (1) toward a solution to (2). We start by defining the following functional spaces:

\[
A = \left\{ v(x, y), \ v \in L^2(\Omega \times Y_f), \ \text{div}_y(v) = 0, \ \text{div}_x(v) = 0, \ v\cdot n = 0 \text{ on } \partial \Omega \times Y_f, \ \bar{v} \cdot n = 0 \text{ on } \partial \Omega \right\},
\]

\[
A_\infty = \left\{ v(x, y), \ v \in A \text{ and } \text{curl}_y(v) \in L^\infty(\Omega \times Y_f) \right\},
\]

where \( \text{div}_y \) and \( \text{div}_x \) denote respectively the divergence in the \( y \) and in the \( x \) variables, namely \( \text{div}_y(v) = \partial_{y_1} v_1 + \partial_{y_2} v_2 \) and \( \text{div}_x(v) = \partial_{x_1} v_1 + \partial_{x_2} v_2 \). Moreover, \( \bar{v} \) denotes the integral of \( v \) over \( Y_f \), namely \( \bar{v}(x) = \int_{Y_f} v(x, y) \, dy \). Finally, \( n \) denotes the exterior normal vector to \( \partial Y_f \) or to \( \partial \Omega \).

Now, we give an existence result for the limit system (2):

**Theorem 1.4.** Take \( v_0 \in A_\infty \) and \( f \in L^1((0, \infty); A_\infty) \). Then, there exists a global solution to the system (2) such that

\[
v \in C([0, \infty); A) \cap L^\infty((0, \infty); A_\infty). \tag{13}
\]

This result is similar to the existence result for the incompressible Euler system by V.-I. Yudovich [15]. However, unlike Yudovich solutions, the uniqueness of the solutions constructed in Theorem 1.4 is not known.

Now, we focus on the convergence result. We have to assume that \( u^0_\varepsilon \) is bounded in \( L^3(\Omega_\varepsilon) \), \( \text{div}(u^0_\varepsilon) = 0 \), \( u^0_\varepsilon \cdot n = 0 \) on \( \partial \Omega_\varepsilon \), \( \varepsilon \text{ curl}(u^0_\varepsilon) \) is in \( L^\infty \) (which implies the existence and uniqueness for the initial system) and that \( u^0_\varepsilon \) two-scale converges strongly to \( v_0 \) where \( v_0 \in A_\infty \). Moreover, we assume that \( f^\varepsilon \) is divergence-free, that it is bounded in \( L^1((0, \infty); L^3(\Omega_\varepsilon)) \), that \( \text{curl} f^\varepsilon \) is bounded in \( L^1((0, \infty); L^\infty(\Omega_\varepsilon)) \) and that \( f^\varepsilon \) two-scale converges strongly to \( f \), namely,

\[
\lim_{\varepsilon \to 0} \left\| u^0_\varepsilon(x) - v_0 \left( x, \frac{x}{\varepsilon} \right) \right\|_{L^2(\Omega_\varepsilon)} = 0, \tag{14}
\]

\[
\lim_{\varepsilon \to 0} \left\| f^\varepsilon(t, x) - f(t, x, \frac{x}{\varepsilon}) \right\|_{L^1((0, \infty); L^2(\Omega_\varepsilon))} = 0, \tag{15}
\]

where \( v_0 \) and \( f \) satisfy the hypotheses of Theorem 1.4. Here, we only take the two-scale convergence in the \( x \) variable, then we have:
Theorem 1.5. Under the above conditions there exists a sequence \( u^\varepsilon \) of solutions to the initial system (1). Moreover, extracting a subsequence if necessary \( u^\varepsilon \) two-scale converges to \( v \) where \( v \) is a solution to the limit system (2).

In Section 2, we deal with the existence theory for the limit system (Theorem 1.4) and prove results quite similar to the results known for the incompressible Euler system. In Section 3, we deal with the convergence proof (Theorem 1.5).

2. Existence of solutions

The existence of global solutions for the initial system can be deduced from classical results concerning the existence for the Euler system in 2D [15]. Indeed, the assumptions we made on \( u^\varepsilon_0 \) and \( f^\varepsilon \) yield the existence and uniqueness of a solution to the Euler system.

The rest of this section will focus on the existence of solutions to the limit system. For the limit system, and to our knowledge, no general existence result is known. We can cite a result of A. Mikelić and L. Paoli [11] where the existence is proved under the condition \( \text{curl}_y v_0 = 0 \) which reduces the system to a linear one. The difficulty in proving the existence of global solutions to the limit system is the passage to the limit in the nonlinear terms due to the lack of compactness in the \( x \) variable. To overcome this difficulty we will prove that if \( v_n \) is a sequence of solutions to (2) such that \( v_n(t = 0) \) converges strongly then oscillations cannot occur. This type of ideas was used by the authors in [6] to prove an existence result for some Oldroyd type fluids. To do so, we need an other formulation of the problem which allows us to keep track of the oscillations.

2.1. An auxiliary problem

To distinguish between functions which are periodic in \( \mathcal{Y} \) and those which are not, we recall the definition of the periodic torus \( T = \mathbb{R}^2 / \mathbb{Z}^2 \). We also recall that \( L^2 \) functions defined in the torus are the same as the \( L^2 \) functions defined in the cube \( \mathcal{Y} \), namely \( L^2(\mathcal{Y}) = L^2(T) \). However, \( H^1(T) \subsetneq H^1(\mathcal{Y}) \) where \( H^1 \) denotes the classical Sobolev space. We also denote \( T_f = T - \mathcal{Y}_f \) the periodic domain without the solid part, hence, \( \partial T_f = \partial \mathcal{Y}_f \) and we recall that \( \partial \mathcal{Y}_f = \partial \mathcal{Y} \cup \partial \mathcal{Y}_s \). We start by studying a problem in the cell \( T_f \) and we define:

\[
H = \{ u \in \left( L^2(T_f) \right)^2 \mid \text{div}(u) = 0 \text{ in } T_f \text{ and } u.n = 0 \text{ on } \partial T_f \}. 
\]  

(16)

We insist on the fact that we do not assume that the averages of functions of \( H \) vanish as it is usually done when working in the periodic case.

Let \( \Phi \) be defined from \( H \) to \( H^{-1} \) by:

\[
\Phi : H \rightarrow H^{-1}, \quad u \mapsto \text{curl}(u).
\]  

(17)

It is easy to prove (see Appendix A) that \( \tilde{H} = \ker(\Phi) \) is of dimension \( N + 1 \) where \( N \) is the number of holes. Moreover, we can construct \( V^1, V^2, \ldots, V^{N+1} \) a basis of \( \ker(\Phi) \).
such that for each $i$, $1 \leq i \leq N + 1$, $V^i = \nabla^\perp \psi^i$, where $\Delta \psi^i = 0$ and $\psi^i |_{h_j}$ is a constant depending only on $i$ and the hole $h_j$. Besides,

- if $i = 1$ or $i = 2$, then $\psi^i \in L^2(Y_f)$,
- if $i \geq 3$, then $\psi^i \in L^2(T_f)$.

Moreover, it is possible to choose $V^1$ and $V^2$ such that, we can find $\tilde{\psi}^1, \tilde{\psi}^2 \in L^2(T_f)$ and $\psi^1(y_1, y_2) = \tilde{\psi}^1(y_1, y_2) - y_2$, $\psi^2(y_1, y_2) = \tilde{\psi}^2(y_1, y_2) + y_1$. This basis allows us to give the following decomposition of every vector $u$:

**Proposition 2.1.** Let $u \in L^2(T_f)^2$ then there exist $\tilde{u} \in H^1$, $N + 1$ constants $a_1, a_2, \ldots, a_n$ and a potential $p \in L^2(T_f)$ such that

$$u = \tilde{u} + \hat{u} + \nabla p = \tilde{u} + \sum_{i=1}^{N+1} a_i V^i + \nabla p$$

(18)

where $\tilde{u} = \nabla^\perp \Delta^{-1}\text{curl}(u)$. Moreover, the decomposition is unique and is orthogonal in $L^2(T_f)^2$.

In the above proposition $\Delta^{-1}f$ stands for the solution of the Laplace problem in $T_f$ with Dirichlet boundary condition, namely $g = \Delta^{-1}f$ is the unique solution of

$$\begin{cases}
\Delta g = f & \text{in } T_f, \\
g = 0 & \text{on } \partial h_i, \forall i.
\end{cases}$$

(19)

The proof of this proposition is very simple. We define $p$ as the unique weak solution in $H^1(T_f)$ of the following Laplace equation:

$$\begin{cases}
\Delta p = \text{div}(u) & \text{in } T_f, \\
\frac{\partial p}{\partial n} = -u \cdot n & \text{on } \partial h_i, \forall i,
\end{cases}$$

(20)

which can be rewritten in the following weak sense

$$\int_{T_f} \nabla p \cdot \nabla \phi \, dx = \int_{T_f} u \cdot \nabla \phi \, dx,$$

(21)

for all $\phi \in H^1(T_f)$.

Then defining $\hat{u}$ as in the proposition, we see that $u - \tilde{u} - \nabla p \in H$ and $u - \tilde{u} - \nabla p \in \ker(\Phi)$. We also notice that if we take $u \in H$ then $p = 0$. This allows us to define an orthogonal decomposition of $H$, namely $H = \tilde{H} + \hat{H}$ where $\tilde{H} = \ker(\Phi)$. 
2.2. Reformulation of the problem

We will rewrite the system (2) using the above decomposition. For all \( t, x \in \Omega \) and \( i, 1 \leq i \leq N + 1 \), we define the functions \( a_i(t, x) \) and \( b_i(t, x) \) such that

\[
v(t, x, y) = \tilde{v}(t, x, y) + \sum_{i=1}^{N+1} a_i(t, x) V_i(y) \tag{22}\]

and

\[
f(t, x, y) = \tilde{f}(t, x, y) + \sum_{i=1}^{N+1} b_i(t, x) V_i(y) + \nabla_y p(x, y). \tag{23}\]

In the sequel, \( a \) will denote the \((N + 1)\)-vector \((a_1, a_2, \ldots, a_{N+1})\) and \( \text{div}(a) \) will denote:

\[
\text{div}(a) = \partial_{x_1} a_1 + \partial_{x_2} a_2. \tag{24}\]

We also recall that

\[
\text{div}_x \int_Y v \ dy = \text{div}_x \left( a_1(x)e_1 + a_2(x)e_2 \right) = \text{div}(a)
\]

and \( \nabla_{N+1} q = (\partial_{x_3} q, \partial_{x_4} q, 0, \ldots, 0) \). We also set \( \partial_{x_i} q = 0 \) if \( i \geq 3 \).

**Proposition 2.2.** There exist linear maps \( E^i, F_{ij} \) and bilinear maps \( E_i \) for all \( i, j \), \( 1 \leq i, j \leq N + 1 \), defined on the following spaces:

\[
E^i : H \cap H^1 \to H, \\
F_{ij} : H \to \mathbb{R}, \\
E_i : H \times H \to \mathbb{R}, \tag{25}
\]

such that \( v \in L^\infty(0, T; \mathcal{A}_\infty) \) is a solution of (2) if and only if \((\tilde{v}, a_1, a_2, \ldots, a_{N+1})\) is a solution of

\[
\begin{aligned}
&\partial_t \tilde{v} + \nabla^\perp \Delta^{-1} \text{curl}[	ilde{v}, \nabla \tilde{v}] + \sum_j a_j E^j(\tilde{v}) = \tilde{f} \quad \text{in } \Omega \times T_f, \\
&\partial_t a_i + \sum_j F_{ij}(\tilde{v}) a_j + E_i(\tilde{v}, \tilde{v}) = b_i \quad \text{in } \Omega, \ 3 \leq i \leq N + 1, \\
&\partial_t a_i + \sum_j F_{ij}(\tilde{v}) a_j + E_i(\tilde{v}, \tilde{v}) + K_{i1} \partial_{x_1} q + K_{i2} \partial_{x_2} q = b_i \quad \text{in } \Omega, \ 1 \leq i \leq 2, \\
&\text{div}(a) = 0,
\end{aligned} \tag{26}
\]

where \( K \) is the permeability matrix which will be defined in Appendix A.
**Proof.** Before starting the proof, we remark that $E^i, F_{ij}$ and $E_i$ only act on the $y$ variable. Now, taking the curl of the momentum equation and then applying the operator $\nabla^\perp \Delta^{-1}$, we get:

$$\partial_t \tilde{v} + \nabla^\perp \Delta^{-1} \text{curl}[v, \nabla v] = \tilde{f}. \quad (27)$$

Using the decomposition of $v = \tilde{v}(t, x, y) + \sum_{i=1}^{N+1} a_i(t, x)V^i(y)$, we get:

$$\nabla^\perp \Delta^{-1} \text{curl}(v, \nabla v) = \nabla^\perp \Delta^{-1} \text{curl}(\tilde{v}, \nabla \tilde{v})$$

$$+ \sum_{i=1}^{N+1} a_i \nabla^\perp \Delta^{-1} \text{curl}(\tilde{v}, \nabla V^i + V^i \nabla \tilde{v}) \quad (28)$$

where, we have used that

$$\text{curl}(\tilde{v}, \nabla \tilde{v}) = \text{curl}\left(\frac{1}{2} |\tilde{v}|^2 + \text{curl}(\tilde{v} \times \tilde{v})\right) = 0 \quad (29)$$

with $\tilde{u} = \sum_{i=1}^{N+1} a_i(t, x)V^i(y)$. Next, denoting

$$E^i(\tilde{v}) = \nabla^\perp \Delta^{-1} \text{curl}(\tilde{v}, \nabla V^i + V^i \nabla \tilde{v}),$$

we get the first part of the system (26). Besides, multiplying the momentum equation by $V^i$ and integrating over $\mathbb{T}_f$, we get the second equation for $3 \leq i \leq N + 1$. Indeed,

$$\int_{\mathbb{T}_f} \tilde{v} V^i \, dy = \int_{\mathbb{T}_f} \nabla^\perp \psi. \nabla^\perp \psi^i \, dy = \int_{\mathbb{T}_f} \psi \Delta \psi^i = 0. \quad (30)$$

where we have integrated by part and used that $\psi, \nabla \psi^i$ are periodic and that $\psi = 0$ on $\partial \mathbb{T}_f$. Moreover,

$$E^i(\tilde{v}, \tilde{v}) = \int_{\mathbb{T}_f} (\tilde{v}, \nabla \tilde{v}) V^i \, dy = - \int_{\mathbb{T}_f} (\tilde{v}, \nabla V^i) \tilde{v} \, dy \quad (31)$$

and

$$F_{ij}(\tilde{v}) = \int_{\mathbb{T}_f} (\tilde{v}, \nabla V^j + V^j, \nabla \tilde{v}) V^i \, dy$$

$$= \int_{\mathbb{T}_f} (\tilde{v}, \nabla V^j) V^i - (V^j, \nabla \tilde{v}) \tilde{v} \quad (32)$$

$$= \int_{\mathbb{T}_f} (\tilde{v}, \nabla V^j) V^i \quad (33)$$
since $V^j \nabla V^i$ is a gradient. Notice that if $3 \leq i, j \leq N + 1$, then $F_{ij}(\tilde{v}) = -F_{ji}(\tilde{v})$.

For $i = 1$ or $2$, we have to multiply the equation by $u^1$ or $u^2$ and the definition of $E_i(\tilde{v}, \hat{v})$ and $F_{ij}(\tilde{v})$ should be replaced by:

$$E_i(\tilde{v}, \hat{v}) = -\int_{\Omega} (\tilde{v}, \nabla u^i) \tilde{v} \, dy,$$

and

$$F_{ij}(\tilde{v}) = \int_{\Omega} (\tilde{v}, \nabla V^j) u^i.$$ (35) (36)

Using the more compact decomposition $v = \tilde{v} + \hat{v}$ and $f = \tilde{f} + \hat{f}$, we get the following proposition (we recall that $v = \int_{\Omega} v = \int_{\Omega} \tilde{v} = e_1 e_1 + e_2 V^2$).

**Proposition 2.3.** There exist bilinear maps $E^0$, $F$ and $E_0$ defined on the following spaces:

$$E^0 : \tilde{H} \cap H^1 \times \hat{H} \rightarrow \tilde{H},$$

$$F : \tilde{H} \times \hat{H} \rightarrow \hat{H},$$

$$E_0 : \tilde{H} \times \tilde{H} \rightarrow \tilde{H},$$

such that $v \in L^\infty(0, T; A^\infty)$ is a solution of (2) if and only if $(\tilde{v}, \hat{v})$ is a solution of

$$\begin{cases} \partial_t \tilde{v} + \nabla^\perp \Delta^{-1} \text{curl}(\tilde{v} \nabla \tilde{v}) + E^0(\tilde{v}, \hat{v}) = \tilde{f} & \text{in } \Omega \times T_f, \\ \partial_t \hat{v} + F(\tilde{v}, \hat{v}) + E_0(\tilde{v}, \hat{v}) = -\partial_{x_1} q u^1 + \partial_{x_2} q u^2 + \hat{f} & \text{in } \Omega \times T_f, \\ \text{div}_x(\tilde{v}) = 0 & \text{in } \Omega \times T_f. \end{cases}$$ (37) (38)

and $(\tilde{v}, \hat{v}) \in L^\infty((0, T) \cap L^\infty(0, T; A^\infty) \times \tilde{H} \times \hat{H}) \times L^\infty((0, T); L^2(\Omega; \hat{H}))$.

In the above system $q$ is just the Lagrange multiplier associated to the constraint $\text{div}_x(\tilde{v}) = 0$. The proof of the above proposition is very simple. We denote,

$$E^0(\tilde{v}, \hat{v}) = \sum_i a_i E_i(\tilde{v}),$$

and

$$E_0(\tilde{v}, \hat{v}) = \sum_i E_i(\tilde{v}, \hat{v}) V^i.$$ (39) (40)

We also use that $\sum_{i,j=1}^2 K_{ij} \partial_{x_i} q V^j = \partial_{x_1} q u^1 + \partial_{x_2} q u^2$ (see Appendix A). Moreover, the conservation of the energy yields the following conservation properties for all $\tilde{v} \in \tilde{H}$ and $\hat{v} \in \hat{H}$, we have:
\[
\int_{\mathcal{Y}_f} E_0(\tilde{v}, \hat{v}).\tilde{v} + E_0(\tilde{v}, \hat{v}).\hat{v} \, dy = 0,
\]
(41)

\[
\int_{\mathcal{Y}_f} F(\tilde{v}, \hat{v}).\hat{v} \, dy = 0.
\]
(42)

2.3. Compactness

As usual in proving the existence of weak solutions, we have to prove the compactness of a sequence of solutions. We take a sequence \(v^n\) of solution of the limit system (2) in \(L^\infty(0, T; A_\infty)\) with the initial data \(v^n(t = 0) = v_0^n\) bounded in \(A_\infty\) and the force \(f^n\) bounded in \(L^1((0, \infty); A_\infty)\) such that

\[
v_0^n \to v_0 \text{ in } L^2(\Omega \times \mathcal{Y}_f) \quad \text{and} \quad f_n \to f \text{ in } L^1((0, \infty); L^2(\Omega \times \mathcal{Y}_f)).
\]
(43)

We want to prove that \(v_n\) converges to a solution of (2). The only problem is the passage to the limit in the nonlinear terms and we will use a method based on defect measures to prove some compactness which will allow us to pass to the limit in the products. Using Proposition 2.1, we can decompose \(v^n\) as

\[
v^n = \tilde{v}^n + \hat{v}^n = \tilde{v}^n + \sum_{i=1}^{N+1} a^n_i V_i.
\]

Then, using the energy bound, we can extract a subsequence (still denoted by \(v^n\)) which converges weakly-* to \(v = \tilde{v} + \sum_{i=1}^{N+1} a_i V_i\) in \(L^\infty((0, T); L^2(\Omega \times \mathcal{Y}_f))\) for all \(T > 0\). The fact that \(\text{curl}_y v^n\) is bounded yields some compactness in the \(y\) variable and the use of the equation yields some compactness in \(t\), however, we have only an \(L^2\) bound in \(x\). Hence, we introduce the following defect measures:

\[
|\tilde{v}^n|^2 \to |\tilde{v}|^2 + \alpha,
\]
(44)

where \(\alpha \in \mathcal{M}((0, T) \times \Omega \times \mathcal{Y}_f)\). Since \(\tilde{v}^n \in L^\infty((0, T) \times \Omega \times \mathcal{Y}_f)\), we deduce that \(\alpha \in L^\infty((0, T) \times \Omega \times \mathcal{Y}_f)\), we also denote \(v = \int_{\mathcal{Y}_f} \alpha \, dy\). Moreover, we define \(\beta\) by:

\[
|\hat{v}^n|^2 \to |\hat{v}|^2 + \beta,
\]
(45)

where \(\beta \in L^\infty((0, T); \mathcal{M}(\Omega; L^\infty(\mathcal{Y}_f)))\) and we denote \(\mu + \mu_s = \int_{\mathcal{Y}_f} \beta \, dy\), where \(\mu\) in continuous with respect to the Lebesgue measure and \(\mu_s\) is singular.

Multiplying the system (38) written for \((\tilde{v}^n, \hat{v}^n)\) by \(\tilde{v}^n\) and \(\hat{v}^n\) and integrating in \(y\), we get:

\[
\partial_t \left( \int_{\mathcal{Y}_f} |\tilde{v}^n|^2 \, dy + |\hat{v}^n|^2 \right) = \int_{\mathcal{Y}_f} f^n v^n - \partial_{x_1} (q^n a^n_1) - \partial_{x_2} (q^n a^n_2).
\]
(46)

Now, passing to the limit, we get:
\[ \partial_t \left( \int_Y (\tilde{v})^2 \, dy + |a|^2 + v + \mu + \mu_s \right) = \int_Y f \, v - \partial_x_1 (qa_1) - \partial_x_2 (qa_2), \]  

(47)

where, we have used that \( f^n \) converges strongly to \( f \) and that \( \text{div}(K \nabla_x q^n) \) is bounded in \( L^1(0, T; W^{-1,1}) \) from which we can deduce that \( q^n \) in bounded in \( L^1(0, T; H^{-1}) \) and then using that \( a^n \) is bounded in \( L^\infty(0, T; L^2) \) and that \( \partial_t a^n \) is bounded in \( L^\infty(0, T; H^{-1}) \) we deduce that \( q^n a^n \) converges weakly to \( qa \).

On the other hand, passing to the limit in (38), we get:

\[ \begin{cases} 
\partial_t \tilde{v} + \nabla^\perp \Delta^{-1} \text{curl}[\tilde{v}^n \cdot \nabla \tilde{v}^n] + E_0(\tilde{v}^n, \hat{v}^n) = \tilde{f} & \text{in } \Omega \times \mathbb{T}_f, \\
\partial_t \hat{v} + F(\tilde{v}^n, \hat{v}^n) + E_0(\tilde{v}^n, \tilde{v}^n) = -\partial_x_1 q w^1 - \partial_x_2 q w^2 + \tilde{f} & \text{in } \Omega \times \mathbb{T}_f, 
\end{cases} \]

(48)

where here and below, \( \overline{A^n} \) denotes the weak limit of \( A^n \). Multiplying the first equation by \( \tilde{v} \) and the second one by \( \hat{v} \), and integrating in \( y \), we get:

\[ \partial_t \left( \int_Y (\tilde{v})^2 \, dy + |a|^2 \right) + \mathcal{W} = \int_Y f \, v \, dy - \partial_x_1 (qa_1) - \partial_x_2 (qa_2), \]

(49)

where \( \mathcal{W}(t, x) \) is given by:

\[ \mathcal{W} = \int_Y \left( \nabla^\perp \Delta^{-1} \text{curl}[\tilde{v}^n \cdot \nabla \tilde{v}^n] + E_0(\tilde{v}^n, \hat{v}^n) \right) \tilde{v} \, dy + \left( F(\tilde{v}^n, \hat{v}^n) + E_0(\tilde{v}^n, \tilde{v}^n) \right) \hat{v} \, dy. \]

(50)

Taking the difference between (47) and (49), we get:

\[ \partial_t (v + \mu + \mu_s) = -\mathcal{W}. \]

(51)

It remains to estimate the different terms appearing in \( \mathcal{W} \). We have:

\[ |\mathcal{W}_1| = \left| \int_Y \nabla^\perp \Delta^{-1} \text{curl}[\tilde{v}^n \cdot \nabla \tilde{v}^n] \tilde{v} \, dy \right| \]

(52)

\[ = \left| \int_Y \nabla^\perp \Delta^{-1} \text{curl}[\tilde{v}^n \cdot \nabla (\tilde{v}^n - \tilde{v})] \tilde{v} \, dy \right| \]

\[ \leq C \int_Y |\tilde{v}| |\nabla_y \tilde{v}| \, dy \]

\[ \leq C \nu (1 + |\log \nu|), \]

(53)

where we have used the following proposition (see for instance [16]).
Proposition 2.4. If $\alpha \in L^\infty(T_f)$ and $\text{curl}\, \tilde{v} \in L^\infty$ then for all $p$, we have $\|\nabla_\gamma \tilde{v}\|_{L^p(T_f)} \leq C p$ where $C$ is independent of $p$ and the following estimate holds:

$$\int_{Y_f} \alpha |\nabla_\gamma \tilde{v}| \, dy \leq C \nu \left(1 + |\log \nu| \right).$$

(54)

The proof of this proposition is a consequence of the following estimate:

$$\int_{Y_f} \alpha |\nabla_\gamma \tilde{v}| \, dy \leq C M^{1/p} \left( \int \alpha \right)^{1-1/p} \left( \int |\nabla_\gamma \tilde{v}|^p \right)^{1/p} = C p \left( \frac{M}{\nu} \right)^{1/p} \nu$$

(55)

$$\leq C \nu \left(1 + |\log \nu| \right),$$

(56)

where $M = \text{Sup}\{\alpha, \, x \in \Omega, \, y \in Y_f\}$ and where we have optimized in $p$.

Next, we have:

$$|W_2| = \left| \int_{Y_f} E^0(\tilde{v}^n, \hat{v}^n) \tilde{v} + \tilde{E}_0(\tilde{v}^n, \hat{v}^n) \tilde{v} \, dy \right|$$

(57)

$$= \left| \int_{Y_f} E^0(\tilde{v}^n - \tilde{v}, \hat{v}^n - \hat{v}) \tilde{v} + \tilde{E}_0(\tilde{v}^n - \tilde{v}, \hat{v}^n - \hat{v}) \tilde{v} \, dy \right|$$

(58)

$$\leq C (v + \mu) + C \sup_y |\hat{v}| \nu.$$

(59)

In the same way, we have:

$$|W_3| = \left| \int_{Y_f} \tilde{F}(\tilde{v}^n, \hat{v}^n) \tilde{v} \right|$$

(60)

$$= \left| \int_{Y_f} \tilde{F}(\tilde{v}^n - \tilde{v}, \hat{v}^n - \hat{v}) \tilde{v} \right|$$

(61)

$$\leq C \sup_y |\hat{v}| \sqrt{\nu \mu}.$$

(62)

Adding the different contributions, we get:

$$\partial_t (v + \mu + \mu_s) \leq C \nu \left(1 + |\log \nu| \right) + C (v + \mu) + C \sup_y |\hat{v}| v + C \sup_y |\hat{v}| \sqrt{\nu \mu}.$$
This yields the compactness for the limit system from which we can prove the existence of solutions to the limit system by some regularizing procedure. We will not detail this regularizing procedure here since we will see that the convergence proof of next section will yield the existence of solutions to the limit system under some extra integrability conditions on the initial data. Then, we can use those solutions to end the proof of Theorem 1.4.

3. Convergence result

In this section, we prove Theorem 1.5, i.e., we prove that extracting a subsequence, \( u^\varepsilon(t, x) \) two-scale converges to \( u(t, x, y) \) a solution to the limit system. We start by recalling the uniform estimates the sequence \( u^\varepsilon(t, x) \) satisfies. From the existence proof, we know that \( u^\varepsilon(t, x) \) is bounded in \( L^\infty((0, \infty); L^2(\Omega, \mathbb{R}^N)) \) and that \( \varepsilon \text{curl} u^\varepsilon(t, x) \) is bounded in \( L^\infty((0, \infty); L^\infty(\Omega)) \). Rewriting (1) as

\[
\partial_t u^\varepsilon - \varepsilon u^\varepsilon \times \text{curl} u^\varepsilon = -\nabla (p^\varepsilon + \varepsilon \frac{|u^\varepsilon|^2}{2}) + f^\varepsilon(t, x)
\]

and projecting on divergence-free vectors, we get:

\[
\partial_t u^\varepsilon - \varepsilon \mathcal{P}(u^\varepsilon \times \text{curl} u^\varepsilon) = f^\varepsilon(t, x)
\]

from which, we deduce that \( u^\varepsilon \) is bounded in \( L^\infty((0, \infty); L^3(\Omega)) \). We also deduce that \( \nabla (p^\varepsilon + \varepsilon \frac{|u^\varepsilon|^2}{2}) \) is bounded in \( L^\infty((0, \infty); L^3(\Omega, \mathbb{R})) \).

Extracting a subsequence if necessary, we can assume that \( u^\varepsilon \) two-scale converges to some function \( u(t, x, y) \), that \( |u^\varepsilon|^2 \) two-scale converges to \( |u(t, x, y)|^2 + \alpha \) for some positive measure \( \alpha \) which is in \( L^\infty((0, \infty); L^3(\Omega, \mathbb{R})) \) and that \( u^\varepsilon \otimes u^\varepsilon \) two-scale converges to \( u(t, x, y) \otimes u(t, x, y) + \beta \) for some measure \( \beta \) which is in \( L^\infty((0, \infty); L^3(\Omega, \mathbb{R})) \).

We also denote \( v = \int_{\mathcal{Y}_j} \alpha \, dy \). Then, we deduce that \( \varepsilon \nabla (u^\varepsilon \otimes u^\varepsilon) \) two-scale converges to \( \nabla_v (u \otimes u + \beta) \).

Moreover, it is easy to see that \( \varepsilon \text{curl}(u^\varepsilon) \) two-scale converges to \( \text{curl}_v(u) \), from which we can deduce that \( \text{curl}_v(u) \) is bounded in \( L^\infty((0, \infty); L^\infty(\Omega \times \mathcal{Y})) \). We also deduce that \( \text{div}_v u = 0, u(x, y), n = 0 \) on \( \Omega \times \partial \mathcal{Y} \) and that \( \text{div}_v \int_{\mathcal{Y}_j} u = 0, \int_{\mathcal{Y}_j} v(x, y) \, dy, n = 0 \) on \( \partial \Omega \) from the divergence-free condition \( \text{div} u^\varepsilon = 0 \) and the fact that \( u^\varepsilon, n = 0 \) on \( \partial \Omega^\varepsilon \).

We refer the reader to [2] and [10] for the precise proofs of similar results.

Now, we can also extend the pressure term \( p^\varepsilon + \varepsilon \frac{|u^\varepsilon|^2}{2} \) to the whole domain \( \Omega \) by setting \( P^\varepsilon = p^\varepsilon + \varepsilon \frac{|u^\varepsilon|^2}{2} \) in \( \Omega^\varepsilon \) and \( P^\varepsilon = \frac{1}{|\mathcal{Y}_j|} \int_{\mathcal{Y}_j} p^\varepsilon + \varepsilon \frac{|u^\varepsilon|^2}{2} \, dy \) in \( \varepsilon \mathcal{Y}_j \), \( \forall k \in K \).

Extracting a subsequence if necessary, we can assume that \( P^\varepsilon \) converges weakly to some \( q(t, x) \) in \( L^\infty((0, \infty); W^{1,3}(\Omega)) \) and the convergence is strong in the space variable. Moreover, \( \nabla P^\varepsilon \) two-scale converges to \( \nabla_x q + \nabla_y p(t, x, y) \) for some \( p \in L^\infty((0, \infty); L^3(\Omega \times \mathbb{T})) \).
Passing to the limit in (1), we deduce that
\[ \partial_t u + \nabla_y (u \otimes u + \beta) = -\nabla_x q + \nabla_y \tilde{p} + f \] (66)
for some \( \tilde{p} \in L^\infty((0, \infty); L^{3/2}(\Omega \times \mathcal{Y}_f)) \). Multiplying by \( u \) and integrating over \( \mathcal{Y}_f \), we deduce that
\[ \partial_t \int_{\mathcal{Y}_f} |u|^2 - \int_{\mathcal{Y}_f} \beta : \nabla_y u \, dy = -\text{div}_x \left( q \int_{\mathcal{Y}_f} u \, dy \right) + \int_{\mathcal{Y}_f} uf \, dy. \] (67)
On the other hand, we have:
\[ \partial_t \frac{|u^\varepsilon|^2}{2} = -\text{div}\left[ u^\varepsilon \left( p^\varepsilon + \varepsilon \frac{|u^\varepsilon|^2}{2} \right) \right] + f^\varepsilon \cdot u^\varepsilon. \] (68)
Passing to the limit, we get:
\[ \partial_t \left[ \int_{\mathcal{Y}_f} \frac{|u|^2}{2} + \alpha \right] = -\text{div}_x \left( q \int_{\mathcal{Y}_f} u \, dy \right) + \int_{\mathcal{Y}_f} uf \, dy. \] (69)
Finally, we get:
\[ \partial_t v = \int_{\mathcal{Y}_f} \beta : \nabla_y u \, dy, \] (70)
using that \( u = \tilde{u} + \hat{u} \), we deduce that
\[ \left| \int_{\mathcal{Y}_f} \beta : \nabla_y u \, dy \right| \leq \left| \int_{\mathcal{Y}_f} \beta : \nabla_y \tilde{u} \, dy \right| + \left| \int_{\mathcal{Y}_f} \beta : \nabla_y \hat{u} \, dy \right| \] (71)
\[ \leq C \left[ v (1 + |\log v|) \right] + C \sup_y |\nabla_y \hat{u}| v. \] (72)
Hence, we infer that
\[ \partial_t v \leq C \left[ v (1 + |\log v|) \right] + \left( \int_{\mathcal{Y}_f} |u| \, dy \right) v \] (73)
from which we deduce that \( v = 0 \) since it is equal to 0 at \( t = 0 \).
4. Remarks on the Navier–Stokes case

For the Navier–Stokes case (5), the natural boundary condition on the holes is the Dirichlet boundary condition, namely
\[ u = 0 \quad \text{on} \quad \partial \Omega_\varepsilon. \]
We refer the reader to [5] for a discussion about other possible boundary conditions. For the Dirichlet case, the limit system becomes:
\[
\begin{align*}
\partial_t v + v \cdot \nabla_y v - v \Delta_y v &= -\nabla_y p(x, y) - \nabla_x q(x) + f(t, x, y), \\
\text{div}_y(v) &= 0, \\
\text{div}_x(\int_{Y_f} v(x, y) \, dy) &= 0, \\
v(x, y) &= 0 \quad \text{on} \quad \Omega \times \partial h_1, \\
(\int_{Y_f} v(x, y) \, dy).n &= 0 \quad \text{on} \quad \partial \Omega, \\
v|_{t=0} &= v_0.
\end{align*}
\]
(74)

Multiplying the first equation by \( v \) and integrating by parts, we get the following estimates \( v \in L^\infty(0, T; L^2(\Omega \times Y_f)) \cap L^2((0, \infty); L^2(\Omega; H^1(Y_f))) \) and it seems this is the only global estimate we can get.

Now, we want to explain why we think that recovering (74) from (5) globally in time is a very difficult problem and should not hold in general. The first difficulty is the boundary condition on \( \partial \Omega \). Indeed, the Dirichlet boundary condition on the outer boundary \( \partial \Omega \) becomes \( \int_{Y_f} v(x, y) \, dy.n = 0 \) and we have the same difficulties as in the inviscid limit of the Navier–Stokes system in a bounded domain (see [9]). This is of course related to the presence of a Prandtl boundary layer. This is not the only difficulty and we will show it, by an example in the three dimensional case with a periodic boundary condition in \( x \), namely \( \Omega = T^3 \) and \( \varepsilon = 1/n \) for \( n \in \mathbb{N} \).

If we know that there exist two different weak solutions to the Navier–Stokes system in the cell \( T_f = T^3 - Y_\varepsilon \), then we can construct solutions to the limit system which oscillate between both weak solutions. This proves the non compactness for the limit system.

Assume that \( v_1(t, y) \) and \( v_2(t, y) \) are two periodic solutions to the 3D Navier–Stokes system in \( T_f \) with Dirichlet boundary condition on \( \partial Y_\varepsilon \) such that \( v_1(0, y) = v_2(0, y) = v_0(y) \) and \( v_0 \) is smooth.

If \( A_n \) is a measurable set of \( \Omega = T^3 \) and \( B_\varepsilon = (A_n)^c \) then
\[ u_n(t, x, y) = 1_{A_n}(x)v_1(t, y) + 1_{B_\varepsilon}(x)v_2(t, y) \]
is a solution to (74) provided that \( \text{div}_x(\int_{Y_f} u_n(x, y) \, dy) = 0 \). This is the case if we assume, for instance, that
\[
\int v_1 \, dy = \int v_2 \, dy \quad \forall t \geq 0.
\]
(75)
We take \( A_n \) such that \( 1_{A_n}(x) \) converges weakly to the constant \( \alpha \). Then, \( u_n(t, x, y) \) converges weakly to \( \alpha v_1 + (1 - \alpha) v_2 \).

Hence, we have the following proposition of non compactness of the limit system:
Proposition 4.1. Assume that the Navier–Stokes system in \( T_f \) has two different weak solutions \( v_1(t, y) \) and \( v_2(t, y) \) with the same initial data and such that

\[
\int v_1 \, dy = \int v_2 \, dy \quad \forall t \geq 0.
\]

Then for all \( \alpha \in (0, 1) \), there exists a sequence of weak solutions to the limit system (74) with a fixed initial data and which converges weakly to \( \alpha v_1 + (1 - \alpha) v_2 \). This yields the non compactness of the limit system.

In other words, one essentially needs to know the uniqueness of the solutions to the 3D Navier–Stokes system in order to decide about the compactness of the limit problem.

Remark 4.2. (1) The condition (75) may seem to be very restrictive. However, it always holds if the cell \( Y_f \) is symmetric with respect to its center \((1/2, 1/2)\) and we restrict ourselves to antisymmetric solutions. More precisely, in the antisymmetric case, Proposition 4.1 holds for antisymmetric solutions without the condition (75) since it holds that \( \int v_1 \, dy = \int v_2 \, dy = 0 \). Hence, the compactness of the limit system implies the uniqueness for the Navier–Stokes system.

(2) It is not obvious from the above analysis whether the limit system is weak compact or not, even if we assume the non uniqueness for the Navier–Stokes system. Indeed, this is related to whether there exists an \( \alpha \in (0, 1) \) such that \( \alpha v_1 + (1 - \alpha) v_2 \) is not a solution of the 3D Navier–Stokes system or not. Let us just mention that the fact that \( \alpha v_1 + (1 - \alpha) v_2 \) is a solution of the 3D Navier–Stokes system for all \( \alpha \in (0, 1) \) is equivalent to the fact that \( v_1 - v_2 \) is a solution to the following system in \( T_f \):

\[
\begin{align*}
\partial_t w + v_1 \nabla w + w \cdot \nabla v_2 - \nu \Delta w &= -\nabla \mathbf{q}, \\
\text{div}(w) &= 0, \\
w \cdot \nabla w &= 0.
\end{align*}
\]

However, we were unable to prove that \( w = 0 \) is the only solution to (76) with 0 initial data.

(3) The third remark concerns the case we take an other boundary condition instead of the Dirichlet boundary condition. If we consider, for instance, (5) in the 2D case, with the following boundary condition: \( u_n = 0 \) and \( \text{curl} u^n = 0 \) on \( \partial \Omega_e \), then we can prove exactly the same results as in the Euler case and the limit system is:

\[
\begin{align*}
\partial_t v + v \cdot \nabla x v - \nu \Delta x v &= -\nabla x p(x, y) - \nabla x q(x) + f(t, x, y), \\
\text{div}_x(v) &= 0, \\
\text{div}_x \left( \int_{Y_f} v(x, y) \, dy \right) &= 0, \\
v(x, y).n &= 0 \quad \text{and} \quad \text{curl}_x v = 0 \quad \text{on} \ \Omega \times \partial h_1, \\
\left( \int_{Y_f} v(x, y) \, dy \right).n &= 0 \quad \text{on} \ \partial \Omega.
\end{align*}
\]

We do not detail this result here.
Appendix A

Here, we want to construct an orthogonal basis \( \{V_1, V_2, \ldots, V_{N+1}\} \) of \( \ker(\Phi) \) satisfying the conditions stated in Section 2.1. We recall the following classical result (see for instance [7]).

**Proposition 4.3.** Let \( \Omega \) be a bounded domain (not necessary simply connected). If \( v \) is a vector field such that \( \text{curl}(v) = 0 \) and for all oriented closed curve \( C \) in \( \Omega \)
\[
\int_C v \cdot dl = 0
\]
then there exists a \( \psi \) such that \( v = \nabla \psi \).

Let \( v \) be in \( \hat{\mathbf{H}} = \ker(\Phi) \), we define \( v^\perp \) to be equal to \((-v_2, v_1)\). Hence using that \( v \in \mathbf{H} \), we deduce that \( v^\perp \in (L^2(\mathbb{T}_f))^2 \) that \( v^\perp \tau = 0 \) on \( \partial h_j \) for all \( j \) where \( \tau \) is a tangential vector to \( \partial h_j \) and that \( \text{curl} v^\perp = 0 \). Moreover, using that \( \text{curl} v = 0 \), we deduce that \( \text{div} v^\perp = 0 \).

Now, if we forget for a while the periodicity of \( v^\perp \) and consider it as being defined on \( Y_f \), we see that we can apply Proposition 4.3 and get the existence of some \( \psi \) in \( L^2(Y_f) \) such that \( v^\perp = -\nabla \psi \) which can also be rewritten \( v^\perp = \nabla \psi \). Indeed, the circulation around any one of the holes is equal to 0. Moreover, we have that \( \psi|_{\partial h_j} = c_j \) is a constant depending only on the hole \( h_j \). Moreover, \( \psi \) is determined up to a constant which can be chosen such that \( \sum_{1 \leq j \leq N} c_j = 0 \). Using that \( v^\perp \) is periodic, we deduce that
\[
\nabla \psi|_{y_1=0} = \nabla \psi|_{y_1=1} \quad \text{and} \quad \nabla \psi|_{y_2=0} = \nabla \psi|_{y_2=1}.
\]

Hence, there exists two constants \( d_1 \) and \( d_2 \) such that \( \tilde{\psi} = \psi + d_1 y_1 + d_2 y_2 \) is periodic.

Conversely, if we choose \( N \) constants \( c_1, c_2, \ldots, c_N \) such that \( \sum_{1 \leq j \leq N} c_j = 0 \) and two constants \( d_1 \) and \( d_2 \), we can easily prove the existence of a unique \( \psi \) and a unique vector field \( v \in \ker(\Phi) \) satisfying the above relations. This can be done by observing that there exist a unique \( \psi \) satisfying:
\[
\begin{cases}
\Delta \psi = 0 & \text{in } Y_f, \\
\psi|_{\partial h_j} = c_j & \text{on } \partial h_j, \\
\psi|_{y_1=1} = \psi|_{y_1=0} + d_1, & \psi|_{y_2=1} = \psi|_{y_2=0} + d_2.
\end{cases}
\]  

Moreover, an easy integration by part gives
\[
\int \nabla \psi \, dy = \sum_j \int_{\partial h_j} c_j \, dn + \begin{pmatrix} d_1 \\ d_2 \end{pmatrix} = \begin{pmatrix} d_1 \\ d_2 \end{pmatrix},
\]
where we have used that \( \int_{\partial h_j} c_j \, dn = 0 \) since \( c_j \) is a constant. Hence, we can deduce that \( \int \nabla^\perp \psi \, dy = \left( \begin{smallmatrix} d_1 \\ d_2 \end{smallmatrix} \right) \). We denote \( \hat{\mathbf{H}}_0 \) the subspace of \( \hat{\mathbf{H}} \) such that \( \psi \) is periodic, namely \( d_1 = d_2 = 0 \). We also define \( \hat{\mathbf{H}} \) its orthogonal supplement, namely \( \hat{\mathbf{H}} = \hat{\mathbf{H}}_0 + \hat{\mathbf{H}} \).
To construct an orthonormal basis of $\mathring{H}_0$, we can use Gram–Schmidt for instance. Taking $d_1 = d_2 = 0$ and varying $c_i$, we can construct $N - 1$ orthonormal vectors $V^j$, $3 \leq j \leq N + 1$, with $V^j = \nabla \perp \psi^j$ where $\psi^j \in L^2(\mathbb{T}_f)$. Then we can construct $V^1 = \nabla \perp \psi^1$ and $V^2 = \nabla \perp \psi^2$ such that $\tilde{\psi}^1 = \psi^1 + y_2$ and $\tilde{\psi}^2 = \psi^2 - y_1$ are periodic and $\nabla \perp \psi^1$ and $\nabla \perp \psi^2$ are orthogonal to all the $V^j$, $3 \leq j \leq N + 1$. It is easy to see that for $j \geq 3$, we have $\int V^j dy = 0$ and that for $j = 1$, or 2, we have $\int V^j dy = e_j$. Indeed, for all vector $d = (d_1, d_2)$, we have:

$$\int_{\mathbb{T}_f} d \cdot V^i dy = \int_{\mathbb{T}_f} d \cdot \nabla \perp \psi^i = \int_{\mathbb{T}_f} -d_1 \partial_{y_2} \psi^i + d_2 \partial_{y_1} \psi^i = \int_{\partial \mathbb{T}_f} d_1 \psi^i n_2 - d_2 \psi^i n_1, \quad (80)$$

where $n = (n_1, n_2)$ is the exterior normal on $\mathbb{T}_f$. Then, using that for any hole $h_j$, $\psi^i|_{h_j}$ is a constant and that $\int_{h_j} n = 0$, we get that

$$\int_{\mathbb{T}_f} d \cdot V^i dy = \int_{\partial \mathbb{T}_f} d_1 \psi^i n_2 - d_2 \psi^i n_1. \quad (81)$$

For $i \geq 3$, $\psi^i$ is $\mathbb{T}$ periodic and hence $\int_{\mathbb{T}_f} c \cdot V^i dy = 0$. However, if $i = 1$ or $i = 2$, we have:

$$\int_{\mathbb{T}_f} d \cdot V^1 dy = \int_{\partial \mathbb{T}_f} d_1 y_2 n_2 - d_2 y_2 n_1 = d_1, \quad (82)$$

$$\int_{\mathbb{T}_f} d \cdot V^2 dy = \int_{\partial \mathbb{T}_f} -d_1 y_1 n_2 + d_2 y_1 n_1 = d_2. \quad (83)$$

In general $\{V^1, V^2\}$ is not an orthogonal basis of $\mathring{H}$. Hence, we can define $(w^1, w^2)$ its dual basis by taking $w^i$ to be the unique solution of:

$$\begin{align*}
  w^i + \nabla \pi^i &= e_i, & \text{div}_\mathbb{T}(w^i) &= 0, \\
  w^i \cdot n &= 0 & \text{on } \partial \mathbb{T}_f, \\
  (w^i, \pi^i) &\in L^2(\mathbb{T}_f)^2 \times L^2(\mathbb{T}_f).
\end{align*} \quad (84)$$

A simple computation yields

$$\int w^i \cdot V^j dy = \int (e_i - \nabla \pi^i) \cdot V^j = \int e_i V^j = \delta_{ij}. \quad (85)$$

Finally, we define the Euler permeability tensor which is given by:

$$K_{ij} = \int w^i_j = \int w^i \cdot e_j = \int w^i \cdot w^j.$$
It is easy to notice that $K$ is symmetric and definite positive. Moreover its inverse $M_{ij}$ is given by:

$$M_{ij} = \int V^i V^j.$$ 

References