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An existence theory for a minimum energy problem

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Abstract

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In this paper we deal with existence theory and develop it for the simple case of the minimum energy problem, as described by Pironneau (1984). We shall treat this problem for the differential inequality by introducing the penalized differential equation and then taking limits of the equations resulting from the penalized approximation.

Keywords: Energy problem, approximation, convexity, Hilbert space, homogenization, Sobolev space, inner product.

In an optimization problem there are ordinarily two important questions, often independent of one another: existence of a solution and characterization of a solution. The existence question usually involves some kind of compactness argument (even convexity arguments often rely on the fact that closed bounded convex sets in a reflexive Banach space are weakly compact) and the characterization question involves calculating derivatives or, more generally, calculating the variations of some functionals. In some instances this calculation is not easy and the theory of calculus of variation was certainly developed in order to understand this kind of question.

In an optimal shape problem where the variable is a domain and usually some partial differential equation is involved, there is another phenomenon that was discovered in [8] fifteen years ago (it was later called homogenization): generalized domains appear which are the analogue of a mixture of two different materials and the effective properties of these mixtures have to be understood (they are not obtained by averaging certain quantities in more than one dimension).

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The same kinds of idea were independently developed in the (former) USSR [6,7]. These ideas have also been applied in a different context in [3–5]. At present, we shall deal with a simple case called the minimum energy problem where homogenization does not show up. Here follows the statement of our main problem.

Let Ω be a bounded open set in $\mathbb{R}^n$ with a smooth boundary Γ. In Ω we consider the operator $A: V = H_0^1 \to V'$ is linear continuous and symmetric satisfying the coercivity condition, i.e.,

$$(A\psi, \psi) = a(\psi, \psi) \geqslant \alpha \|\psi\|^2, \quad \forall \psi \in V, \ \alpha > 0,$$

i.e., $A = -\nabla \cdot \nabla$; let K be a nonempty, closed, convex subset of a Hilbert space V, i.e.,

$$K = \{\phi \mid \phi \in H_0^1, \ \phi \geqslant 0 \text{ almost everywhere (a.e.) in } \Omega\}. \tag{1}$$

Let V be a Hilbert space, whose inner product and norm are denoted by $((\cdot, \cdot))$ and $\|\cdot\|$. Let V' be the dual of V, the pairing between V and V' being denoted by $(\cdot, \cdot)$. Let us define the Sobolev space:

$$H^m(\Omega) = \{\phi \in L^2(\Omega);\ \mathrm{D}^\alpha \phi \in L^2(\Omega),\ |\alpha| \leqslant m\}. \tag{2}$$

For $\phi, \psi \in L^2(\Omega)$:

$$(\phi, \psi)_{L^2(\Omega)} = \int_\Omega \phi(x)\psi(x)\,\mathrm{d}x, \tag{3}$$

and for $\phi, \psi \in H^m(\Omega)$:

$$(\phi, \psi)_{H^m(\Omega)} = \sum_{|\alpha| \leqslant m} (\mathrm{D}^\alpha \phi, \mathrm{D}^\alpha \psi)_{L^2(\Omega)}. \tag{4}$$

The problem we want to consider first consists of finding an optimal domain Ω_ϵ whicn is minimizer of the following performance criterion:

$$\min_{\Omega_\epsilon \in \Theta} E(\Omega_\epsilon) = \frac{1}{2}\int_{\Omega_\epsilon} |\nabla \phi_\epsilon|^2\,\mathrm{d}x + \frac{1}{2\epsilon}\int_{\Omega_\epsilon} |\phi_\epsilon^-|^2\,\mathrm{d}x - \int_{\Omega_\epsilon} f\phi_\epsilon\,\mathrm{d}x, \tag{5}$$

where $\phi_\epsilon(\Omega_\epsilon)$ is the solution of

$$-\Delta\phi_\epsilon + \frac{1}{\epsilon}\phi_\epsilon^- = f, \quad \text{in } \Omega_\epsilon, \tag{6}$$

(where $V^- = -\sup(-v, 0)$).

$$\phi_\epsilon|_\Gamma = 0, \tag{7}$$

$$\Theta = \left\{\Omega_\epsilon \subset \sigma:\ \Omega_\epsilon \text{ open}, \int_{\sigma - \Omega_\epsilon} \mathrm{d}x = 1\right\}, \tag{8}$$

where σ is a fixed measurable set in $\mathbb{R}^2$.

The special condition of this problem is that $E(\Omega_\epsilon)$ is the energy of the system. The constraints in (8) on the measure of Ω_ϵ are necessary to make the problem nontrivial. We assume, obviously, that the measure of σ is greater than 1.

Now we shall show that the problem given in (5)–(8) has at least one solution Ω_ϵ which minimizes (5) while satisfying (6) and (8); it can be found by solving the following associated problem:

$$\min_{\phi_\epsilon \in U_\epsilon} E(\phi_\epsilon) = \frac{1}{2}\int_\sigma |\nabla\phi_\epsilon|^2 \, dx + \frac{1}{2\epsilon}\int_\sigma |\phi_\epsilon^-|^2 \, dx - \int_\sigma f\phi_\epsilon \, dx, \tag{9}$$

where

$$U_\epsilon = \{\phi_\epsilon \mid \phi_\epsilon \in H_0^1(\sigma): \text{mes}\{x: \phi_\epsilon(x) = 0\} \geqslant 1\}, \tag{10}$$

and by setting

$$\Omega_\epsilon = \sigma - \bigcap_{\psi_\epsilon = \phi_\epsilon \text{ a.e.}} \overline{\{x: \psi_\epsilon(x) = 0\}}. \tag{11}$$

Before proving the result, first we prove the equivalence between (9) and (5). For this, let us suppose that Ω_ϵ^0 is the solution of (5) and $\phi(\Omega_\epsilon^0)$ the corresponding solution of (6). Extending ϕ_ϵ by zero outside Ω_ϵ^0 yields an admissible function for (9); so if $\hat{\phi}_\epsilon$ is a solution of (9), we have

$$E(\hat{\phi}_\epsilon) \leqslant E(\Omega_\epsilon^0). \tag{12}$$

To prove the converse, let us denote by $\hat{\Omega}_\epsilon$ the set defined by (11) with $\hat{\phi}_\epsilon$. As

$$W = \{\phi_\epsilon \in H_0^1(\sigma): \phi_\epsilon = 0 \text{ in } \sigma - \hat{\Omega}_\epsilon\} \tag{13}$$

is a subset of U_ϵ, we have

$$E(\hat{\phi}_\epsilon) = \min_{U_\epsilon} E(\phi_\epsilon). \tag{14}$$

Now $E(\phi_\epsilon)$ is the energy of (6); so $\hat{\phi}_\epsilon$ satisfies (6). This proves the equivalence between (9) and (5).

Now let us prove that (9) has a solution. Since E is strictly convex weakly semicontinuous in $H^1(\sigma)$, all we have to do to find $\phi_\epsilon \in U_\epsilon$ minimizing (9) is to prove that U_ϵ is weakly close, i.e.,

$$\phi_\epsilon^n \to \phi_\epsilon, \quad \text{weak in } H^1(\sigma), \qquad \text{mes}\{x: \phi_\epsilon^n(x) = 0\} \geqslant 1, \tag{15}$$

which implies that $\text{mes}\{x: \phi_\epsilon(x) = 0\} \geqslant 1$. From [9] we know that

$$|\phi_\epsilon^n - \phi_\epsilon| \to 0, \tag{16}$$

which implies that $\limsup \text{mes}\{x: \phi_\epsilon^n(x) = 0\} \leqslant \text{mes}\{x: \phi_\epsilon(x) = 0\}$.

So (15) holds, and U_ϵ is weakly closed. Hence ϕ_ϵ is a solution of (9)–(11), i.e., Ω_ϵ is minimizer of (5). This proof was adapted from [10]. Now we shall discuss the limiting process of our problem, i.e., we shall prove that the optimal Ω which is solution of the following problem:

$$\min_{\Omega \in \theta} E(\Omega) = \frac{1}{2}\int_\Omega |\nabla\phi(\Omega)|^2 \, dx - \int_\Omega f\phi \, dx, \tag{17}$$

where $\phi(\Omega)$ is the solution of

$$a(\phi, \omega - \phi) \geqslant (f, \omega - \phi), \tag{18}$$

$$\Theta = \left\{\Omega \subset \sigma: \Omega \text{ open}, \int_{\sigma - \Omega} dx = 1\right\}, \tag{19}$$

can be found by solving

$$\min_{\phi \in U} E(\phi) = \frac{1}{2}\int_\sigma |\nabla\phi|^2 \, dx - \int_\sigma f\phi \, dx, \tag{20}$$

where

$$U = \{\phi \mid \phi \text{ is solution of (18)}, \phi \in H_0^1(\sigma)\text{: mes}\{x\text{: } \phi(x) = 0\} \geq 1\}, \tag{21}$$

and by setting

$$\Omega = \sigma - \bigcap_{\psi=\phi \text{ a.e.}} \overline{\{x\text{: } \psi(x) = 0\}}. \tag{22}$$

We know from [2] that

$$\phi_\epsilon \to \phi^0, \quad \text{in } H^1(\sigma) \text{ weakly, as } \epsilon \to 0, \tag{23}$$

and also

$$\phi_\epsilon \to \phi^0, \quad \text{in } L^2(\sigma) \text{ strongly, as } \epsilon \to 0. \tag{24}$$

Since the function ϕ_ϵ is a solution of (9), let us state

$$I_\epsilon(\phi_\epsilon) = \frac{1}{2}\int_\sigma |\nabla\phi_\epsilon|^2 \, dx + \frac{1}{2\epsilon}\int_\sigma |\phi_\epsilon^-|^2 \, dx - \int_\sigma f\phi_\epsilon \, dx. \tag{25}$$

By taking limits as $\epsilon \to 0$ of both sides of (25), we can show [1] that I_ϵ tends to I, i.e.,

$$I(\phi) = \frac{1}{2}\int_\sigma |\nabla\phi|^2 \, dx - \int_\sigma f\phi \, dx. \tag{26}$$

Thus (9) (when $\epsilon \to 0$) becomes

$$\lim_{\epsilon \to 0} \min_{\phi_\epsilon \in U_\epsilon} E(\phi_\epsilon) = \frac{1}{2}\int_\sigma |\nabla\phi^0|^2 \, dx - \int_\sigma f\phi^0 \, dx = E(\phi^0). \tag{27}$$

We prove that the left-hand side of this equality equals $\min_{\psi \in U} E(\psi)$, that is, ϕ^0 is a minimizer for $\psi \to E(\psi)$. Consider the set

$$\bigcap_{\epsilon > 0} U_\epsilon; \tag{28}$$

it is weakly dense in the H^1-topology. Note also that $\phi \to E(\phi)$ is a weakly continuous functional on H^1. Suppose that ϕ^0 is not a minimizer for E on U, then there exists $\hat{\phi} \in U$ so that

$$E(\hat{\phi}) < E(\phi^0).$$

Given $\alpha > 0$, we can find $\epsilon > 0$ so that

(i) $|E(\phi^0) - E(\phi_\epsilon)| < \alpha$, since E is continuous;

(ii) there is $\theta^\epsilon \in U_\epsilon$ so that $|E(\hat{\phi}) - E(\theta^\epsilon)| < \alpha$, since the set (28) is dense in U.

Call $E(\phi^0) - E(\hat{\phi}) = a$. Then (i) and (ii) imply that $E(\theta^\epsilon) < E(\phi_\epsilon)$, provided that α is chosen so that $a - 2\alpha > 0$. This contradicts the fact that ϕ_ϵ is a minimizer for E on U_ϵ; ϕ^0 is then minimizer for E on U. Thus ϕ^0 is minimizer for the energy function (27); also, then Ω as defined in (22) is a minimizer of (17) because of the equivalence between problems of the type of (9) and (5), as we proved above.

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