Hilbert’s Nullstellensatz Is in the Polynomial Hierarchy

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We show that if the Generalized Riemann Hypothesis is true, the problem of deciding whether a system of polynomial equations in several complex variables has a solution is in the second level of the polynomial hierarchy (in fact, this problem is in RP^{NP}). The best previous bound was PSPACE. © 1996 Academic Press, Inc.

1. Introduction

In its weak form, Hilbert’s Nullstellensatz states that a system

\[ f_1(x) = 0, \ldots, f_s(x) = 0 \] (1)

of polynomial equations in \( n \) unknowns has no solution over \( \mathbb{C} \) if and only if there are polynomials \( g_1, \ldots, g_s \in \mathbb{C}[X_1, \ldots, X_n] \) such that \( \sum_{i=1}^{s} f_i g_i = 1 \). For this reason, the problem of deciding whether (1) is satisfiable has also been called Hilbert’s Nullstellensatz (HN). This problem (and similar problems over the reals) has generated a lot of interest due to its importance in algebraic geometry and its potential applications. For instance, it is the basic step in algorithms for the decision and quantifier elimination problems in the first-order theory of \( \mathbb{C} \). This opens up applications in, e.g., geometric theorem proving and robot motion planning. Note also that if the \( f_i \)'s can have arbitrary complex coefficients, HN is the canonical NP-complete problem in the Blum–Shub–Smale model of computation (Blum et al., 1989).

In this paper we consider systems of equations with integral coefficients only; i.e., we assume that \( f_1, \ldots, f_s \in \mathbb{Z}[X_1, \ldots, X_n] \). Our model of

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computation is the Turing machine model. It is a folklore result that HN is NP-hard (there is a simple reduction from the satisfiability of boolean formulas in conjunctive normal form to HN). In terms of general complexity classes, the best upper bound previously known was PSPACE. Here we show that HN is in the second level of the polynomial hierarchy (in \( \Pi_2 \)). In fact, we show that HN is in RP\(^\mathsf{NP} \), where RP is the class of problems solved by polynomial-time probabilistic Turing machines with one-sided error (i.e., the algorithm always gives a correct answer for a positive instance of the problem, and for negative instances the probability of mistake is bounded by some constant \( \varepsilon < 1 \)). This is a stronger result since RP\(^\mathsf{NP} \subseteq \Pi_2 \). Some background on these complexity classes can be found, e.g., in (Balcazar et al., 1988).

Previous algorithms were based on a direct application of effective Nullstellensätze: if one knows an upper bound on the degree of the \( g_i \)'s then the satisfiability of (1) is equivalent to the satisfiability of a certain linear system (of exponential size). Our algorithm is fundamentally different. The Nullstellensatz is used only indirectly, to give a correctness proof. The basic idea is to study the satisfiability of (1) in \( \mathbb{Z}/p\mathbb{Z} \). If the system is satisfiable modulo \( p \) for "many" primes \( p \) then it is satisfiable in \( \mathbb{C} \); otherwise it is not satisfiable. This is established in Section 4. We need a certain result on the roots of univariate polynomials which is established in Section 5. Its proof requires the Generalized Riemman Hypothesis (GRH). Assuming these properties, we show in Section 2 that HN is in \( \Pi_2 \). This requires some complexity-theoretic techniques (approximation of \#P functions).

Of course now the tantalizing question is whether HN \( \in \) NP. (To the author’s knowledge, this is an open problem even for sparse univariate polynomials.) The modular techniques used in this paper suggest a natural approach to this question. Unfortunately, as shown in Section 6 this approach cannot establish that HN \( \in \) NP, even for sparse univariate polynomials.

1.1. Notations

Let \( S \) be a system of the form (1), where the \( f_i \)'s have degree \( d_i \leq d \) and coefficients of size at most \( L \). (The size of an integer \( a \) is \( \log |a| \);\(^1 \) by convention the size of 0 is 0.) By definition, the total degree of \( S \) is \( \sigma = 2 + \sum_{i=1}^{s} d_i \). The size of this system is the bit size of a representation of \( S \) in a suitable binary encoding scheme. In this paper we use a sparse representation. This means that we do not charge for monomials with a coefficient equal to 0. Sometimes the opposite choice is made (see, e.g., Giusti and Heintz, 1993; Heintz and Morgenstern, 1993). With that dense representation, the decision algorithms mentioned above are downsized.

\(^1\) Throughout the paper log stands for \( \log_2 \).
from PSPACE to LOGSPACE. For univariate systems, there is an equally important representational issue, namely, whether exponents are coded in unary or binary. If binary (or sparse) encoding is used then polynomials can have exponential degree in the system’s size (for instance, it takes about \( n \) bits to code the monomial \( x^{2^n} \)). This is not an important issue for us because one can always represent polynomials of exponential degree by introducing intermediate variables and using “repeated squaring.” In fact, one could assume that the \( f_i \)’s are of degree \( d_i \leq 2 \) and have all their coefficients in \( \{-2, \ldots, 2\} \) without loss of generality (i.e., the general case of HN is polynomial-time many-one reducible to this special case).

As usual we denote by \( \pi(x) \) the number of primes in \( \{2, \ldots, x\} \). Let \( R_S \) be the set of prime numbers such that \( S \) is satisfiable in \( \mathbb{F}_p = \mathbb{Z}/p\mathbb{Z} \). \( \pi_S(x) \) denotes the cardinality of the set \( R_S(x) = R_S \cap \{1, 2, \ldots, x\} \). Given a polynomial \( f \in \mathbb{Z}[X] \), we use the abbreviations \( R_f, R_f(x) \), and \( \pi_f(x) \) for \( R_{f=0}, R_{f=0}(x) \), and \( \pi_{f=0}(x) \).

2. Position in the Polynomial Hierarchy

Theorem 1 is the crucial fact which makes it possible to locate HN in the polynomial hierarchy.

**Theorem 1.** There exist constants \( c_1, c_2, c_3 \in \mathbb{N} \) such that if \( A = d^c s(\lceil \log s \rceil + L) \) and \( x_0 \geq L^{c/2(\log o)^y} \), the following two properties hold:

- If (1) is not satisfiable in \( \mathbb{C} \) then \( \pi_S(x_0) \leq A \).
- If (1) is satisfiable in \( \mathbb{C} \) then \( \pi_S(x_0) \geq B = 8A(\log A + 3) \).

**Proof.** In Theorem 5 we show that if \( S \) is not satisfiable then there are at most \( d^c s(\log s + L) \) primes \( p \) such that \( S \) is satisfiable in \( \mathbb{F}_p \). Hence the bound \( \pi_S(x_0) \leq A \) holds for every \( x_0 \).

In Theorem 8, we show that there are absolute constants \( c_4, c_5, c_6 \) such that

\[
\pi_S(x) \geq \pi(x) - c_4 \frac{x^{1/2} \log x}{L \cdot 2^{(\log o)y}} - L \cdot 2^{(\log o)^y} x^{1/2}
\]

if \( S \) is satisfiable. The result follows from the theorem of prime numbers: \( \pi(x) \sim x/\ln x \). \( \blacksquare \)

The rationale for the setting of \( B \) in this theorem will become clear in the proof of Theorem 2. It is already clear that HN is in \( \text{P}^\text{NP} \): in order to decide whether \( S \) is satisfiable we just have to compute \( \pi_S(x_0) \) (note that Theorem 1 provides a bound on the size of \( x_0 \) which is polynomial in the system’s size). If \( \pi_S(x_0) \leq A \) then \( S \) is not satisfiable, otherwise it is satisfi-
This counting problem can be solved in \#P with the help of an oracle in NP. This oracle decides, given an integer \( p \), whether \( p \) is prime and \( S \) is satisfiable modulo \( p \). The first task is feasible since the set of prime numbers is known to be in NP (Pratt, 1975), and for the second one we just have to guess a solution.

In fact, we do not need exact counting since there is a large gap between \( A \) and \( B \). Stockmeyer (1985) has shown that approximate counting can be performed in the polynomial hierarchy (in \( \Delta_3 \)), and this result relativizes to an arbitrary oracle. Hence \( HN \) is in the polynomial hierarchy. By taking a closer look at Stockmeyer’s argument, one can prove the following result.

**Theorem 2.** Hilbert’s Nullstellensatz is in \( \text{RP}^{\#P} \) (and therefore in \( \Pi_2 \)).

**Proof.** As mentioned in the Introduction, the second part of the claim follows from \( \text{RP}^{\#P} \subseteq \Pi_2 \). However, for the clarity of exposition, we will show first that \( HN \in \Pi_2 \).

Stockmeyer’s result is based on a lemma of Sipser (1983) on universal hashing: there exists a \( \Sigma_2 \) predicate Hash(\( E \), \( m \)) which has the following property. If a set \( E \subseteq \{0, 1\}^k \) has at most \( 2^{m-2} \) elements then Hash(\( E \), \( m \)) is true; but if \( |E| \geq m2^m \), Hash(\( E \), \( m \)) is false. The \( \Pi_2 \) predicate \( \neg\text{Hash}(E, m) \), which expresses that \( E \) is not hashable into \( \{0, 1\}^m \), has the form

\[
\forall f_1 \cdots f_m C(f_1, \ldots, f_m)
\]

(2)

where \( C(f_1, \ldots, f_m) \) is the predicate:

\[
\exists x, x_1, \ldots, x_m \in E \bigwedge_{i=1}^{m} [f_i(x) = f_i(x_i) \land x \neq x_i].
\]

Here \( f_i : \{0, 1\}^k \to \{0, 1\}^m \) is a hash function, i.e., a function of the form \( f_i(x) = A_ix \), where \( A_i \) is a binary matrix and arithmetic is performed modulo 2 (actually the exact form of \( f_i \) does not really matter for our purposes).

We are going to apply this result to \( E = R_S(x_0) \) (given \( x_0 \), one can easily code the elements of \( R_S(x_0) \) by binary strings of equal length \( k \)). The membership of \( x, x_1, \ldots, x_m \) in \( E \) can be expressed by \( \Sigma_1 \) predicates. When these predicates are substituted in (2), the blocks of existential quantifiers can be merged, and we still have a \( \Pi_2 \) predicate. This predicate will be of polynomial size by Theorem 1. The satisfaction of \( \neg\text{Hash}(E, m) \) will be equivalent to the satisfiability of \( S \) if \( A \leq 2^{m-2} \leq m2^m \leq B \). Let \( m \) be the unique integer such that \( A \leq 2^{m-2} < 2A \): \( m < \log A + 3 \), hence the condition \( m2^m \leq B \) is satisfied with the choice of \( B \) made in Theorem 1.

To see that \( HN \in \text{RP}^{\#P} \), consider a variation of this \( \Pi_2 \) algorithm where the matrices \( A_1, \ldots, A_m \) are randomly chosen. It follows from the
analysis above that this randomized algorithm always gives a correct answer for satisfiable systems. In fact, Sipser’s (1983) lemma states that, if \(|E| \leq 2^{m-2}\) and \(A_1, \ldots, A_m\) are chosen at random, the probability of collision is at most 1/2 ("collision" meaning that \(C(f_1, \ldots, f_m)\) is true).

Hence this randomized algorithm fails with probability at most 1/2 for unsatisfiable systems. □

3. The Complexity of Primitive Elements

This section is of a technical nature. We establish some bounds on the complexity of primitive elements for use in Section 4.

The following result is an effective version of the primitive element theorem. Recall that the norm of a polynomial \(P = \sum_{k=0}^{d} a_k X^k \in \mathbb{Z}[X]\) is \(N(P) = \left(\sum_{k=0}^{d} |a_k|^2\right)^{1/2}\).

\[\text{THEOREM 3.} \quad \text{Let } \alpha_1 \text{ and } \alpha_2 \text{ be roots of two squarefree polynomials } P_1, P_2 \in \mathbb{Z}[X] \text{ of degree } d_1 \text{ and } d_2, \text{ and maximum norm } N. \text{ Let } d = \max\{d_1, d_2\}. \text{ There exists a squarefree polynomial } R \in \mathbb{Z}[X] \text{ of degree at most } n_1 n_2 \text{ and a root } \beta \text{ of } R \text{ such that } \alpha_i = Q_i(\beta)/a_i \text{ (i = 1, 2), } \text{where } Q_i \in \mathbb{Z}[X] \text{ and } a_i \in \mathbb{Z}, |a_i| \leq c \cdot N^{2d}. \text{ Here } c \text{ is a universal constant, } \deg(Q_i) < n_1 n_2 \text{ and } N(R) \leq c \cdot N^{2d}.\]

In fact \(R\) depends only on \(P_1\) and \(P_2\). See (Canny, 1988; Loos, 1982) for proofs of this result. We need a generalization to several polynomials. This is tedious but straightforward.

\[\text{LEMMA 1.} \quad \text{Let } \alpha_1, \ldots, \alpha_n \text{ be roots of } n \text{ squarefree polynomials } P_1, \ldots, P_n \in \mathbb{Z}[X] \text{ of degree } 2 \leq d_i \leq d, \text{ and norm } N(P_i) \leq N. \text{ There exists a squarefree polynomial } R_n \in \mathbb{Z}[X] \text{ of degree at most } d^n \text{ and a root } \beta_n \text{ of } R_n \text{ such that } \alpha_i = Q_m(\beta_n)/a_m, \text{ where } Q_m \in \mathbb{Z}[X] \text{ and } a_m \in \mathbb{Z}, |a_m| = N^{d^m}. \text{ Here } c' \text{ is a universal constant, } \deg(Q_m) < \sum_{i=2}^{d^m} d^i \text{ and } \]

\[N(R_n) \leq c' \cdot N^{2^{n-1}d^{n-1}d^{n-1}/2} \quad (3)\]

\[\text{Proof.} \quad \text{By induction. Let } R_{n-1} \in \mathbb{Z}[X] \text{ be a polynomial of degree at most } d^{n-1} \text{ such that } \alpha_1, \ldots, \alpha_{n-1} \text{ can be represented as } \alpha_i = Q_{i,n-1}(\beta_{n-1})/a_{i,n-1}, \text{ where } R_{n-1}(\beta_{n-1}) = 0, Q_{i,n-1} \in \mathbb{Z}[X] \text{ and } a_{i,n-1} \in \mathbb{Z}. \text{ In order to obtain } \beta_n, \text{ we can apply Theorem 3 to } \beta_{n-1} \text{ and } \alpha_n: \text{ there exists } R_n \in \mathbb{Z}[X] \text{ and a root } \beta_n \text{ of } R_n \text{ such that each element } \gamma \in \{\beta_{n-1}, \alpha_n\} \text{ can be expressed as } \gamma = Q(\beta_n)/a, \text{ where } a \in \mathbb{Z} \text{ and } Q \in \mathbb{Z}[X]. \text{ Each } \alpha_i (i = 1, \ldots, n-1) \text{ can now be expressed as} \]

\[ \alpha_i = Q_{i,n-1}(Q(\beta_n)/a)/a_{i,n-1}. \] (4)

Let \( N_{n-1} \approx N \) be (an upper bound on) the norm of \( R_{n-1} \) and \( N_n \) the norm of \( R_n \). By Theorem 3, \( N_n \leq c \cdot N_{n-1}^{2\alpha_{n-1}} \). Equation (3) follows from this inductive relation.

By induction hypothesis, \( \deg(Q_{i,n-1}) < \sum_{i=2}^{n-1} d^i \). Since \( \deg(Q) < d^n \), using (4) one can write

\[ \alpha_i = Q_{i,n}(\beta)/a_n \text{ with } \deg(Q_n) < \sum_{i=2}^{n-1} d^i \] and

\[ a_n = a^{\deg(Q_{i,n})}a_{i,n-1}. \]

The degree of \( Q_n \) can be reduced to at most \( d^n - 1 \) by computing \( \text{rem}(Q_n, R_n) \) (this will slightly increase \( a_n \)). One can give somewhat better bounds if instead of treating the list of \( \alpha_i \)'s iteratively, one computes the primitive element by “divide-and-conquer” (i.e., if the first and second half of the list are processed separately, and the 2 primitive elements are put together at the end). For instance, the exponent \( n(n-1)/2 \) in (3) can be replaced by \( O(n) \).

**Theorem 4.** Let \( x_1, \ldots, x_n \) be \( n \) algebraic numbers which are roots of polynomials \( A_i \in \mathbb{Z}[X] \) of degree at most \( d \) with coefficients of size at most \( L \). There exists a primitive element \( r \) for \( x_1, \ldots, x_n \) which is a root of an irreducible polynomial \( B \in \mathbb{Z}[X] \) of degree at most \( d^n \). The coefficients of \( B \) are of size at most \( L \cdot d^{nO(1)} \). Moreover, each \( x_i \) can be represented as \( x_i = Q_i(r)/a_i \), where \( Q_i \in \mathbb{Z}[X] \) and \( \log |a_i| = L \cdot d^{nO(1)} \).

**Proof.** The primitive element can be obtained in three steps:

1. Make the \( A_i \)'s squarefree by computing \( P_i = A_i/gcd(A_i, A_i') \).
2. Apply Lemma 1 to \( P_1, \ldots, P_n \); this gives \( a_i, Q_i \), and a polynomial \( R \in \mathbb{Z}[Z] \).
3. The primitive element \( r \) is a root of \( R \). So \( B \) is an irreducible factor of \( R \).

The stated bounds follow from Lemma 3. Indeed, steps 1 and 3 are very cheap compared to step 2. This follows from the bound on polynomial factors in (Mignotte, 1982).

**4. Solutions Modulo \( p \)**

One can take care easily of unsatisfiable systems with the effective Hilbert Nullstellensatz. The case of satisfiable systems is more involved and requires the Generalized Riemann Hypothesis.
4.1. Unsatisfiable Systems

If (1) has no solution in \( \mathbb{C} \) then by Hilbert’s Nullstellensatz there exist \( a \in \mathbb{Z}, a \neq 0 \) and polynomials \( g_1, \ldots, g_s \in \mathbb{Z}[X_1, \ldots, X_n] \) such that

\[
a = g_1 f_1 + \cdots + g_s f_s.
\]

A bound on the size of \( a \) can be easily obtained if one has an a-priori bound on the degrees \( D_i \) of the \( g_i \)'s. It is shown in (Kollár, 1988) that one can take \( D_i = \max\{3, d\}^r \). This leads to a \( s^{O(1)} d^{O(r^2)} L \) bound on the size \( a \). This bound was improved to \( d^{O(r^2)}(\log s + L) \) in (Krick and Pardo, 1994). Theorem 5 follows easily (one could also use the simpler \( d^{O(r^2)} \) bound).

Theorem 5. If (1) has no solution in \( \mathbb{C} \) then \( R_S \) is finite and \( |R_S| \leq d^{O(r^2)}(\log s + L) \).

Recall that \( R_S \) is the set of prime numbers \( p \) such that (1) has a solution in \( \mathbb{Z}/p\mathbb{Z} \).

Proof. If (1) has no solution in \( \mathbb{C} \) then (5) holds in \( \mathbb{C} \) and thus also in \( \mathbb{Z}/p\mathbb{Z} \). This implies that (1) has no solution in \( \mathbb{Z}/p\mathbb{Z} \) if \( a \mod p \neq 0 \). The result follows since \( a \) cannot have more than \( \log a \) prime factors. \( \blacksquare \)

4.2. Satisfiable Systems

The first step toward the analysis of satisfiable systems will be to show (in Theorem 7) that these systems have algebraic solutions which have “short” descriptions. For this we need the following theorem. It follows from the quantifier elimination results in (Fichtas et al., 1990).

Theorem 6. Let \( \Phi \) be a prenex formula in the first-order theory of \( \mathbb{C} \). Let \( r \) be the number of quantifier blocks, \( n \) the total number of variables, and \( \sigma(\Phi) \) the total degree of \( \Phi \), defined as

\[
\sigma(\Phi) = 2 + \sum_{i=1}^{s} \deg F_i,
\]

where \( F_1, \ldots, F_s \) are the polynomials occurring in \( \Phi \). \( \Phi \) is equivalent to a quantifier-free formula \( \Psi \) in which all polynomials have degree at most

\[
2^{\sigma(\Phi)(\log n)^{O(1)}}.
\]

The number of polynomials occurring in \( \Psi \) is \( O(\sigma(\Phi)^d) \).

\[
\text{HILBERT'S NULLSTELLENSATZ}
\]
Moreover, when the constants in $\Phi$ are integers of bit size at most $L$, the constants in $\Psi$ are integers of bit size at most $L \cdot 2^{n^{O((\log \sigma) / \log \log n)O(1)}}$.

**Theorem 7.** There are absolute constants $c_1$ and $c_2$ such that if $(1)$ has a solution over $\mathbb{C}$ then there exists a solution $x = (x_1, \ldots, x_n)$ such that each $x_i$ is a root of a polynomial of degree at most $2^{(n \log \sigma)\gamma_1}$ with coefficients of bit size at most $L \cdot 2^{(n \log \sigma)\gamma_2}$.

We shall first prove this result in a special case.

**Lemma 2.** Theorem 7 holds for systems that have a finite number of solutions.

Proof. Let $S$ be the solution set of $(1)$ and $S_i \subseteq \mathbb{C}$ the projection of $S$ on the $i$th coordinate axis. By Theorem 6, $S_i$ can be defined by a quantifier-free formula in which polynomials $P_{i1}, \ldots, P_{im_i}$ of degree at most $2^{(n \log \sigma)\gamma_1}$, and coefficients of size at most $L \cdot 2^{(n \log \sigma)\gamma_2}$ appear (moreover, $m_i = \sigma^{O(1)}$; $c_1$ and $c_2$ are absolute constants). If $S$ is finite then each $S_i$ is finite. Hence each element of $S_i$ is a root of some $P_{ij}$. The result follows since by definition the components of any solution $x \in S$ must be in $S_1, \ldots, S_n$.

Proof of Theorem 7. By induction on $n$; the constants will satisfy $c_1 = c_1'$ and $c_2 \geq c_2'$. If $(1)$ has finitely many solutions the result holds by Lemma 2.

Assume now that $(1)$ has infinitely many solutions. Then at least one $S_i$ must be infinite. This implies that $\mathbb{C} \setminus S_i$ is finite and that its elements are chosen among the roots of $m_i = \sigma^{O(1)}$ polynomials of degree at most $2^{(n \log \sigma)\gamma_1}$. Hence $|\mathbb{C} \setminus S_i| \leq 2^{(n \log \sigma)\gamma_2}$, for some absolute constant $\gamma_2$. This guarantees the existence of an integer $\alpha \in S_i$ such that $0 \leq \alpha = 2^{(n \log \sigma)\gamma_1}$.

Since $\alpha$ is of polynomial size, this integer can be substituted to $x_i$ in $(1)$ without blowing up the system’s size too much. More precisely, we obtain a new satisfiable system in $n - 1$ variables where the polynomials are of degree at most $d$ and have coefficients of size at most $L + d \log \alpha \leq L + d(n \log \sigma)\gamma_2$. By induction hypothesis this system has a solution whose components are roots of polynomials of degree at most $2^{((n-1) \log \sigma)\gamma_1}$. They have coefficients of size bounded by

$$B = [L + d(n \log \sigma)\gamma_2]2^{((n-1) \log \sigma)\gamma_2}.$$ 

For the induction hypothesis to hold in dimension $n$, we need to have $B \leq L \cdot 2^{(n \log \sigma)\gamma_2}$. Assuming without loss of generality that $L$ and $d(n \log \sigma)\gamma_2$ are both larger than 2, we have

$$B \leq L \cdot 2^{((n-1) \log \sigma)\gamma_2 \log [d(n \log \sigma)\gamma_2]}.$$

Recalling that \( d \) is \( \sigma \), it suffices to have
\[
(n \log \sigma)^2 - ((n - 1) \log \sigma)^2 \geq \log \sigma + c_3 \log(n \log \sigma).
\]
It is not hard to see that this constraint is satisfied if the absolute constant \( c_3 \) is large enough.

**Lemma 3.** Let \( x = (x_1, \ldots, x_n) \) be a vector of algebraic numbers solution of (1). Let \( r \) be a primitive element for \( x_1, \ldots, x_n \); there exist polynomials \( Q_1, \ldots, Q_n \in \mathbb{Z}[x] \) and \( a \in \mathbb{N} \) such that \( x_i = Q_i(r)/a \). Let \( R \in \mathbb{Z}[X] \) be an irreducible polynomial such that \( R(r) = 0 \). If \( R \) has a root in \( F_p \) and \( a \mod p \neq 0 \), (1) is satisfiable in \( F_p \).

**Proof.** For \( i \in \{1, \ldots, s\} \), let
\[
g_i(X) = a^{d_i}f_i(Q_1(X)/a, \ldots, Q_n(X)/a) \tag{6}
\]
Note that \( g_i \in \mathbb{Z}[X] \). These polynomials must be multiple of \( R \) since \( R \) is irreducible and \( g_i(r) = 0 \). Hence there are polynomials \( A_1, \ldots, A_s \in \mathbb{Z}[X] \) such that
\[
g_i(X) = R(X)A_i(X). \tag{7}
\]
If \( a \mod p \neq 0 \), (6) and (7) must also hold in \( F_p \). This implies that if \( x_0 \) is a root of \( R \) in \( F_p \), \( (Q_1(x_0)/a, \ldots, Q_n(x_0)/a) \) is a solution of (1) in \( F_p \).

**Theorem 8.** There are absolute constants \( c_4, c_5, c_6 \) such that if (1) is satisfiable,
\[
\pi_S(x) \geq \frac{\pi(x) - c_6 \eta x^{1/2} \log x}{L \cdot 2^{(\log \eta)^{1/2} x^{1/2}}} - L \cdot 2^{(\log \eta)^{1/2} x^{1/2}}.
\]

**Proof.** We just need explicit estimates on \( a \) and the \( Q_i \)'s in order to apply Lemma 3. The algebraic numbers \( x_1, \ldots, x_n \) are roots of polynomials \( P_1, \ldots, P_n \) whose degree and coefficient size can be bounded by Theorem 7. The complexity of the primitive element \( r \) can then be estimated by Theorem 4. The number of primes \( p \) in \( \pi_S(x) \) can be estimated by Corollary 1 (here we use the fact that the bit size of the discriminant of a polynomial is polynomially bounded in its degree and the bit size of its coefficient). From this estimate one has to subtract the prime factors of \( a \); there are at most \( \log a \) such primes.

5. Roots of Univariate Polynomials

Let \( f \in \mathbb{Z}[X] \) be an irreducible polynomial of degree \( n \); let \( \Delta \) be the discriminant of \( f \). For a prime \( p \),
\[ W(p) = |\{ k; 0 \leq k \leq p - 1, f(k) \equiv 0 \pmod{p}\}| \]

denotes the number of roots of \( f \) in \( F_p \). Let \( S(x) = \sum_{p \leq x} (1 - W(p)) \), where \( \sum' \) denotes summation on those primes \( p \) which do not divide \( \Delta \). The following bound is due to Adleman and Odlyzko (1983, proof of Lemma 3); see also (Weinberger, 1984).

**Theorem 9.** \( |S(x)| = O(x^{1/2} \log(\Delta x^n)) \).

This result relies on GRH through an effective version of the Chebotarev density theorem (Lagarias and Odlyzko, 1977) (in fact, only the effective prime ideal theorem is used). A bound on \( r_f(x) \) follows easily.

**Corollary 1.** There exists an absolute constant \( c \) such that
\[
\pi_f(x) \geq \frac{1}{n} \left[ \pi(x) - \log \Delta - c \cdot x^{1/2} \log(\Delta x^n) \right].
\]

**Proof.** By Theorem 9,
\[
\sum'_{p \leq x} W(p) = \sum'_{p \leq x} 1 - c \cdot x^{1/2} \log(\Delta x^n)
\]
for some universal constant \( c \). Since \( \Delta \) has at most \( \log \Delta \) prime factors, \( \sum_{p \leq x} 1 \approx \pi(x) - \log \Delta \). Let \( r_f(p) = 1 \) if \( f \) has a root in \( F_p \), and \( r_f(p) = 0 \) otherwise. Since \( f \) is irreducible in \( \mathbb{Z}[x] \) this polynomial cannot be identically 0 in \( F_p \); hence \( W(p) = nr(p) \). The result now follows from (8) since \( \pi_f(x) = \sum_{p \leq x} r_f(p) = \sum_{p \leq x} W(p)/n \).

**Remark 1.** Corollary 1 provides a lower bound of \( 1/n \) on the density of \( R_f \) in the set of prime numbers. Infact, by the Chebotarev density theorem, the exact value of the density is \( |C|/|G| \), where \( G \) is the Galois group of \( f \) and \( C \) is the set of permutations \( g \in G \) which have at least one fixed point. It can be shown that \( |C| = 1 \) when \( |G| = n \).

Hence the \( 1/n \) lower bound cannot be improved in the worst case. However, “generic” polynomials with large coefficients have the full symmetric group as Galois group. In this case, \( |C|/|G| \geq 1/2 \) (and \( |C|/|G| \approx 1 - 1/e \approx 0.63 \) for polynomials of high degree which have \( S_n \) as Galois group).

Theorem 8 provides a lower bound on the density of \( R_S \) which can be exponentially small in the size of a satisfiable system. The following example

\[ 2 \text{ For any transitive subgroup } G \text{ of the symmetric group (and in particular for the Galois group of an irreducible polynomial) and any } i \in \{1, \ldots, n\}, \text{ there are exactly } |G|/n \text{ permutations } g \in G \text{ such that } g(i) = i. \text{ Hence only the identity can have fixed points when } |G| = n. \]
HILBERT’S NULLSTELLENSATZ shows that in some cases the density of \( R_S \) can really be exponentially small (however, by Remark 1 one can expect its density to be at least \( 1/2 \) for most satisfiable systems).

**Example 1.** Let \( S_p \) be the system

\[
\begin{align*}
    x^p & = 1 \\
    y(x - 1) & = 1,
\end{align*}
\]

where \( p \) is a prime number. The density of \( R_{S_p} \) is \( 1/(p - 1) \).

The \( p - 1 \) solutions of \( S_p \) over \( \mathbb{C} \) are of the form \((x, 1/(x - 1))\), where \( x \) is a \( p \)th root of unity different from 1. One can write \( X^p - 1 = (X - 1)_p(X) \), where \( (X) = \sum_{k=0}^{p-1} X^k \) is the cyclotomic polynomial of order \( p \). Given a prime \( q \), \( S_p \) is satisfiable in \( F_q \) if and only if \( C_p \) has a root different from 1 in \( F_q \). There is a single value of \( q \) \((q = p)\) for which \( C_p(1) = 0 \) in \( F_q \). It is well known that \( C_p \) is irreducible over \( \mathbb{Z} \) and that its Galois group is (isomorphic to) \( \mathbb{Z}/(p - 1)\mathbb{Z} \). Hence by Remark 1, the density of \( R_{S_p} \) is indeed \( 1/(p - 1) \).

This density is exponentially small in the size of \( S_p \), which is \( O(\log p) \). Recall also that, as pointed out in subsection 1.1, \( S_p \) can be replaced by an equivalent system \( S'_p \) of size \( O(\log p) \) where all polynomials have degree 2. Since \( R_{S'_p} = R_{S_p} \), the density of \( R_{S'_p} \) is again exponentially small in the system’s size.

### 6. A LOWER BOUND FOR UNSATISFIABLE SYSTEMS

In Theorem 5 we have given an upper bound on the size of \(|R_S|\) which may be exponential in the size \(|S|\) of an unsatisfiable system. It is of interest to find out whether this bound can be improved. For instance, if one had a polynomial upper bound \((\text{say}, |S|^c)\) then, under GRH, Hilbert’s Nullstellensatz would be in NP. (A certificate for a satisfiable system would simply consist of a list \( p_1, \ldots, p_{|S|+1} \) of distinct primes of polynomial size such that \( S \) is satisfiable modulo \( p_i \), together with a list of corresponding solutions; the existence of such primes follows from Theorem 8.) Unfortunately, there is no such polynomial upper bound. We shall see in Theorem 10 that \(|R_S|\) can really be exponential in \(|S|\). First, we need the following observation, which is due to Noam Elkies.

**Lemma 4.** Let \( \pi_n \) be the product of the first \( n \) prime numbers: for \( n \equiv 2 \), \( \pi_n^2 - 1 \) has at least \( 2^n \) distinct prime factors.
Proof. The factorization of the polynomial \( x^N - 1 \) \((N \geq 2)\) over the integers is

\[
x^N - 1 = \prod_{d \mid N} \Psi_d(x),
\]

where

\[
\Psi_d(x) = \prod_{1 \leq k \leq d, \gcd(k, d) = 1} (x - e^{2i\pi k/d})
\]

is the cyclotomic polynomial of order \( d \). These two properties of cyclotomic polynomials will be useful.

1. If a prime number \( p \) does not divide \( N \) then the polynomials \( \Psi_d \) in (9) have pairwise distinct roots in \( F_p \). Indeed, if \( \Psi_{d_1} \) and \( \Psi_{d_2} \) had a common root \( r \) in \( F_p \), the derivative of \( x^N - 1 \) would vanish at \( r \). However, the derivative \((Nx^N - 1)\) has no roots in \( F_p \) since \( N \mod p \neq 0 \).

2. \( \Psi_d(x) \geq 2 \) for any integer \( x \geq 3 \). Indeed, each factor \((x - e^{2i\pi k/d})\) in (10) has modulus larger than 1. Hence \(|\Psi_d(x)| > 1\), and in fact \(\Psi_d(x) \geq 2\) since \(\Psi_d\) has integer coefficients.

Let \( p \) be a prime factor of \( \Psi_d(N) \), where \( d \mid N \). Since \( p \) is a fortiori a prime factor of \( N^N - 1 \), \( p \) does not divide \( N \) (\( N \) and \( N^N - 1 \) are relatively prime). Hence by property 1, \( p \) is not a factor of any other \( \Psi_{d'}(N) \), where \( d' \mid N \). Setting \( N = \pi_n \) gives the desired result since in this case there are \( 2^n \) factors in (9). (Each factor \( \Psi_d(N) \) gives at least one new prime factor since \( \Psi_d(N) \geq 2 \) by property 2.)

The motivation for this lemma came from a conjecture of Shub and Smale (1996) on the length of computations for \( k! \). The “naïve” method for computing \( k! \) requires \( \Theta(k \log k) \) operations (additions and multiplications; subtractions are also allowed). Elkies has pointed out that a version of the elliptic curve factoring method suggests that there should exist computations of length growing slower than any power of \( k \), indeed no faster than \( \exp[\log(k)^{1/2}] \).

**Theorem 10.** For \( n \geq 1 \), let \( S_n \) be the system

\[
x^{\pi_n} - 1 = 0
\]

\[
x - \pi_n = 0,
\]

where \( \pi_n \) is the product of the first \( n \) prime numbers. This system is unsatisfiable over \( \mathbb{C} \), and \( |R_{S_n}| \geq 2^n \).
HILBERT'S NULLSTELLENSATZ

Proof. $S_n$ is obviously unsatisfiable over $\mathbb{C}$. The prime factors of $\pi_n$ are in $R_{S_n}$, and by Lemma 4 there are at least $2^n$ of them.

By the theorem of prime numbers, the $n$th prime is $O(n \log n)$. Hence the bit size of $\pi_n$ is also $O(n \log n)$. Therefore $|R_{S_n}|$ really is exponential in $|S_n|$.

7. Final Remarks

Some of the techniques in this paper might be useful in practice. For instance, one can try and solve (1) modulo several randomly drawn primes. If there are “many” positive answers, then one can conclude that (1) is satisfiable over $\mathbb{C}$ with high probability. Precise bounds on the proportion of positive answers that should be obtained can be worked out using only the effective Nullstellensatz (see subsection 4.1). By Remark 1, one can expect that trying a “small” number of random primes will be enough to establish that a satisfiable system is indeed satisfiable (with high probability), if the defining polynomial $R$ for the primitive element has the full symmetric group as Galois group (in this case, (1) is solvable modulo $p$ for at least one half of all primes $p$). For other satisfiable systems, one may have to try an exponential number of primes. Another related shortcoming is that one cannot establish (at least if we try only a polynomial number of primes) that a system is not satisfiable, even in a probabilistic sense. The reason is again that the density of “good” primes for a satisfiable system may be exponentially small (see section 5). It would be interesting to have a rigorous average-case analysis of this method.

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