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Suppleness of the sheaf of algebras of generalized functions on manifolds

Stevan Pilipović, Milica Žigić *

Department of Mathematics, University of Novi Sad, Trg Dositeja Obradovića 4, Novi Sad, Serbia

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ABSTRACT

We show that the sheaves of algebras of generalized functions $\Omega \to \mathcal{G}(\Omega)$ and $\Omega \to \mathcal{G}^\infty(\Omega)$, Ω are open sets in a manifold X, are supple, contrary to the non-suppleness of the sheaf of distributions.

1. Introduction and definitions

The aim of this paper is to give a complete answer to the question concerning suppleness of sheaves of certain generalized function algebras. This question is discussed in [21] and here it is completely solved. Note that Bros and Iagolnitzer [3] conjectured that the analytic singular support (analytic wavefront set) for distributions is decomposable. Bengel and Schapira [1] have studied this decomposition by considering Cousin's problem with bounds in a tuboid. In [8] authors have studied microlocal decomposition for ultradistributions and ultradifferentiable functions. They used the Laubin decomposition of delta distribution [19] for the proof in this setting. We consider in this paper the algebra $\mathcal{G}$ of generalized functions containing the Schwartz distributions space $\mathcal{D}'$ as a subspace so that all the linear operations on $\mathcal{D}'$ are preserved within $\mathcal{G}$. We refer to [2,4,5,9,10,20] for the theory of generalized function algebras and applications to non-linear and linear problems with non-smooth coefficients. Such algebras are also called Colombeau algebras, since he was the first one who introduced and analyzed such algebras. The geometric theory of algebras of generalized functions [10] is further developed in papers [13,15–18,22]. In these papers applications to general relativity show the strong impact of the new approach developed by the authors through the analysis of PDE on manifolds with singular metrics and, in particular, in Lie group analysis of differential equations (see [9–12]). A version of this theory, which is the object of the present article, is initiated in [6,14]. The sheaf properties of generalized function algebras are investigated in [7,21].

In this paper we are interested in an important sheaf property, the suppleness. It is known that the sheaves of Schwartz distributions $\Omega \to \mathcal{D}'(\Omega)$ and of smooth functions $\Omega \to C^\infty(\Omega)$, where Ω varies through all open sets of a manifold X, are not supple. The extensions of these sheaves $\Omega \to \mathcal{G}(\Omega)$ and $\Omega \to \mathcal{G}^\infty(\Omega)$, Ω are open sets of a manifold X, which are actually sheaves of algebras of generalized functions, are supple. The proof of this assertion is the subject of this paper.

1.1. Generalized functions on $\mathbb{R}^d$

We recall the main definitions. Let Ω be an open set in $\mathbb{R}^d$ and $\mathcal{E}(\Omega)$ be the space of nets of smooth functions. Then the set of moderate nets $\mathcal{E}_M(\Omega)$, respectively of negligible nets $\mathcal{N}(\Omega)$, consists of nets $(f_\varepsilon)_{\varepsilon\in(0,1)} \in \mathcal{E}(\Omega)$ with the properties

* Corresponding author. Fax: +381 21 6350458.
E-mail addresses: stevan.pilipovic@dmi.uns.ac.rs (S. Pilipović), milica.zigic@dmi.uns.ac.rs (M. Žigić).

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$$(\forall K \Subset \Omega)\ (\forall n \in \mathbb{N})\ (\exists a \in \mathbb{R}) \quad \Big(\sup_{x\in K}\big|f_\varepsilon^{(n)}(x)\big| = O\big(\varepsilon^a\big)\Big),$$

$$\text{respectively, } (\forall K \Subset \Omega)\ (\forall n \in \mathbb{N})\ (\forall b \in \mathbb{R}) \quad \Big(\sup_{x\in K}\big|f_\varepsilon^{(n)}(x)\big| = O\big(\varepsilon^b\big)\Big).$$

(O is the Landau symbol "big O" and $K \Subset \Omega$ means that K is compact in Ω or that $\bar{K}$ is compact in Ω.) Both spaces are algebras and the latter is an ideal of the former.

The algebra of generalized functions $\mathcal{G}(\Omega)$ is defined as the quotient $\mathcal{G}(\Omega) = \mathcal{E}_M(\Omega)/\mathcal{N}(\Omega)$. This is also a differential algebra. If the nets $(f_\varepsilon)_\varepsilon$ consist of constant functions on Ω (i.e. supremums over the compact set K reduce to the absolute value), then one obtains the corresponding spaces $\mathcal{E}_M$ and $\mathcal{N}_0$. They are algebras, $\mathcal{N}_0$ is an ideal in $\mathcal{E}_M$ and, as a quotient, one obtains the algebra of generalized complex numbers $\bar{\mathbb{C}} = \mathcal{E}_M/\mathcal{N}_0$ (or $\bar{\mathbb{R}}$). It is a ring, not a field.

The embedding of the Schwartz distributions in $\mathcal{E}'(\Omega)$ is realized through the sheaf homomorphism $\mathcal{E}'(\Omega) \ni f \mapsto [(f * \phi_\varepsilon|_\Omega)_\varepsilon] \in \mathcal{G}(\Omega)$, where the fixed net of mollifiers $(\phi_\varepsilon)_\varepsilon$ is defined by $\phi_\varepsilon = \varepsilon^{-d}\phi(\cdot/\varepsilon)$, $\varepsilon < 1$, where $\phi \in \mathcal{S}(\mathbb{R}^d)$ satisfies

$$\int \phi(t)\,dt = 1, \qquad \int t^m \phi(t)\,dt = 0, \quad m \in \mathbb{N}_0^n,\ |m| > 0.$$

($t^m = t_1^{m_1} \cdots t_n^{m_n}$ and $|m| = m_1 + \cdots + m_n$.) In fact $\mathcal{E}'(\Omega)$ is embedded into the space $\mathcal{G}_c(\Omega)$ of compactly supported generalized functions. This sheaf homomorphism, extended onto $\mathcal{D}'$, gives the embedding of $\mathcal{D}'(\Omega)$ into $\mathcal{G}(\Omega)$.

The algebra of generalized functions $\mathcal{G}^\infty(\Omega)$ is defined in [20] as the quotient of $\mathcal{E}_M^\infty(\Omega)$ and $\mathcal{N}(\Omega)$, where $\mathcal{E}_M^\infty(\Omega)$ consists of nets $(f_\varepsilon)_{\varepsilon\in(0,1)} \in \mathcal{E}(\Omega)^{(0,1)}$ with the properties

$$(\forall K \Subset \Omega)\ (\exists a \in \mathbb{R})\ (\forall n \in \mathbb{N}) \quad \Big(\sup_{x\in K}\big|f_\varepsilon^{(n)}(x)\big| = O\big(\varepsilon^a\big)\Big).$$

Note that $\mathcal{G}^\infty$ is a subsheaf of $\mathcal{G}$.

1.2. Generalized functions on a manifold

We will recall the main definitions and assertions following [10]. Let X be a smooth Hausdorff paracompact manifold. We denote by $\mathcal{U} = \{(V_\alpha, \psi_\alpha)\colon\ \alpha \in \Lambda\}$ an atlas on X, Λ is the index set.

We use $\mathcal{P}(X,E)$ to denote the space of linear differential operators $\Gamma(X,E) \to \Gamma(X,E)$, where E is a vector bundle on X and $\Gamma(X,E)$ is the space of smooth sections of the vector bundle E over X. Particularly, if $E = X \times \mathbb{R}$ we write $\mathcal{P}(X)$ instead of $\mathcal{P}(X,E)$. We denote by $\mathfrak{X}(X)$ the space of smooth vector fields on X.

Let $\mathcal{E}(X) := (C^\infty(X))^{(0,1)}$ and $(u_\varepsilon)_\varepsilon \in \mathcal{E}(X)$. Then the following statements are equivalent:

1. $(\forall K \Subset X)$ $(\forall P \in \mathcal{P}(X))$ $(\exists N \in \mathbb{N})$ $(\sup_{p\in K}|Pu_\varepsilon(p)| = O(\varepsilon^{-N}))$;
2. $(\forall K \Subset X)$ $(\forall k \in \mathbb{N}_0)$ $(\exists N \in \mathbb{N})$ $(\forall \xi_1,\dots,\xi_k \in \mathfrak{X}(X))$ $(\sup_{p\in K}|L_{\xi_1}\cdots L_{\xi_k}u_\varepsilon(p)| = O(\varepsilon^{-N}))$ (L_{ξ_i} is the Lie derivative);
3. For any chart (V,ψ): $(u_\varepsilon \circ \psi^{-1})_\varepsilon \in \mathcal{E}_M(\psi(V))$.

Denote by $\mathcal{E}_M(X)$ the subset of $\mathcal{E}(X)$ defined by any of the conditions 1, 2 or 3. We call it the space of moderate nets on the manifold X. The space of negligible nets is defined as

$$\mathcal{N}(X) := \Big\{(u_\varepsilon)_\varepsilon \in \mathcal{E}_M(X)\colon\ \forall K \Subset X\ \forall m \in \mathbb{N}\colon\ \sup_{p\in K}\big|u_\varepsilon(p)\big| = O\big(\varepsilon^m\big)\Big\}.$$

An algebra of generalized functions on the manifold X is defined as the quotient space $\mathcal{G}(X) := \mathcal{E}_M(X)/\mathcal{N}(X)$. Elements of $\mathcal{G}(X)$ are written as $u = [(u_\varepsilon)_\varepsilon] = (u_\varepsilon)_\varepsilon + \mathcal{N}(X)$. As one can expect, $\mathcal{E}_M(X)$ is a differential algebra (with respect to Lie derivatives) and $\mathcal{N}(X)$ is a differential ideal in it. Moreover, $\mathcal{E}_M(X)$ and $\mathcal{N}(X)$ are invariant with respect to any $P \in \mathcal{P}(X)$. Thus $Pu := [(Pu_\varepsilon)_\varepsilon]$ is a well-defined element of $\mathcal{G}(X)$.

Let $u \in \mathcal{G}(X)$ and let X' be an open set on a manifold X. The restriction of a generalized function u, denoted by $u|_{X'} \in \mathcal{G}(X')$, is represented by $(u_\varepsilon|_{X'})_\varepsilon + \mathcal{N}(X')$. The support of a generalized function u, denoted by $\operatorname{supp} u$, is defined as the complement of the union of open sets $X' \subseteq X$ such that $u|_{X'} = 0$.

The algebra $\mathcal{G}^\infty(X)$ is defined as a subalgebra of $\mathcal{G}(X)$ satisfying $u \in \mathcal{G}^\infty(X)$ if there exists a representative $(u_\varepsilon)_\varepsilon$ of u so that for any chart (U,φ), $(u_\varepsilon \circ \varphi^{-1})_\varepsilon \in \mathcal{G}^\infty(\varphi(U))$.

Now we recall the sheaf properties of the space $\mathcal{G}(X)$ (see [10]) and $\mathcal{G}^\infty(X)$.

A generalized function u on X allows the following local description via the correspondence: $\mathcal{G}(X) \ni u \mapsto (u_\alpha)_{\alpha\in A}$, where $u_\alpha := u \circ \psi_\alpha^{-1} \in \mathcal{G}(\psi_\alpha(V_\alpha))$. We call u_α the local expression of u with respect to the chart (V_α,ψ_α). Then $\mathcal{G}(X)$ can be identified with the set of all families $(u_\alpha)_\alpha$ of generalized functions $u_\alpha \in \mathcal{G}(\psi_\alpha(V_\alpha))$ satisfying the transformation law

$$u_\alpha|_{\psi_\alpha(V_\alpha\cap V_\beta)} = u_\beta \circ \psi_\beta \circ \psi_\alpha^{-1}|_{\psi_\alpha(V_\alpha\cap V_\beta)}$$

for all $\alpha, \beta \in A$ with $V_\alpha \cap V_\beta \neq \emptyset$.

It is well known that $\Omega \to \mathcal{G}(\Omega)$, Ω are open sets in X, is a fine and soft sheaf of $\mathbb{K}$-algebras on X. Thus, $\mathcal{G}$ is defined directly as a quotient sheaf of the sheaves of moderate modulo negligible sections. Similarly, $\Omega \to \mathcal{G}^\infty(\Omega)$, Ω open in X, is a fine and soft sheaf.

2. Supple sheaves

Recall [23], if $\mathcal{F}$ is a sheaf over the differential manifold X and $U \subset X$ open than a continuous map $f: U \to \mathcal{F}$ such that $\pi \circ f = id$ is called a section of $\mathcal{F}$ over U. The set of sections of $\mathcal{F}$ over U is denoted by $\Gamma(U, \mathcal{F})$.

Definition 2.1. Let $\mathcal{F}$ be a sheaf over the topological space X. Then, $\mathcal{F}$ is a supple sheaf if for all $f \in \Gamma(U, \mathcal{F})$, U open in X, the following is true: If $\operatorname{supp} f = Z = Z_1 \cup Z_2$, where Z_1 and Z_2 are arbitrary closed sets of X, then there exist $f_1, f_2 \in \Gamma(U, \mathcal{F})$ such that

$$\operatorname{supp} f_1 \subseteq Z_1, \qquad \operatorname{supp} f_2 \subseteq Z_2 \quad \text{and} \quad f = f_1 + f_2.$$

It will be shown that the sheaf of algebras $\Omega \to \mathcal{G}(\Omega)$, Ω varies over all open sets of the manifold X, is supple, but it is not flabby. It is well known that $\mathcal{D}'$ is not supple. We give an example which shows this.

Example 2.2. Consider

$$f(x) = \sum_{n=1}^{\infty} \left(\delta\left(x + \frac{1}{n^2}\right) - \delta\left(x - \frac{1}{n^2}\right) \right) + \delta(x),$$

where δ is the delta distribution. The support of this distribution is

$$Z = \left\{ \frac{1}{n^2}\colon n \in \mathbb{N} \right\} \cup \left\{ -\frac{1}{n^2}\colon n \in \mathbb{N} \right\} \cup \{0\}.$$

One can see that the closed set Z is a union of two closed sets

$$Z_1 = \left\{ \frac{1}{n^2}\colon n \in \mathbb{N} \right\} \cup \{0\} \quad \text{and} \quad Z_2 = \left\{ -\frac{1}{n^2}\colon n \in \mathbb{N} \right\} \cup \{0\}.$$

Then distributions f_1 and f_2 (in order to satisfy Definition 2.1) should be of the form

$$f_1 = \sum_{n=1}^{\infty} \delta\left(x - \frac{1}{n^2}\right) + C_1 \delta(x) \quad \text{and} \quad f_1 = \sum_{n=1}^{\infty} \delta\left(x + \frac{1}{n^2}\right) + C_2 \delta(x),$$

since $\operatorname{supp} f_1 \subseteq Z_1$, $\operatorname{supp} f_2 \subseteq Z_2$ and $f = f_1 + f_2$. However, it is known that f_1 and f_2 are not distributions since they are infinite sums of shifted delta distributions so that their supports have zero as the accumulation point.

We will prove the next theorem:

Theorem 2.3. *$\Omega \to \mathcal{G}(\Omega)$, Ω open in X, is a supple sheaf.*

Proof. For the set A, we will denote by A^ε the set $A^\varepsilon = \{x \in \mathbb{R}^d\colon d(x, A) < \varepsilon\}$, $\varepsilon < 1$, where d is a distance on X. The notation $L(x, r)$ stands for the open ball of radius $r > 0$ centered in $x \in X$, i.e. $L(x, r) = \{y \in X\colon d(x, y) < r\}$.

We divide the proof of this theorem into two parts (I) and (II) and use the following two simple assertions:

1. Let A be a measurable set in $\mathbb{R}^d$. Then there exists a generalized function $\eta = [(\eta_\varepsilon)_\varepsilon] \in \mathcal{G}(\mathbb{R}^d)$ such that

   $$\eta_\varepsilon(x) := \begin{cases} 1, & x \in A, \\ 0, & x \in \mathbb{R}^d \setminus A^\varepsilon \end{cases}$$

   and $0 \leqslant |\eta_\varepsilon(x)| \leqslant 1$, $x \in \mathbb{R}^d$. More precisely, η_ε is defined to be $1_A * \phi_\varepsilon$, where 1_A is the characteristic function of A, ϕ is a compactly supported smooth function so that $\int_{\mathbb{R}^d} \phi(t)\,dt = 1$ and $\phi_\varepsilon(x) = 1/\varepsilon^d \phi(x/\varepsilon)$.
2. Let $\delta > 0$ and Z_1 and Z_2 arbitrary closed sets of $\mathbb{R}^d$. Then there exists a closed set $\widetilde{Z_1^\delta} \supset Z_1$ such that $Z_1 \cap Z_2 = \widetilde{Z_1^\delta} \cap Z_2$ and $d(x, Z_1) \leqslant \delta$, $x \in \widetilde{Z_1^\delta}$.
   We define

   $$\widetilde{Z_1^\delta} = \{x \in \mathbb{R}^d\colon d(x, Z_1) \leqslant \delta \wedge d(x, Z_1) \leqslant d(x, Z_2)\}.$$

(I) Now we show that $\Omega \to \mathcal{G}(\Omega)$, Ω open in $\mathbb{R}^d$, is a supple sheaf.

It is enough to prove the assertion for $U = \mathbb{R}^d$ and $f \in \mathcal{G}(\mathbb{R}^d) = \Gamma(\mathbb{R}^d, \mathcal{G})$ with the property $\operatorname{supp} f = Z$ and let $Z = Z_1 \cup Z_2$, where Z_1 and Z_2 are arbitrary closed sets. Let $\delta > 0$ and define $\widetilde{Z_1^\delta}$ as in Assertion 2. Next by Assertion 1, let $\eta_\varepsilon \in C^\infty(\mathbb{R}^d)$, $\varepsilon \in (0,1)$, such that

$$\eta_\varepsilon(x) = \begin{cases} 1, & x \in \widetilde{Z_1^\delta}, \\ 0, & x \in X \setminus (\widetilde{Z_1^\delta})^\varepsilon, \ \varepsilon \in (0,1), \end{cases}$$

with $(\eta_\varepsilon)_\varepsilon \in \mathcal{E}_M(\mathbb{R}^d)$.

Let $f_1 = [(f_\varepsilon \eta_\varepsilon)_\varepsilon]$ and $f_2 = [(f_\varepsilon(1-\eta_\varepsilon))_\varepsilon]$. Then $f_1, f_2 \in \mathcal{G}(\mathbb{R}^d)$ and $f = [(f_\varepsilon \eta_\varepsilon)_\varepsilon] + [(f_\varepsilon(1-\eta_\varepsilon))_\varepsilon]$. So, we have to show that $\operatorname{supp}(f_1) \subseteq Z_1$ and $\operatorname{supp}(f_2) \subseteq Z_2$.

We show the inclusion $\operatorname{supp}(f_1) \subseteq Z_1$ by showing that for any point $x \notin Z_1$ there exists a neighborhood X' of x such that $(f_{1\varepsilon}|_{X'})_\varepsilon \in \mathcal{N}(X')$ (according to the definition of the support this means $x \notin \operatorname{supp} f_1$). Let $x \in \mathbb{R}^d \setminus Z_1$. Then we have $A = d(x, Z_1) > 0$ and there are two possibilities: $x \in \widetilde{Z_1^\delta}$ and $x \notin \widetilde{Z_1^\delta}$. If $x \in \widetilde{Z_1^\delta}$ then the ball $X' = L(x, \frac{A}{2})$ has no intersection with Z_1 and Z_2 because in the set $\widetilde{Z_1^\delta}$ we have $d(x, Z_1) \leqslant d(x, Z_2)$. So X' does not intersect the set $Z = Z_1 \cup Z_2$. From

$$\left|f_{1\varepsilon}(y)\right| = \left|f_\varepsilon(y)\eta_\varepsilon(y)\right| \leqslant \left|f_\varepsilon(y)\right| \quad \text{for all } y \in X'$$

and $(f_\varepsilon|_{X'})_\varepsilon \in \mathcal{N}(X')$, we have $(f_{1\varepsilon}|_{X'})_\varepsilon \in \mathcal{N}(X')$. Finally, if $x \notin \widetilde{Z_1^\delta}$ then $B = d(x, \widetilde{Z_1^\delta}) > 0$ (since $\widetilde{Z_1^\delta}$ is a closed set). Let $X' = L(x, \frac{B}{2})$. Then

$$f_{1\varepsilon}(y) = f_\varepsilon(y)\eta_\varepsilon(y) = f_\varepsilon(y) \cdot 0 = 0, \quad y \in X',$$

where $\varepsilon < \frac{B}{2}$. Again, we have $(f_{1\varepsilon}|_{X'})_\varepsilon \in \mathcal{N}(X')$.

Similarly, we show the second inclusion $\operatorname{supp}(f_2) \subseteq Z_2$. Let $x \notin Z_2$. Our aim is to show that $x \notin \operatorname{supp} f_2$. There are two possibilities: $x \notin Z_1$ and $x \in Z_1$. If $x \notin Z_1$, we also have $x \notin Z_2$ and so $x \notin Z$. Then there exists a neighborhood W of x such that $(f_\varepsilon|_W)_\varepsilon \in \mathcal{N}(W)$, since $\operatorname{supp} f \subseteq Z$. We take $X' = W$. Then, clearly,

$$\left|f_{2\varepsilon}(y)\right| = \left|f_\varepsilon(y)\big(1 - \eta_\varepsilon(y)\big)\right| \leqslant \left|f_\varepsilon(y)\right|, \quad y \in X',\ \varepsilon < 1.$$

Since $(f_\varepsilon|_{X'})_\varepsilon \in \mathcal{N}(X')$ we also have that $(f_{2\varepsilon}|_{X'})_\varepsilon \in \mathcal{N}(X')$. The second possibility is $x \in Z_1$. Let $H = d(x, Z_2) > 0$ and $H' = \min\{H, \delta\}$. Note that $d(x, Z_1) = 0$ since $x \in Z_1$. Then for $X' = L(x, \frac{H'}{2})$ we have $X' \subseteq \widetilde{Z_1^\delta}$ (since $H' \leqslant \delta$ and for all $y \in X'$ holds $d(y, Z_1) < \frac{H'}{2} \leqslant d(y, Z_2)$). So X' has no intersection with Z_2 (since $H' \leqslant H$). We have

$$f_{2\varepsilon}(y) = f_\varepsilon(y)\big(1 - \eta_\varepsilon(y)\big) = f_\varepsilon(y) \cdot 0 = 0, \quad y \in X',\ \varepsilon < 1.$$

Again, $(f_{2\varepsilon}|_{X'})_\varepsilon \in \mathcal{N}(X')$. This finishes the proof of the suppleness of $\Omega \to \mathcal{G}(\Omega)$, Ω open in $\mathbb{R}^d$.

(II) Let $Z \subseteq X$ be closed. Let $Z = Z_1 \cup Z_2$ where Z_1 and Z_2 are closed and let $f \in \mathcal{G}(X) = \Gamma(X, \mathcal{G})$ such that $\operatorname{supp}(f) \subseteq Z$. Cover X by a family of chart neighborhoods $\mathcal{U}$ and let $\{\tilde{\chi}_\alpha\colon \alpha \in \Lambda\}$ be a partition of unity subordinated to $\mathcal{U}$. Set

$$\chi_\alpha = \frac{\tilde{\chi}_\alpha}{(\sum_{\alpha\in\Lambda} \tilde{\chi}_\alpha^2)^{1/2}}.$$

So, we obtain the family of functions $\{\chi_\alpha\colon \alpha \in \Lambda\}$ such that $\{\operatorname{supp}(\chi_\alpha)\colon \alpha \in \Lambda\}$ is locally finite and $\sum_{\alpha\in\Lambda} \chi_\alpha^2 = 1$. Hence, one can write

$$f = \sum_{\alpha\in\Lambda} \chi_\alpha^2 f = \sum_{\alpha\in\Lambda} \chi_\alpha(\chi_\alpha f). \tag{1}$$

For the functions $\chi_\alpha f \alpha \in \Lambda$ we have $\operatorname{supp}(\chi_\alpha f)$ is closed in some $U_\alpha \in \mathcal{U}$, where the results hold (due to part (I) of this proof). Precisely, one can see $\operatorname{supp}(\chi_\alpha f) \subseteq Z \cap U_\alpha$ as

$$\operatorname{supp}(\chi_\alpha f) = \big(Z_1 \cap \big(\operatorname{supp}(\chi_\alpha f)\big)\big) \cup \big(Z_2 \cap \big(\operatorname{supp}(\chi_\alpha f)\big)\big) \subseteq (Z_1 \cap U_\alpha) \cup (Z_2 \cap U_\alpha),$$

where the sets $Z_1 \cap (\operatorname{supp}(\chi_\alpha f))$ and $Z_2 \cap (\operatorname{supp}(\chi_\alpha f))$ are closed.

Applying part (I) of this proof to $\chi_\alpha f$, $\alpha \in \Lambda$ we obtain $f_1^\alpha, f_2^\alpha \in \mathcal{G}(U_\alpha)$ such that

$$\chi_\alpha f = f_1^\alpha + f_2^\alpha, \qquad \operatorname{supp}\big(f_1^\alpha\big) \subseteq Z_1 \cap U_\alpha \subseteq Z_1, \qquad \operatorname{supp}\big(f_2^\alpha\big) \subseteq Z_2 \cap U_\alpha \subseteq Z_2.$$

According to (1) $f = \sum_{\alpha\in\Lambda} \chi_\alpha(f_1^\alpha + f_2^\alpha) = \sum_{\alpha\in\Lambda} \chi_\alpha f_1^\alpha + \sum_{\alpha\in\Lambda} \chi_\alpha f_2^\alpha$.

Set $f_1 = \sum_{\alpha\in\Lambda} \chi_\alpha f_1^\alpha$ and $f_2 = \sum_{\alpha\in\Lambda} \chi_\alpha f_2^\alpha$. Then $f_1, f_2 \in \mathcal{G}(X)$ and $\operatorname{supp} f_1 \subseteq Z_1$, $\operatorname{supp} f_2 \subseteq Z_2$. □

Theorem 2.4. *$\Omega \to \mathcal{G}^\infty(\Omega)$, Ω open in X, is a supple sheaf.*

Proof. Suppleness of the sheaf $\Omega \to \mathcal{G}^\infty(\Omega)$ can be proved using the same ideas as in the proof of Theorem 2.3. Let $Z \subseteq X$ be closed. Let $Z = Z_1 \cup Z_2$ where Z_1 and Z_2 are closed and let $f \in \mathcal{G}^\infty(X)$ such that $\operatorname{supp}(f) \subseteq Z$. Now we have to construct generalized functions $f_1, f_2 \in \mathcal{G}^\infty(X)$ such that $\operatorname{supp} f_1 \subseteq Z_1$, $\operatorname{supp} f_2 \subseteq Z_2$. In order to obtain $f_1, f_2 \in \mathcal{G}^\infty(X)$ we will take $\hat{\eta}$ to be a generalized function from $\mathcal{G}^\infty(X)$ (see Assertion 1). We will replace ε by $|\ln\varepsilon|^{-1}$ and then the generalized function $\hat{\eta} = [(\hat{\eta}_\varepsilon)_\varepsilon]$ will be

$$\hat{\eta}_\varepsilon(x) = \begin{cases} 1, & x \in A, \\ 0, & x \in \mathbb{R}^d \setminus A_{|\ln\varepsilon|^{-1}}. \end{cases}$$

This finishes the proof, since for $f = [(f_\varepsilon)_\varepsilon] \in \mathcal{G}^\infty(X)$ generalized functions $\hat{f}_1 = [(f_\varepsilon \hat{\eta}_\varepsilon)_\varepsilon]$ and $\hat{f}_2 = [(f_\varepsilon(1-\hat{\eta}_\varepsilon))_\varepsilon]$ are in $\mathcal{G}^\infty(X)$ and following the proof of Theorem 2.3 (replacing ε by $|\ln\varepsilon|^{-1}$) we obtain $\operatorname{supp} \hat{f}_1 \subseteq Z_1$ and $\operatorname{supp} \hat{f}_2 \subseteq Z_2$. □

Let us remark at the end that $\mathcal{G}(X)$ and $\mathcal{G}^\infty(X)$ are not flabby sheaves (see [21, Remark on p. 95]). If we take $X = \mathbb{R}$ and $X' = (0, \infty)$ then one cannot extend the generalized function $[(\varepsilon^{-1/x})_\varepsilon]$, defined on $(0, \infty)$, to the whole space $\mathbb{R}$.

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