Bethe–Salpeter scattering amplitude in Minkowski space

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A B S T R A C T

The off-mass shell scattering amplitude, satisfying the Bethe–Salpeter equation for spinless particles in Minkowski space with the ladder kernel, is computed for the first time.

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1. Introduction

Obtaining the solutions of Bethe–Salpeter (BS) equation in its original Minkowski space formulation [1] has raised an increasing interest in the recent years [2,5,3,4,6]. On one hand, the Wick rotation itself is not directly applicable for computing electromagnetic form factors [7,8] due to the existence of singularities in the complex momentum plane whose contributions are in general unknown. The Euclidean solutions are still used in the context of BS–Schwinger–Dyson equations and it is claimed they provide reliable results for bound state form factors at the price of a numerical *tour de force* [9–15]. On the other hand, the off-shell BS scattering amplitude – mandatory for important physical applications – like computing the transition e.m. form factor $\gamma^* d \to np$, or solving the three-body BS–Faddeev equations – requires a full Minkowski solution which has not yet been obtained.

A method based on the Nakanishi representation [16] of the BS amplitude was developed in [3,4] allowing to compute for the first time the bound state Minkowski amplitude and latter on [7,8] the corresponding form factors. Although this approach could be naturally extended to the scattering states, we have found that the problem could be solved in a simplest and more straightforward way. The aim of this Letter is to presents a direct solution of the original BS equation in Minkowski space providing the scattering length, elastic and inelastic phase shifts and the first results for the half-off-shell BS amplitude.

2. Method for solving the equation

The inhomogeneous BS equation is graphically represented in Fig. 1. In Minkowski space it reads:

$$F(p, p_s; P) = K(p, p_s; P) - i \int \frac{d^4 p'}{(2\pi)^4} \frac{K(p, p'; P) F(p', p_s; P)}{[(\frac{P}{2} + p')^2 - m^2 + i\epsilon][(\frac{P}{2} - p')^2 - m^2 + i\epsilon]}. \tag{1}$$

We denote by p the relative (off-shell) four-momentum variable of the equation, p_s the scattering momentum and P the total momentum of the state with $P^2 = M^2$, the squared total mass of the system. We will consider hereafter spinless particles ($m = 1$) interacting by the one-boson exchange kernel K:

$$K(p, p'; P) = -\frac{16\pi m^2 \alpha}{(p - p')^2 - \mu^2 + i\epsilon}, \tag{2}$$

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Fig. 1. Bethe–Salpeter equation for a scattering state.

where α is the dimensionless coupling constant of the Yukawa potential $V(r) = -\alpha \exp(-\mu r)/r$.

The difficulty in computing the off-shell amplitude $F(p, p_s; P)$ in the entire domain of its arguments is due the singular character of the amplitude itself as well as in the integrand of the BS equation in Minkowski space (1). These singularities are integrable in the mathematical sense, due to $i\epsilon$ in the denominators of propagators, but their integration is a quite delicate task and requires the use of appropriate analytical as well as numerical methods.

To avoid these problems, Eq. (1) was first solved on-shell [17] by rotating the integration contour $p_0 \to ip_4$ and taking into account the contributions of the crossed singularities, which are absent in the bound state case. Few other methods were also developed in the same line [18]. Until now, the off-shell amplitude has been computed only for a separable kernel [19]. A method similar to the one developed in [3,4] has been proposed in [6] for solving the scattering states although the numerical solutions are not yet available.

The amplitude $F(p, p_s; P)$ depends on the three four-momenta p, p_s, P. For a given incident momentum $\vec{p}_s$ and written in the center of mass frame $\vec{P} = 0$, $P_0 = M = 2\varepsilon_{p_s} = 2\sqrt{m^2 + p_s^2}$, F depends on three scalar variables $|\vec{p}|$, $|\vec{p}_s|$ and $z = \cos(\vec{p}, \vec{p}_s)$. It will be hereafter denoted by $F(p_0, p, z; p_s)$, setting abusively $p = |\vec{p}|$, $p_s = |\vec{p}_s|$.

In this Letter we will restrict to sketch our solution method and to present the first results for the S-wave on-mass and half-off-mass shell amplitude $F_0(p_0, p)$ obtained by the following partial wave decomposition of Eq. (1):

$$F_0(p_0, p) = \frac{1}{32\pi} \int_{-1}^{1} dz \, F(p_0, p, z; p_s) \tag{3}$$

The amplitude we consider here is a particular case of the so-called full off-shell amplitude $F_0(p_0, p, p_{0s}, p_s; M)$. The latter, in addition to the variables p_0, p depends also on the off-shell independent variables p_{0s}, p_s, now with $p_{0s} \neq \varepsilon_{p_s}$. The total mass $M = \sqrt{s}$ is neither equal to $2\varepsilon_{p_s}$ nor related to p_{0s}. By "off-shell amplitude" we will hereafter mean half-off-shell amplitude. The method we have developed is also applicable to the full off-shell amplitude, though its dependence on two extra variables p_{0s}, p_s requires much more extensive numerical calculations.

The on-shell amplitude $F_0^{on} \equiv F_0(p_0 = 0, p = p_s)$ determines the phase shift according to:

$$\delta_0 = \frac{1}{2i} \log\left(1 + \frac{2ip_s}{\varepsilon_{p_s}} F_0^{on}\right). \tag{4}$$

Several steps must be accomplished before obtaining a soluble equation for F_0 which takes into account the four sources of singularities of the BS equation:

(i) The propagators in the r.h. side of (1) have two poles, each of them represented as sum of principal value and δ-function. Their product gives rise to terms having respectively 0, 1 and 2 δ's. After partial wave decomposition, the 4D equation (1) is reduced into a 2D one. Integrating analytically over p_0' the δ contributions and eliminating the principal values singularities by subtractions, one is left with an S-wave equation in the form:

$$F_0(p_0, p) = F_0^B(p_0, p) + \frac{i\pi^2 p_s}{8\varepsilon_{p_s}} W_0^S(p_0, p, 0, p_s) F_0(0, p_s)$$

$$+ \frac{\pi}{2M} \int_0^\infty \frac{dp'}{\varepsilon_{p'}(2\varepsilon_{p'} - M)} \left[p'^2 W_0^S(p_0, p, a_-, p') F_0(|a_-|, p') - \frac{2p_s^2 \varepsilon_{p'}}{\varepsilon_{p'} + \varepsilon_{p_s}} W_0^S(p_0, p, 0, p_s) F_0(0, p_s) \right]$$

$$- \frac{\pi}{2M} \int_0^\infty \frac{p'^2 dp'}{\varepsilon_{p'}(2\varepsilon_{p'} + M)} W_0^S(p_0, p, a_+, p') F_0(a_+, p')$$

$$+ \frac{i}{2M} \int_0^\infty \frac{p'^2 dp'}{\varepsilon_{p'}} \int_0^\infty dp_0' \left[\frac{W_0^S(p_0, p, p_0', p') F_0(p_0', p') - W_0^S(p_0, p, a_-, p') F_0(|a_-|, p')}{p_0'^2 - a_-^2} \right]$$

$$- \frac{i}{2M} \int_0^\infty \frac{p'^2 dp'}{\varepsilon_{p'}} \int_0^\infty dp_0' \left[\frac{W_0^S(p_0, p, p_0', p') F_0(p_0', p') - W_0^S(p_0, p, a_+, p') F_0(a_+, p')}{p_0'^2 - a_+^2} \right] \tag{5}$$

where $a_\mp = \varepsilon_{p'} \mp \varepsilon_{p_s}$ and W_0^S is the S-wave kernel – suitably symmetrized on p_0' variable to restrict its integration domain to $[0, \infty]$ – given by

$$W_0^S(p_0, p, p_0', p') = W_0(p_0, p, p_0', p') + W_0(p_0, p, -p_0', p')$$

with:

$$W_0(p_0, p, p_0', p') = -\frac{\alpha m^2}{\pi p p'} \left\{ \frac{1}{\pi} \log \left| \frac{(\eta+1)}{(\eta-1)} \right| - iI(\eta) \right\}, \qquad I(\eta) = \begin{cases} 1 & \text{if } |\eta| \leqslant 1, \\ 0 & \text{if } |\eta| > 1, \end{cases} \tag{6}$$

and

$$\eta = \frac{1}{2pp'} \left[(p_0 - p_0')^2 - p^2 - p'^2 - \mu^2 \right].$$

The inhomogeneous (Born) term F_0^B reads:

$$F_0^B(p_0, p) = \frac{\pi^2}{4} W_0(p_0, p, 0, p_s)$$

The details of the derivation of Eq. (5) as well as its generalization to the full off-shell amplitude are quite lengthy and will be given in a forthcoming publication. The origin of the different terms appearing in (5) is however quite clear. The non-integral term in the first line, follows from the integrated (2D) product of the two δ-function mentioned above. The one-dimensional integrals – second and third lines – results from one δ-function terms, after integration over p_0'. The last two lines come from the principal values (PV) alone. The differences appearing in the squared brackets correspond to removing the pole singularities at $2\epsilon_{p'} = M$ (second line) and $p_0' = a_\pm$ (third and forth lines) according to the well-known subtraction technique eliminating singularity:

$$PV \int_0^\infty \frac{f(x') \, dx'}{x'^2 - a^2} = \int_0^\infty dx' \left[\frac{f(x') - f(a)}{x'^2 - a^2} \right].$$

In l.h. side the integrand at $x' = a$ is singular that complicates the numerical calculation of integral, whereas r.h. side does not contain this singularity.

(ii) The propagator of the exchanged particle (2) has two poles which, after partial wave decomposition, turn into logarithmic singularities in kernel (6). Their positions are found analytically and the numerical integration over p_0' is split into intervals between two consecutive singularities. Inside each of these intervals an appropriate change of variable is made to make regular the integrand of Eq. (5). We proceed in a similar way for the p' integration.

(iii) The inhomogeneous (Born) term F_0^B has also logarithmic singularities in both variables p_0, p which are analytically known.

(iv) The amplitude F_0 itself has many singularities, among which those originated by the Born term F_0^B are the strongest ones. This makes difficult representing F_0 on a basis of regular functions. To circumvent this problem we made the replacement $F_0 = F_0^B f_0$, where f_0 is a smoother function. After that, the singularities of the inhomogeneous Born term are casted into the kernel and integrated using the same procedure as in (ii).

We obtain in this way a non-singular equation for f_0 which we solve by standard methods. The off-mass shell BS amplitude F_0 in Minkowski space is thus safely computed.

3. Numerical results

Our first check was to solve the bound state problem by dropping the inhomogeneous term in (1) and setting $M = 2m - B$. The binding energy B thus obtained, coincides within four-digit accuracy, with the one calculated in our previous work [3].

The S-wave off-shell scattering amplitude F_0 was then calculated and the phase shifts extracted according to (4). An independent equation, similar to the one obtained in [17], has also been derived, which provides the phase shifts and the Minkowski off-shell amplitude for the particular value $p_0 = \varepsilon_{p_s} - \varepsilon_p$ restricted to the interval $0 \leqslant p \leqslant p_s$. The values found by these two independent methods are consistent to each other.

The BS relativistic formalism accounts naturally for the meson creation in the scattering process, when the available kinetic energy allows it. The inelasticity threshold corresponding to n-particle creation is given by

$$p_s^{(n)} = m \sqrt{\left(\frac{\mu}{m}\right) n + \frac{1}{4}\left(\frac{\mu}{m}\right)^2 n^2}. \tag{7}$$

Below the first inelastic threshold, $p_s^{(1)} = \sqrt{m\mu + \mu^2/4}$, the phase shifts are real. This unitarity condition is not automatically fulfilled in our approach, but appears as a consequence of handling the correct solution and provides a stringent test of the numerical method. Above $p_s^{(1)}$, the phase shift obtains an imaginary part which behaves like

$$\text{Im}(\delta_0) \sim \left(p_s - p_s^{(1)}\right)^2 \tag{8}$$

in the threshold vicinity. Higher inelasticity thresholds, corresponding to creation of 2, 3, etc. intermediate mesons at $p_s^{(n)}$, are also taken into account in our calculations.

The low energy parameters were computed and found to be consistent with a quadratic fit to the effective range function $p \cot \delta(p) = -\frac{1}{a_0} + \frac{1}{2} r_0 p^2$. The BS scattering length a_0 as a function of the coupling constant α is given in Fig. 2 for $\mu = 0.50$. It is compared to the non-relativistic (NR) values provided by the Schrödinger equation with the Yukawa potential. The singularities correspond to the appearance of the first bound state at $\alpha_0 = 1.02$ for BS and $\alpha_0 = 0.840$ for NR. As one can see, the differences between a relativistic and
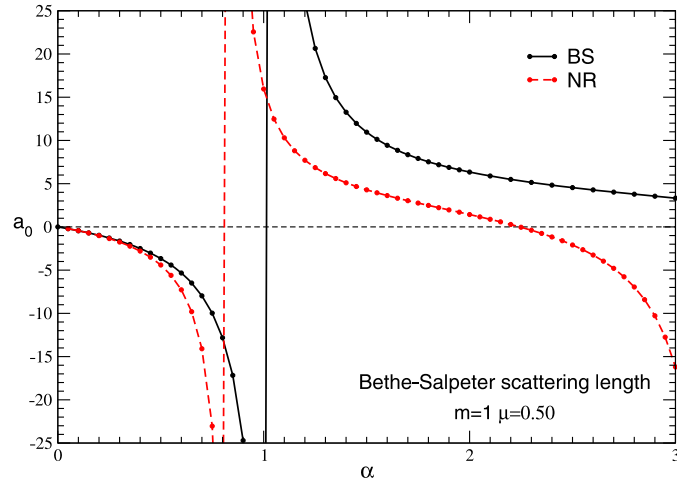
Fig. 2. BS scattering length a_0 versus the coupling constant α (solid), compared to the non-relativistic results (dashed) for $\mu = 0.5$.

Table 1
Scattering length values obtained with Bethe–Salpeter equation as a function of the coupling constant α for $m = 1$ and different values of the exchanged mass $\mu = 0.15$, $\mu = 0.50$ and $\mu = 1.00$.

α	$\mu = 0.15$	$\mu = 0.50$	$\mu = 1.00$
0.01	−0.460D+00	−0.403D−01	−0.100D−01
0.05	−0.270D+01	−0.209D+00	−0.510D−01
0.10	−0.692D+01	−0.438D+00	−0.104D+00
0.20	−0.346D+02	−0.971D+00	−0.217D+00
0.30	0.795D+02	−0.164D+01	−0.339D+00
0.40	0.272D+02	−0.250D+01	−0.474D+00
0.50	0.214D+02	−0.366D+01	−0.621D+00
0.60	0.128D+02	−0.534D+01	−0.784D+00
0.70	0.866D+01	−0.798D+01	−0.965D+00
0.80	0.373D+01	−0.128D+02	−0.117D+01
0.90	−0.457D+01	−0.247D+02	−0.140D+01
1.00	−0.281D+02	−0.103D+03	−0.166D+01
1.10	0.900D+03	0.620D+02	−0.195D+01
1.50	0.247D+02	0.110D+02	−0.379D+01
2.0	0.174D+02	0.634D+01	−0.111D+02
2.5	0.144D+02	0.454D+01	0.568D+02

a non-relativistic treatments of the same problem are not of kinematical origin since even for processes involving zero energy they can be substantially large, especially in presence of bound state. It is worth noticing that only in the limit $\alpha \to 0$ the two curves are tangent to each other and in this region the results are given by the Born approximation

$$a_0^B = -\frac{1}{\mu}\frac{m}{\mu}\alpha \tag{9}$$

which is the same for the NR and the BS equation. Beyond this region both dynamics are not compatible. Some selected numerical values of the scattering length are listed in Table 1 for different values of the exchanged mass μ. As one can see by direct inspection, the scaling properties of the non-relativistic equation [20], in particular the relation between the scattering length corresponding to different values of μ and coupling constants

$$a_0\left(\frac{\mu}{m}, \alpha\right) = \frac{1}{\mu} a_0\left(1, \frac{\alpha}{\frac{\mu}{m}}\right) \tag{10}$$

are no longer valid except in the Born approximation region.

Fig. 3 (left panel) shows the real phase shifts calculated with BS (solid line) and NR (dashed line) equations and the same parameters as in Fig. 2. For this value of α there exists a bound state and, according to the Levinson theorem, the phase shift starts at 180°. One can see that the difference between relativistic and non-relativistic results is considerable even for relatively small incident momentum. The right panel shows the imaginary part of the phase shift. It appears starting from the first inelastic meson-production threshold $p_s^{(1)} = 0.75$ and displays the expected quadratic behavior. Simultaneously the modulus squared of the S-matrix (displayed in dashed line) starts differing from unity. The results of this figure contain the contributions of the second $p_s^{(2)} = 1.118$ and third $p_s^{(3)} = 1.435$ meson creation thresholds as well. Corresponding numerical values are given in Table 2.

Finally, we display in Fig. 4 the real (left panel) and imaginary (right panel) parts of the off-shell scattering amplitude $F_0(p_0, p)$ as a function of p_0 and p calculated for $\alpha = 0.5$, $p_s = \mu = 0.5$. Its real part shows a non-trivial structure with a ridge and a gap resulting from the singularities of the inhomogeneous term. Its on-shell value $F_0^{on} = F_0(0, p_s) = 0.753 + i0.292$, determining the phase
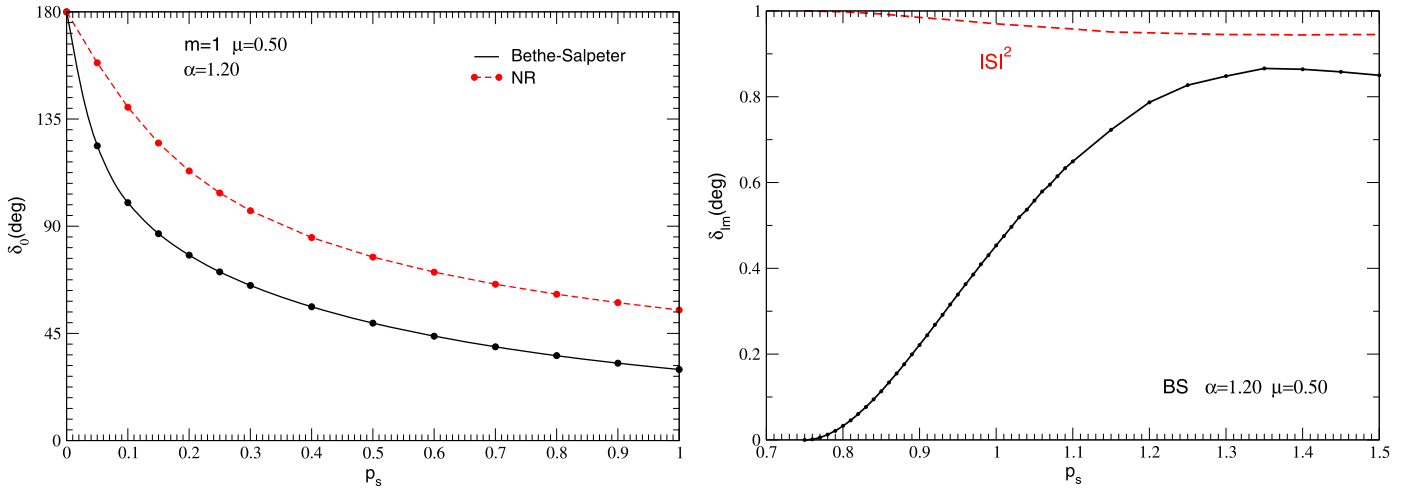
Fig. 3. Real (left panel) and imaginary (right panel) phase shift (degrees) for $\alpha = 1.2$ and $\mu = 0.50$ calculated via BS equation (solid) compared to the non-relativistic results (dashed).

Table 2
Real and imaginary parts of the phase shift (degrees) calculated by BS Eq. (1) vs. incident momentum p_s for $\alpha = 1.2$ and $\mu = 0.5$. Corresponding first inelastic threshold is $p_s^{(1)} = 0.75$.

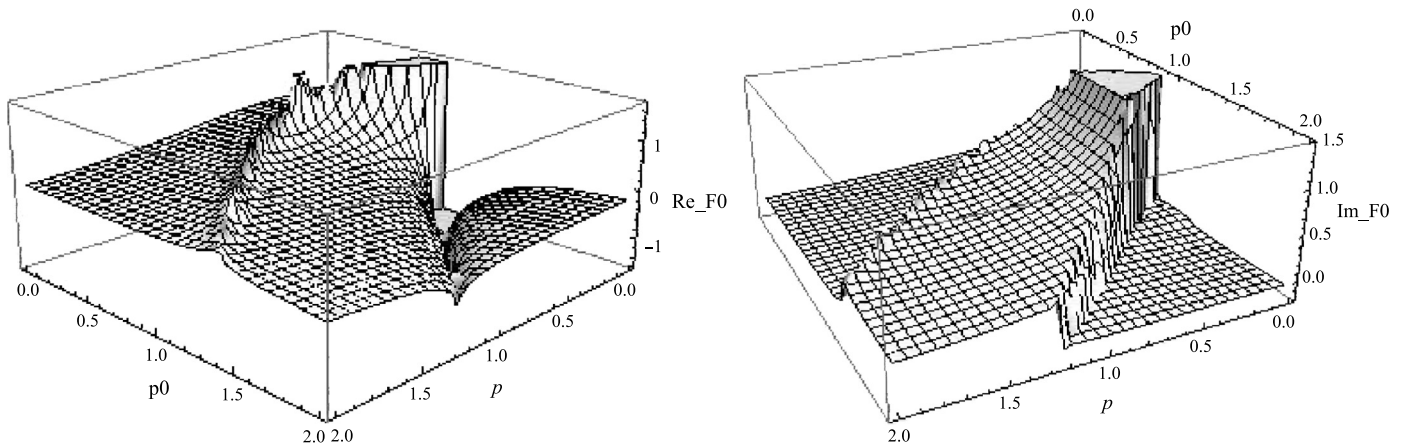| p_s	0.05	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0	1.3	1.5
$\text{Re}[\delta]$	124	99.9	77.8	65.1	56.2	49.3	43.9	39.4	35.7	32.5	29.7	22.8	19.3
$\text{Im}[\delta]$	0	0	0	0	0	0	0	0	0.033	0.221	0.453	0.848	0.852

Fig. 4. Real (left panel) and imaginary (right panel) parts of the off-shell amplitude $F(p_0, p; p_s)$ for $\alpha = 0.5$, $p_s = \mu = 0.5$.

shift $\delta = 21.2°$, corresponds to a single point on theses two surfaces. Our calculation, shown in Fig. 4, provides the full amplitude $F_0(p_0, p)$ in a two-dimensional domain.

Computing this quantity, and related on-shell observables, is the main result of this work. Together with the bound state solution in Minkowski space [3], they pave the way for a consistent relativistic description of composite systems in the framework of BS equation.

4. Conclusion

We have presented the first results of the BS off-shell scattering amplitude in Minkowski space. The different kinds of singularities of the original BS equation are properly treated. A regular equation is obtained and solved by standard methods. The results presented here were limited to S-wave in the spinless case and the ladder kernel but they can be extended to any partial wave. Coming on mass shell, the elastic phase shifts where accurately computed. They considerably differ, even at zero energy, from the non-relativistic ones. Above the meson creation threshold, an imaginary part of the phase shift appears and has also been calculated. The off-shell BS scattering amplitude thus obtained can be further used to calculate the transition form factor. In its full off-shell form, it can be used as input in the three-body BS–Faddeev equations.

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