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Abstract Logics as Dialgebras

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Abstract

The aim of this report is to propose a line of research that studies the connections between the theory of consequence operators as developed in [1] and [4] and the theory of dialgebras. The first steps in this direction are taken in this report, namely some of the basic notions of the theory of consequence operators - such as *abstract logics* - are translated into notions of the theory of dialgebras, and internal characterizations of the corresponding classes of objects are presented. Moreover it is shown that the class of coalgebras that corresponds to abstract logics of empty signature is a covariety.

Key words: Coalgebra, dialgebra, abstract logic, closure system functor.

1 Structure and content of this report

Abstract logics and other related notions are presented in the first section together with some basic properties.

In the second section it is shown how abstract logics of empty similarity type can be turned into coalgebras of the *closure system functor* $\mathcal{C}$, which is a contravariant endofunctor on **Set**. This way of translating abstract logics into $\mathcal{C}$-coalgebras closely resembles the way in which topological spaces are turned into coalgebras of the filter functor presented in [2].

Analogously to what happens in the case of topological spaces, not all $\mathcal{C}$-coalgebras come from abstract logics, so a characterization of the class of the $\mathcal{C}$-coalgebras which come from abstract logics is presented. It is also shown that morphisms between $\mathcal{C}$-coalgebras which come from abstract logics precisely correspond to *strict* morphisms of abstract logics. Being strict is a property that guarantees a good logical behaviour (see Proposition 2.13 below).

It is shown that the class of $\mathcal{C}$-coalgebras which come from abstract logics is “closed under bisimulations” (Proposition 3.14). Finally, it is shown that the class of $\mathcal{C}$-coalgebras that come from abstract logics is a covariety.

The third section is about how to extend the content of the second section to abstract logics of nonempty algebraic signature τ. The idea is to turn abstract logics into special *dialgebras* (see [3]) which have an algebraic part and a coalgebraic part, and let their algebraic part account for τ. Some properties which can be immediately extended to the nonempty signature case are also mentioned. Finally, some questions about future work are presented in the fourth section.

2 Abstract logics

The notions of closure operator and closure system are well known in Universal Algebra. A detailed presentation of their properties can be found in [4].

Throughout this section, τ is an arbitrary algebraic similarity type, Var is a given set of proposition variables, and $\mathbf{Fm}$ is the τ-algebra of formulas over Var.

Definition 2.1 (Closure operator) Let A be a set. A *closure operator on* A is a map $C : \mathcal{P}(A) \longrightarrow \mathcal{P}(A)$ such that for all $X, Y \subseteq A$,

(i) $X \subseteq C(X)$.

(ii) If $X \subseteq Y$ then $C(X) \subseteq C(Y)$.

(iii) $C(C(X)) \subseteq C(X)$.

If C is a closure operator on A, the elements of the set

$$\mathcal{C} = \{X \subseteq A \mid C(X) = X\}$$

are the *C-closed sets* of A. I will refer to them as *closed sets* whenever there is no ambiguity.

Definition 2.2 (Abstract logic) (cf. [1]) A *τ-abstract logic* is a pair $\mathbf{L} = \langle \mathcal{A}, C \rangle$, where $\mathcal{A}$ is a τ-algebra and C is a closure operator on A. The closed sets of $\mathbf{L}$ are also called the *theories* of $\mathbf{L}$ or the $\mathbf{L}$-*theories*.

Definition 2.3 (Closure system) Let A be a set. A *closure system on* A is a family $\mathcal{C}$ of subsets of A such that $A \in \mathcal{C}$ and $\mathcal{C}$ is closed under arbitrary intersection.

Remark 2.4 Let A be a set. If C is a closure operator on A, then the set $\mathcal{C}$ of the C-closed sets of A is a closure system on A, and I will refer to it as *the closure system associated with* C. Conversely, if $\mathcal{C}$ is a closure system on A, then the map defined by $X \longmapsto \bigcap\{Y \in \mathcal{C} \mid X \subseteq Y\}$ is a closure operator on A. These two correspondences are inverse to one another. Due to this fact, abstract logics can equivalently be defined as pairs $\mathbf{L} = \langle \mathcal{A}, \mathcal{C} \rangle$ such that $\mathcal{A}$ is an algebra and $\mathcal{C}$ is a closure system on A.

Definition 2.5 (Frege relation of an abstract logic) (cf. [1]) Let $\mathbf{L} = \langle \mathcal{A}, C \rangle$ be an abstract logic. The *Frege relation of* $\mathbf{L}$ is the equivalence relation defined as follows:

$$\Lambda_{\mathbf{L}} = \{\langle a, b \rangle \in A \times A \mid C(a) = C(b)\}.$$

The Frege relation of $\mathbf{L}$ might not be a congruence of $\mathcal{A}$.

Definition 2.6 (Tarski relation of an abstract logic) (cf. [1]) Let $\mathbf{L} = \langle \mathcal{A}, \mathcal{C} \rangle$ be an abstract logic. A congruence $\theta \in Con(\mathcal{A})$ is a *congruence of* $\mathbf{L}$ iff $\theta \subseteq \Lambda_{\mathbf{L}}$, i.e.

$$\langle a, b \rangle \in \theta \quad \text{iff for every } T \in \mathcal{C}, \quad a \in T \;\Leftrightarrow\; b \in T.$$

Let $Con(\mathbf{L})$ be the set of the congruences of $\mathbf{L}$. It holds that $Con(\mathbf{L})$, ordered by the inclusion relation, is a complete lattice and a principal ideal of the lattice $Con(\mathcal{A})$. Its generator - i.e. the greatest congruence of $\mathbf{L}$ - is the *Tarski relation of* $\mathbf{L}$ and it is denoted by $\widetilde{\Omega}(\mathbf{L})$.

Abstract logics can be used as models of sentential logics (see [1], [4]) because they induce a consequence relation on the algebra of formulas. None of the facts mentioned in the remainder of this section will be used in the rest of the report, but they will be relevant in further developements of the work presented here.

Definition 2.7 (Consequence relation induced by an abstract logic) (cf. Def. 2.1 of [1]) Let $\mathbf{L} = \langle \mathcal{A}, C \rangle$ be a τ-abstract logic. $\mathbf{L}$ induces the following (local) consequence relation $\models_{\mathbf{L}}$ on $\mathbf{Fm}$: For every $\Gamma \cup \{\phi\} \subseteq Fm$,

$$\Gamma \models_{\mathbf{L}} \phi \text{ iff for every } h \in Hom(\mathbf{Fm}, \mathcal{A}),\; h(\phi) \in C(h[\Gamma]).$$

Lemma 2.8 (cf. 1.5.1 Proposition A of [4]) *Let $\mathcal{A}$ be a set, let C and C' be two closure operators on A, let $\mathcal{C}$ and $\mathcal{C}'$ be the closure systems associated with C and C' respectively. The following are equivalent:*

(i) *For every $X \subseteq A$, $C(X) \subseteq C'(X)$.*

(ii) *$\mathcal{C}' \subseteq \mathcal{C}$.*

Proof. (i) $\Rightarrow$ (ii) If $T \in \mathcal{C}'$, then $C(T) \subseteq C'(T) = T$.

(ii) $\Rightarrow$ (i) If $X \subseteq A$ then $X \subseteq C'(X)$ and so $C(X) \subseteq C(C'(X)) = C'(X)$.$\square$

Corollary 2.9 *Let $\mathcal{A}$ be a τ-algebra, let $\mathcal{C}$ and $\mathcal{C}'$ be closure systems of A such that $\mathcal{C}' \subseteq \mathcal{C}$, and let $\mathbf{L} = \langle \mathcal{A}, \mathcal{C} \rangle$ and $\mathbf{L}' = \langle \mathcal{A}, \mathcal{C}' \rangle$. Then $\models_{\mathbf{L}} \subset \models_{\mathbf{L}'}$.*

Definition 2.10 (Morphism of abstract logics) (cf. [1]) Let $\mathbf{L} = \langle \mathcal{A}, \mathcal{C} \rangle$ and $\mathbf{L}' = \langle \mathcal{A}', \mathcal{C}' \rangle$ be τ-abstract logics, let $h \in Hom(\mathcal{A}, \mathcal{A}')$. h is a *morphism of abstract logics* or a *logical morphism* (Notation: $h \in Hom(\mathbf{L}, \mathbf{L}')$) iff

$$\{h^{-1}[T'] \mid T' \in \mathcal{C}'\} \subseteq \mathcal{C}.$$

Lemma 2.11 *Let $\mathcal{A}$, $\mathcal{A}'$ be τ-algebras, let $h \in Hom(\mathcal{A}, \mathcal{A}')$ and let $\mathcal{C}'$ be a closure system of $\mathcal{A}'$.*

(i) *$\mathcal{C} = \{h^{-1}[T'] \mid T' \in \mathcal{C}'\}$ is a closure system of $\mathcal{A}$.*

(ii) *Let C and C' be the closure operators associated with $\mathcal{C}$ and $\mathcal{C}'$ respectively. Then for every $X \subseteq A$, $C(X) = h^{-1}[C'(h[X])]$.*

(iii) *If $\mathbf{L} = \langle \mathcal{A}, \mathcal{C} \rangle$, then $\models_{\mathbf{L}'} \subseteq \models_{\mathbf{L}}$.*

(iv) *If h is onto, then for every $X \subseteq A$, $h[C(X)] = C'(h[X])]$.*

Definition 2.12 (Strict morphism of abstract logics) Let $\mathbf{L} = \langle \mathcal{A}, \mathcal{C} \rangle$ and $\mathbf{L}' = \langle \mathcal{A}', \mathcal{C}' \rangle$ be abstract logics, and let $h \in Hom(\mathbf{L}, \mathbf{L}')$. h is a *strict logical morphism* iff

$$\mathcal{C} = \{h^{-1}[T'] \mid T' \in \mathcal{C}'\}.$$

Proposition 2.13 *Let $\mathbf{L} = \langle \mathcal{A}, \mathcal{C} \rangle$, $\mathbf{L}' = \langle \mathcal{A}', \mathcal{C}' \rangle$ be τ-abstract logics and let $h \in Hom(\mathbf{L}, \mathbf{L}')$.*

(i) *If h is strict, then $\models_{\mathbf{L}'} \subseteq \models_{\mathbf{L}}$.*

(ii) *If h is onto, then $\models_{\mathbf{L}} \subseteq \models_{\mathbf{L}'}$.*

3 Abstract logics as coalgebras in the empty signature case

Let $\mathbf{L} = \langle A, \mathcal{C} \rangle$ be an abstract logic of empty algebraic similarity type, i.e. A is a set with no further algebraic structure. $\mathbf{L}$ can be turned into a $\mathcal{C}$-coalgebra, where $\mathcal{C}$ is the closure system functor introduced in the next definition:

Definition 3.1 (Closure system functor) Let **Set** be the category of sets and maps. For every set A, let $\mathcal{C}(A)$ be the set of the closure systems of A. For every set map $f : A \longrightarrow B$, let

$$\begin{array}{rcl} \mathcal{C}(f) : \mathcal{C}(B) & \longrightarrow & \mathcal{C}(A) \\ \mathcal{F} & \longmapsto & \{f^{-1}[T] \mid T \in \mathcal{F}\}. \end{array}$$

These two assignments define the *closure system functor* $\mathcal{C}$ on **Set**.

Lemma 3.2 *$\mathcal{C}$ is a contravariant endofunctor on* **Set**.

Proof. By item 1 of Lemma 2.11, if $\mathcal{F} \in \mathcal{C}(B)$, then $\{f^{-1}[T] \mid T \in \mathcal{F}\} \in \mathcal{C}(A)$. For every set A, for every $\mathcal{F} \in \mathcal{C}(A)$,

$$\mathcal{C}(id_A)(\mathcal{F}) = \{id^{-1}[T] \mid T \in \mathcal{F}\} = \mathcal{F},$$

so $\mathcal{C}(id_A) = id_{\mathcal{C}(A)}$. If $f : A \longrightarrow B$, $g : B \longrightarrow C$ and $\mathcal{F} \in \mathcal{C}(C)$, then

$$\begin{array}{rcl} \mathcal{C}(f) \circ \mathcal{C}(g)(\mathcal{F}) & = & \{f^{-1}[g^{-1}[T]] \mid T \in \mathcal{F}\} \\ & = & \{(g \circ f)^{-1}[T] \mid T \in \mathcal{F}\} \\ & = & \mathcal{C}(g \circ f)(\mathcal{F}). \end{array}$$

□

Notation: Let $\mathbf{L} = \langle A, \mathcal{F} \rangle$ be an abstract logic of empty similarity type. For every $a \in A$, let us denote

$$C_{\mathcal{F}}(a) := \{T \in \mathcal{F} \mid a \in T\}.$$

Remark 3.3 Let $\mathbf{L} = \langle A, \mathcal{F} \rangle$ be an abstract logic of empty similarity type. For every $a \in A$, $C_{\mathcal{F}}(a)$ is a closure system on A.

Remark 3.4 Let $\mathbf{L} = \langle A, \mathcal{F} \rangle$ be an abstract logic of empty similarity type. $\mathbf{L}$ can be turned into a $\mathcal{C}$-coalgebra $\langle A, \xi \rangle$ by defining the following coalgebra map:

$$\begin{array}{rcl} \xi : A & \longrightarrow & \mathcal{C}(A) \\ a & \longmapsto & C_{\mathcal{F}}(a). \end{array}$$

Definition 3.5 A $\mathcal{C}$-coalgebra $\langle A, \xi \rangle$ *comes from an abstract logic* iff there is a closure system $\mathcal{F}$ on A such that for every $a \in A$,

$$\xi(a) = C_{\mathcal{F}}(a).$$

Remark 3.6 Let $\langle A, \xi \rangle$ be a $\mathcal{C}$-coalgebra which comes from an abstract logic, and let $\mathcal{F}$ be a closure system on A such that for every $a \in A$, $\xi(a) = C_{\mathcal{F}}(a)$. $\mathcal{F}$ may not be unique. Suppose that $\bigcap \mathcal{F} \neq \emptyset$, so $\emptyset \notin \mathcal{F}$, then consider $\mathcal{F}' = \mathcal{F} \cup \{\emptyset\}$. $\mathcal{F}'$ is a closure system of A, and for every $a \in A$,

$$C_{\mathcal{F}'}(a) = C_{\mathcal{F}}(a).$$

However, the next lemma shows that if $\mathcal{F}, \mathcal{F}' \in \mathcal{C}(A)$ and $C_{\mathcal{F}'}(a) = \xi(a) = C_{\mathcal{F}}(a)$, then $(\mathcal{F} \setminus \mathcal{F}') \cup (\mathcal{F}' \setminus \mathcal{F}) \subseteq \{\emptyset\}$.

Lemma 3.7 *Let $\langle A, \xi \rangle$ be a $\mathcal{C}$-coalgebra which comes from an abstract logic, and let $\mathcal{F}$, $\mathcal{F}'$ be closure systems on A such that for every $a \in A$, $\xi(a) = C_{\mathcal{F}}(a) = C_{\mathcal{F}'}(a)$. Then $\mathcal{F} \setminus \{\emptyset\} = \mathcal{F}' \setminus \{\emptyset\}$, which implies that $\mathcal{F} \cup \{\emptyset\} = \mathcal{F}' \cup \{\emptyset\}$.*

Proof. If $T \in \mathcal{F}$ and $T \neq \emptyset$, then $a \in T$ for some $a \in A$, therefore $T \in C_{\mathcal{F}}(a) = C_{\mathcal{F}'}(a)$, hence $T \in \mathcal{F}'$. The other inclusion goes analogously. □

Not all $\mathcal{C}$-coalgebras come from abstract logics:

Proposition 3.8 *Let $\langle A, \xi \rangle$ be a $\mathcal{C}$-coalgebra. $\langle A, \xi \rangle$ comes from an abstract logic iff the following conditions are satisfied:*

(i) *For every $a \in A$, $a \in \bigcap \xi(a)$.*

(ii) *Either of $\bigcup_{a \in A} \xi(a)$ or $\{\emptyset\} \cup \bigcup_{a \in A} \xi(a)$ belongs to $\mathcal{C}(A)$.*

(iii) *For every $T \in \bigcup_{b \in A} \xi(b)$, for every $a \in A$,*

$$a \in T \quad \Rightarrow \quad T \in \xi(a).$$

Proof. ($\Rightarrow$) Assume that $\langle A, \xi \rangle$ comes from an abstract logic, and let $\mathcal{F}$ be a closure system such that for every $a \in A$,

$$\xi(a) = C_{\mathcal{F}}(a) = \{T \in \mathcal{F} \mid a \in T\},$$

therefore it follows that for every $a \in A$, $a \in \bigcap C_{\mathcal{F}}(a) = \bigcap \xi(a)$, which is item (i), moreover if $T \in \bigcup_{b \in A} \xi(b) = \bigcup_{b \in A} C_{\mathcal{F}}(b) = (\mathcal{F} \setminus \{\emptyset\})$ and $a \in T$, then $T \in \xi(a)$, which is item (iii).

Let us show that $\bigcup_{a \in A} \xi(a)$ is closed under nonempty intersection. Let $\mathcal{X} \subseteq \bigcup_{a \in A} \xi(a)$ and $\bigcap \mathcal{X} \neq \emptyset$, and let us show that $\bigcap \mathcal{X} \in \bigcup_{a \in A} \xi(a)$.

As $\mathcal{X} \subseteq \bigcup_{a \in A} \xi(a) \subseteq \mathcal{F}$, then $\bigcap \mathcal{X} \in \mathcal{F}$, and as $\bigcap \mathcal{X} \neq \emptyset$, $b \in \bigcap \mathcal{X}$ for some $b \in A$, hence

$$\bigcap \mathcal{X} \in C_{\mathcal{F}}(b) = \xi(b) \subseteq \bigcup_{a\in A} \xi(a).$$

Let us prove item (ii). If $\bigcap \bigcup_{a\in A} \xi(a) \neq \emptyset$, then $\bigcap \mathcal{X} \neq \emptyset$ for every $\mathcal{X} \subseteq \bigcup_{a\in A} \xi(a)$, hence $\bigcup_{a\in A} \xi(a)$ is closed under arbitrary intersection, and as $A \in \xi(a) \subseteq \bigcup_{a\in A} \xi(a)$ for every $a \in A$, then $\bigcup_{a\in A} \xi(a) \in \mathcal{C}(A)$.

If $\bigcap \bigcup_{a\in A} \xi(a) = \emptyset$, then - as $\bigcup_{a\in A} \xi(a)$ is closed under nonempty intersection - $\{\emptyset\} \cup \bigcup_{a\in A} \xi(a)$ is closed under arbitrary intersection, hence it is a closure system on A.

($\Leftarrow$) Let $\mathcal{F} = \bigcup_{a\in A} \xi(a)$ and $\mathcal{F}' = \{\emptyset\} \cup \bigcup_{a\in A} \xi(a)$, and let us show that for every $a \in A$, $\xi(a) = C_{\mathcal{F}}(a) = C_{\mathcal{F}'}(a)$.

Let $a \in A$. As $a \in \bigcap \xi(a)$ and $\xi(a) \subseteq \mathcal{F}$, then $\xi(a) \subseteq C_{\mathcal{F}}(a)$. Let $T \in C_{\mathcal{F}}(a)$, so $a \in T \in \bigcup_{b\in A} \xi(b)$. Then, by item (iii), $T \in \xi(a)$. □

Remark 3.9 Let $\langle A, \xi\rangle$ be a $\mathcal{C}$-coalgebra which comes from an abstract logic. The proof of the previous Proposition shows that we can canonically associate an element $\mathcal{F} \in \mathcal{C}(A)$ with $\langle A, \xi\rangle$, namely, $\mathcal{F} = \bigcup_{a\in A} \xi(a)$ if $\bigcap \bigcup_{a\in A} \xi(a) \neq \emptyset$, and $\mathcal{F} = \{\emptyset\} \cup \bigcup_{a\in A} \xi(a)$ if $\bigcap \bigcup_{a\in A} \xi(a) = \emptyset$.

Definition 3.10 (Morphism) Let $\langle A, \xi\rangle$ and $\langle B, \sigma\rangle$ be $\mathcal{C}$-coalgebras. A map $f : A \longrightarrow B$ is a *coalgebra morphism* iff the following diagram commutes:

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \xi\big\downarrow & & \sigma\big\downarrow \\ \mathcal{C}(A) & \xleftarrow{\mathcal{C}(f)} & \mathcal{C}(B). \end{array} \tag{1}$$

Proposition 3.11 *Let $\langle A, \xi\rangle$ and $\langle B, \sigma\rangle$ be $\mathcal{C}$-coalgebras which come from abstract logics, let $\mathcal{F} \in \mathcal{C}(A)$, $\mathcal{G} \in \mathcal{C}(B)$ such that for every $a \in A$ $\xi(a) = C_{\mathcal{F}}(a)$ and for every $b \in B$ $\sigma(b) = C_{\mathcal{G}}(b)$, let $\mathcal{F}' = \mathcal{F} \cup \{\emptyset\}$, $\mathcal{G}' = \mathcal{G} \cup \{\emptyset\}$, let $\mathbf{L} = \langle A, \mathcal{F}'\rangle$, $\mathbf{L}' = \langle B, \mathcal{G}'\rangle$ be abstract logics and let $f : A \longrightarrow B$ be a map. The following are equivalent:*

(i) *f is a coalgebra morphism.*

(ii) *$f : \mathbf{L} \longrightarrow \mathbf{L}'$ is a strict morphism of abstract logics.*

Proof. (i) $\Rightarrow$ (ii) If f is a coalgebra morphism then for every $a \in A$,

$$\begin{aligned} C_{\mathcal{F}}(a) &= \xi(a) \\ &= \mathcal{C}(f) \circ \sigma \circ f(a) \\ &= \mathcal{C}(f)(\sigma(f(a))) \\ &= \{f^{-1}[T'] \mid T' \in C_{\mathcal{G}}(f(a))\}. \end{aligned}$$

Let us show that $\mathcal{F}' = \{f^{-1}[T'] \mid T' \in \mathcal{G}'\}$.

($\subseteq$) If $T \in \mathcal{F}'$ and $T \neq \emptyset$, then $a \in T$ for some $a \in A$, so $T \in C_{\mathcal{F}}(a)$, hence $T = f^{-1}[T']$ for some $T' \in C_{\mathcal{G}}(f(a)) \subseteq \mathcal{G}'$. If $T = \emptyset$, then $T = f^{-1}[\emptyset]$ and $\emptyset \in \mathcal{G}'$.

($\supseteq$) Let $T' \in \mathcal{G}'$. If $T' \cap range(f) \neq \emptyset$ then $f(a) \in T$ for some $a \in A$, so $f^{-1}[T'] \in C_{\mathcal{F}}(a) \subseteq \mathcal{F}'$. If $T' \cap range(f) = \emptyset$ then $f^{-1}[T'] = \emptyset \in \mathcal{F}'$.

(ii) $\Rightarrow$ (i) Assume that $f : \mathbf{L} \longrightarrow \mathbf{L}'$ is a strict morphism of abstract logics, let $a \in A$ and let us show that $\xi(a) = \mathcal{C}(f) \circ \sigma \circ f(a)$, i.e. that

$$C_{\mathcal{F}}(a) = \{f^{-1}[T'] \mid T' \in C_{\mathcal{G}}(f(a))\}.$$

($\subseteq$) Let $T \in C_{\mathcal{F}}(a)$, so $T \in \mathcal{F}$ and $a \in T$. As f is strict, $\mathcal{F} = \{f^{-1}[T'] \mid T' \in \mathcal{G}'\}$, so $T = f^{-1}[T']$ for some $T' \in \mathcal{G}$, and as $a \in T$ then $f(a) \in T'$, so $T' \in C_{\mathcal{G}}(f(a))\}$.
($\supseteq$) If $T' \in C_{\mathcal{G}}(f(a))\}$, then $a \in f^{-1}[T']$ and as f is a morphism of abstract logics $f^{-1}[T'] \in \mathcal{F}'$, so $f^{-1}[T'] \in C_{\mathcal{F}}(a)$. □

Definition 3.12 (Bisimulation) Let $\langle A, \xi\rangle$ and $\langle B, \sigma\rangle$ be $\mathcal{C}$-coalgebras. A relation $R \subseteq A \times B$ is a *bisimulation* between them iff there is some $\mathcal{C}$-coalgebra $\langle R, \zeta\rangle$ such that the following diagram commutes:

$$\begin{array}{ccccc} A & \xleftarrow{\pi_1} & R & \xrightarrow{\pi_2} & B \\ \xi\downarrow & & \zeta\downarrow & & \sigma\downarrow \\ \mathcal{C}(A) & \xrightarrow{\mathcal{C}(\pi_1)} & \mathcal{C}(R) & \xleftarrow{\mathcal{C}(\pi_2)} & \mathcal{C}(B) \end{array} \tag{2}$$

where π_1 and π_2 are the canonical projections restricted to R.

Lemma 3.13 *Let A be a set, let $\mathcal{B}$ and $\mathcal{D}$ be closure systems of A, let*

$$\mathcal{B} \sqcap \mathcal{D} = \{B \cap D \mid B \in \mathcal{B}, D \in \mathcal{D}\}.$$

(i) *$\mathcal{B} \sqcap \mathcal{D}$ is a closure system of A.*
(ii) *$\mathcal{B} \sqcap \mathcal{B} = \mathcal{B}$.*

Proof. (i) As $A \in \mathcal{B}$ and $A \in \mathcal{D}$, then $A = A \cap A \in \mathcal{B} \sqcap \mathcal{D}$. Let $\mathcal{X} \subseteq \mathcal{B} \sqcap \mathcal{D}$, let $\mathcal{X}_1 = \{B \in \mathcal{B} \mid B \cap D \in \mathcal{X}$ for some $D \in \mathcal{D}\}$, $\mathcal{X}_2 = \{D \in \mathcal{D} \mid B \cap D \in \mathcal{X}$ for some $B \in \mathcal{B}\}$, then $\mathcal{X} = \{B \cap D \mid B \in \mathcal{X}_1, D \in \mathcal{X}_2\}$, $\bigcap \mathcal{X}_1 \in \mathcal{B}$ and $\bigcap \mathcal{X}_2 \in \mathcal{D}$, and so $\bigcap \mathcal{X} = \bigcap \mathcal{X}_1 \cap \bigcap \mathcal{X}_2 \in \mathcal{B} \sqcap \mathcal{D}$. The proof of (ii) is immediate. □

Proposition 3.14 *Let $\langle A, \xi\rangle$ and $\langle B, \sigma\rangle$ be $\mathcal{C}$-coalgebras which come from abstract logics, let $\langle R, \zeta\rangle$ be a bisimulation between them. Then $\langle R, \zeta\rangle$ comes from an abstract logic.*

Proof. Let $\mathcal{F} \in \mathcal{C}(A)$, $\mathcal{G} \in \mathcal{C}(B)$ such that $\xi(a) = C_{\mathcal{F}}(a)$ for every $a \in A$ and $\sigma(b) = \mathcal{C}_{\mathcal{G}}(b)$ for every $b \in B$, let us consider

$$\mathcal{H} = \{R \cap (T_1 \times T_2) \mid T_1 \in \mathcal{F} \text{ and } T_2 \in \mathcal{G}\}$$

and let us show that $\mathcal{H} \in \mathcal{C}(R)$ and that $\zeta(\langle a, b\rangle) = C_{\mathcal{H}}(\langle a, b\rangle)$ for every $\langle a, b\rangle \in R$.

As $A \in \mathcal{F}$ and $B \in \mathcal{G}$, then $R = R \cap (A \times B) \in \mathcal{H}$. Let $\mathcal{X} \subseteq \mathcal{H}$, let

$$\mathcal{X}_1 = \{T_1 \in \mathcal{F} \mid R \cap (T_1 \times T_2) \in \mathcal{X} \text{ for some } T_2 \in \mathcal{G}\},$$
$$\mathcal{X}_2 = \{T_2 \in \mathcal{G} \mid R \cap (T_1 \times T_2) \in \mathcal{X} \text{ for some } T_1 \in \mathcal{F}\},$$

so $\mathcal{X} = \{R \cap (T_1 \times T_2) \mid T_1 \in \mathcal{X}_1 \text{ and } T_2 \in \mathcal{X}_2\}$, $\bigcap \mathcal{X}_1 \in \mathcal{F}$ and $\bigcap \mathcal{X}_2 \in \mathcal{G}$, therefore

$\bigcap \mathcal{X} = \bigcap_{T_i \in \mathcal{X}_i}(R \cap (T_1 \times T_2)) = R \cap \bigcap_{T_i \in \mathcal{X}_i}(T_1 \times T_2) = R \cap (\bigcap \mathcal{X}_1 \times \bigcap \mathcal{X}_2) \in \mathcal{H}$.

Let $\langle a, b\rangle \in R$.

$$\begin{array}{rcl} C_{\mathcal{H}}(\langle a,b\rangle) & = & \{T \in \mathcal{H} \mid \langle a,b\rangle \in T\} \\ & = & \{R \cap (T_1 \times T_2) | \ T_1 \in \mathcal{F}, T_2 \in \mathcal{G}, a \in T_1, b \in T_2\} \\ & = & \{R \cap (T_1 \times T_2) | \ T_1 \in C_{\mathcal{F}}(a), T_2 \in C_{\mathcal{G}}(b)\} \\ & = & \{R \cap (T_1 \times T_2) | \ T_1 \in \xi(a), T_2 \in \sigma(b)\}. \end{array}$$

As $\langle R, \zeta\rangle$ is a bisimulation,

$$\begin{array}{rcl} \zeta(\langle a,b\rangle) & = & \mathcal{C}(\pi_1) \circ \xi \circ \pi_1(\langle a,b\rangle) \\ & = & \mathcal{C}(\pi_1)(\xi(a)) \\ & = & \{\pi_1^{-1}[T_1] \mid T_1 \in \xi(a)\} \\ & = & \{R \cap (T_1 \times B) | \ T_1 \in \xi(a)\}. \\ \zeta(\langle a,b\rangle) & = & \mathcal{C}(\pi_2) \circ \sigma \circ \pi_2(\langle a,b\rangle) \\ & = & \mathcal{C}(\pi_2)(\sigma(b)) \\ & = & \{\pi_2^{-1}[T_2] \mid T_2 \in \sigma(b)\} \\ & = & \{R \cap (A \times T_2) | \ T_2 \in \sigma(b)\}. \end{array}$$

By the previous lemma, it is enough to show that

$$C_{\mathcal{H}}(\langle a,b\rangle) = \{R \cap (T_1 \times B) | \ T_1 \in \xi(a)\} \sqcap \{R \cap (A \times T_2) | \ T_2 \in \sigma(b)\},$$

but this immediately follows from the fact that for every $T_1 \in \xi(a)$, $T_2 \in \sigma(b)$,

$$R \cap (T_1 \times T_2) = (R \cap (T_1 \times B)) \cap (R \cap (A \times T_2)).$$

□

Lemma 3.15 *Let A, B be sets let $X \subseteq A$, $\mathcal{X} \subseteq \mathcal{P}(A)$, $Y \subseteq B$, $\mathcal{Y} \subseteq \mathcal{P}(B)$ and let $f : A \longrightarrow B$ be a map. Then*

(i) $f^{-1}[\bigcap \mathcal{Y}] = \bigcap_{Y \in \mathcal{Y}} f^{-1}[Y]$.

(ii) *If f is injective then $f[\bigcap \mathcal{X}] = \bigcap_{X \in \mathcal{X}} f[X]$.*

(iii) *If f is injective then $f^{-1}[f[X]] = X$.*

(iv) *If f is surjective then $f[f^{-1}[Y]] = Y$.*

Proposition 3.16 *Let $\langle A, \xi\rangle$ and $\langle B, \sigma\rangle$ be $\mathcal{C}$-coalgebras, let $f : A \longrightarrow B$ be a surjective coalgebra morphism. If $\langle A, \xi\rangle$ comes from an abstract logic, so does $\langle B, \sigma\rangle$.*

Proof. We have to show that there exists $\mathcal{G} \in \mathcal{C}(B)$ such that $\sigma(b) = C_{\mathcal{G}}(b)$ for every $b \in B$. As $\langle A, \xi\rangle$ comes from an abstract logic, then there exists

$\mathcal{F} \in \mathcal{C}(A)$ such that $\xi(a) = C_{\mathcal{F}}(a)$ for every $a \in A$.

As f is a coalgebra morphism, $\xi = \mathcal{C}(f) \circ \sigma \circ f$, and so for every $a \in A$,

$$C_{\mathcal{F}}(a) = \xi(a) = \mathcal{C}(f)(\sigma(f(a))) = \{f^{-1}[T'] \mid T' \in \sigma(f(a))\}, \tag{3}$$

hence for every $a \in A$,

$$\sigma(f(a)) = \{f[T] \mid T \in C_{\mathcal{F}}(a)\}, \tag{4}$$

for if $T' \in \sigma(f(a))$, then $T = f^{-1}[T'] \in C_{\mathcal{F}}(a)$, and so $T' = f[f^{-1}[T']] = f[T]$ for some $T \in C_{\mathcal{F}}(a)$. Conversely, if $T \in C_{\mathcal{F}}(a)$, then $T = f^{-1}[T']$ for some $T' \in \sigma(f(a))$, so $f[T] = f[f^{-1}[T']] = T' \in \sigma(f(a))$.

Let us consider $\mathcal{G} = \{f[T] \mid T \in \mathcal{F}\} \cup \{\emptyset\}$ and let us show that $\sigma(b) = C_{\mathcal{G}}(b)$ for every $b \in B$.

$$\begin{aligned}
C_{\mathcal{G}}(b) &= C_{\mathcal{G}}(f(a)) \\
&= \{f[T] \mid T \in \mathcal{F} \text{ and } f(a) \in f[T]\} \\
&= \{f[T] \mid T \in C_{\mathcal{F}}(a') \text{ for some } a' \in A \text{ s.t. } f(a') = f(a)\} \\
&= \bigcup\{\{f[T] \mid T \in C_{\mathcal{F}}(a')\} \mid a' \in A \text{ and } f(a') = f(a)\} \\
&= \bigcup\{\sigma(f(a')) \mid a' \in A \text{ and } f(a') = f(a)\} && \text{(by (4))} \\
&= \sigma(f(a)) \\
&= \sigma(b).
\end{aligned}$$

From this it immediately follows that

$$\mathcal{G} \setminus \{\emptyset\} = \bigcup_{b \in B} C_{\mathcal{G}}(b) = \bigcup_{b \in B} \sigma(b). \tag{5}$$

Let us show that $\mathcal{G} \in \mathcal{C}(B)$. As f is surjective, $B = f[A]$ and $A \in \mathcal{F}$, so $B \in \mathcal{G}$.

It holds that

$$\mathcal{F} \setminus \{\emptyset\} = \{f^{-1}[T'] \mid T' \in \mathcal{G} \setminus \{\emptyset\}\}. \tag{6}$$

$$\begin{aligned}
\mathcal{F} \setminus \{\emptyset\} &= \textstyle\bigcup_{a \in A} C_{\mathcal{F}}(a) \\
&= \textstyle\bigcup_{a \in A} \{f^{-1}[T'] \mid T' \in \sigma(f(a))\} && \text{(by (3))} \\
&= \textstyle\bigcup_{b \in B} \{f^{-1}[T'] \mid T' \in \sigma(b)\} && \text{(f is surjective)} \\
&= \{f^{-1}[T'] \mid T' \in \textstyle\bigcup_{b \in B} \sigma(b)\} \\
&= \{f^{-1}[T'] \mid T' \in \mathcal{G} \setminus \{\emptyset\})\}. && \text{(by (5))}
\end{aligned}$$

Let $\mathcal{X} \subseteq \mathcal{G}$. If $\bigcap \mathcal{X} \neq \emptyset$, then $\mathcal{X} \subseteq \mathcal{G} \setminus \{\emptyset\}$, and so by (6) $\mathcal{X}' = \{f^{-1}[T'] \mid T' \in \mathcal{X}\} \subseteq \mathcal{F} \setminus \{\emptyset\}$, therefore $\bigcap \mathcal{X}' \in \mathcal{F}$, and as $\bigcap \mathcal{X}' = \bigcap_{T' \in \mathcal{X}} f^{-1}[T'] = f^{-1}[\bigcap \mathcal{X}]$,

$$\bigcap \mathcal{X} = f[f^{-1}[\bigcap \mathcal{X}]] = f[\bigcap \mathcal{X}'] \in \mathcal{G}.$$

□

Definition 3.17 (Subcoalgebra) Let $B \subseteq A$ and $\langle A, \xi \rangle$ and $\langle B, \sigma \rangle$ be $\mathcal{C}$-coalgebras. $\langle B, \sigma \rangle$ is a *subcoalgebra* of $\langle A, \xi \rangle$ iff the natural inclusion map $i : B \longrightarrow A$ is a coalgebra morphism.

Proposition 3.18 *Let $\langle A, \xi \rangle$ and $\langle B, \sigma \rangle$ be $\mathcal{C}$-coalgebras, and let $\langle B, \sigma \rangle$ be a subcoalgebra of $\langle A, \xi \rangle$. If $\langle A, \xi \rangle$ comes from an abstract logic, so does $\langle B, \sigma \rangle$.*

Proof. We have to show that there exists $\mathcal{G} \in \mathcal{C}(B)$ such that $\sigma(b) = C_{\mathcal{G}}(b)$ for every $b \in B$. As $\langle A, \xi \rangle$ comes from an abstract logic, then there exists $\mathcal{F} \in \mathcal{C}(A)$ such that $\xi(a) = C_{\mathcal{F}}(a)$ for every $a \in A$.

As i is a coalgebra morphism, $\sigma = \mathcal{C}(i) \circ \xi \circ i$, and so for every $b \in B$,

$$\sigma(b) = \mathcal{C}(i)(\xi(b)) = \mathcal{C}(i)(C_{\mathcal{F}}(b)) = \{i^{-1}[T] \mid T \in C_{\mathcal{F}}(b)\} = \{B \cap T \mid T \in C_{\mathcal{F}}(b)\}.$$

Let us consider $\mathcal{G} = \{B \cap T \mid T \in \mathcal{F}\}$ and let us show that $\mathcal{G} \in \mathcal{C}(B)$ and that $\sigma(b) = C_{\mathcal{G}}(b)$ for every $b \in B$.

As $B \subseteq A$, then $B = B \cap A$ and $A \in \mathcal{F}$, so $B \in \mathcal{G}$.

Let $\mathcal{X} \subseteq \mathcal{G}$ and let $\mathcal{X}' = \{T \in \mathcal{F} \mid B \cap T \in \mathcal{X}\}$, so $\bigcap \mathcal{X}' \in \mathcal{F}$, hence $\bigcap \mathcal{X} = \bigcap_{T \in \mathcal{X}'} (B \cap T) = B \cap \bigcap \mathcal{X}' \in \mathcal{G}$.

For every $b \in B$,

$$C_{\mathcal{G}}(b) = \{B \cap T \mid T \in \mathcal{F} \text{ and } b \in T\} = \{B \cap T \mid T \in C_{\mathcal{F}}(b)\} = \sigma(b).$$

□

Definition 3.19 (Sum) Let $\langle A_i, \xi_i \rangle_{i \in I}$ be a family of $\mathcal{C}$-coalgebras and let $A = \coprod_{i \in I} A_i$ be the disjoint sum of $(A_i)_{i \in I}$. A can be endowed with a $\mathcal{C}$-coalgebra structure by defining the coalgebra map as follows:

$$\begin{array}{rcl} \xi : \coprod_{i \in I} A_i & \longrightarrow & \mathcal{C}(\coprod_{i \in I} A_i) \\ \langle a, i \rangle & \longmapsto & \{A\} \cup \{\mathtt{inj}_i[T] \mid T \in \xi_i(a)\}. \end{array}$$

$\langle A, \xi \rangle$ is the *sum* of $\langle A_i, \xi_i \rangle_{i \in I}$.

Lemma 3.20 *Let $\langle A_i, \xi_i \rangle_{i \in I}$ be a family of $\mathcal{C}$-coalgebras and let $\langle A, \xi \rangle$ be the sum of $\langle A_i, \xi_i \rangle_{i \in I}$. Then*

(i) *For every $\bar{a} \in A$ $\xi(\bar{a}) \in \mathcal{C}(A)$.*

(ii) *For every $i \in I$ $\mathtt{inj}_i$ is a coalgebra morphism.*

Proof. (i) Let $\bar{a} \in A$, then $A \in \xi(\bar{a})$ by definition, moreover $\bar{a} = \langle a, i \rangle$ and $a \in A_i$ for some $i \in I$. Let $\mathcal{X} \subseteq \xi(\bar{a})$, and let $\mathcal{X}' = \{T \in \xi_i(a) \mid \mathtt{inj}_i[T] \in \mathcal{X}\}$, so $\bigcap \mathcal{X}' \in \xi_i(a)$ and $\bigcap \mathcal{X} = \bigcap_{T \in \mathcal{X}'} \mathtt{inj}_i[T] = \mathtt{inj}_i[\bigcap \mathcal{X}']$.

(ii) Let $i \in I$, let $a \in A_i$.

$$\begin{aligned}
\mathcal{C}(\mathtt{inj}_i)(\xi(\langle a, i\rangle)) &= \{\mathtt{inj}_i^{-1}[T'] \mid T' \in \xi(\langle a, i\rangle)\} \\
&= \{\mathtt{inj}_i^{-1}[T'] \mid T' = A \text{ or } T' \in \{\mathtt{inj}_i[T] \mid T \in \xi_i(a)\}\} \\
&= \{\mathtt{inj}_i^{-1}[\mathtt{inj}_i[T]] \mid T \in \xi_i(a)\} \\
&= \xi_i(a)
\end{aligned}$$

□

Proposition 3.21 *Let $\langle A_i, \xi_i\rangle_{i\in I}$ be a family of $\mathcal{C}$-coalgebras and let $\langle A, \xi\rangle$ be the sum of $\langle A_i, \xi_i\rangle_{i\in I}$. If $\langle A_i, \xi_i\rangle$ comes from an abstract logic for every $i \in I$, then so does $\langle A, \xi\rangle$.*

Proof. We have to show that there exists $\mathcal{F} \in \mathcal{C}(A)$ such that $\xi(\bar{a}) = C_{\mathcal{F}}(\bar{a})$ for every $\bar{a} \in A$. As $\langle A_i, \xi_i\rangle$ comes from an abstract logic for every $i \in I$, then there exists $\mathcal{F}_i \in \mathcal{C}(A_i)$ such that $\xi_i(a) = C_{\mathcal{F}_i}(a)$ for every $a \in A_i$, $i \in I$.

Let us consider $\mathcal{F} = \{\emptyset, A\} \cup \bigcup_{i\in I}\{\mathtt{inj}_i[T] \mid T \in \mathcal{F}_i\}$ and let us show that $\mathcal{F} \in \mathcal{C}(A)$ and that $\xi(\bar{a}) = C_{\mathcal{F}}(\bar{a})$ for every $\bar{a} \in A$. Let $\mathcal{X} \subseteq \mathcal{F}$. If $\bigcap \mathcal{X} \neq \emptyset$ then $\mathcal{X} \subseteq \{\mathtt{inj}_i[T] \mid T \in \mathcal{F}_i\}$ for some $i \in I$, so let $\mathcal{X}' = \{T \in \mathcal{F}_i \mid \mathtt{inj}_i[T] \in \mathcal{X}\}$, then $\bigcap \mathcal{X}' \in \mathcal{F}_i$ and $\bigcap \mathcal{X} = \bigcap_{T\in\mathcal{X}'} \mathtt{inj}_i[T] = \mathtt{inj}_i[\bigcap \mathcal{X}'] \in \mathcal{F}$.

Let $\bar{a} \in A$, then $\bar{a} = \langle a, i\rangle$ and $a \in A_i$ for some $i \in I$.

$$\begin{aligned}
C_{\mathcal{F}}(\bar{a}) &= \{T' \in \mathcal{F} \mid \bar{a} \in T'\} \\
&= \{A\} \cup \{\mathtt{inj}_i[T] \mid T \in \mathcal{F}_i \text{ and } a \in T\} \\
&= \{A\} \cup \{\mathtt{inj}_i[T] \mid T \in C_{\mathcal{F}_i}(a)\} \\
&= \{A\} \cup \{\mathtt{inj}_i[T] \mid T \in \xi_i(a)\} \\
&= \xi(\bar{a}).
\end{aligned}$$

□

Definition 3.22 (Covariety) Let **K** be a class of $\mathcal{C}$-coalgebras. **K** is a *covariety* iff it is closed under homomorphic images, subcoalgebras and sums.

The next proposition immediately follows from Propositions 3.16, 3.18 and 3.21.

Proposition 3.23 *Let **K** be the class of $\mathcal{C}$-coalgebras which come from abstract logics. **K** is a covariety.*

3.1 Abstract logics as coalgebras of the filter functor

There is an alternative way of turning abstract logics of empty algebraic similarity type into coalgebras. Let $\mathbf{L} = \langle A, \mathcal{C}\rangle$ be an abstract logic of empty algebraic similarity type. Let us stipulate that an *open set* of **L** is of the kind $A \setminus T$ for some $T \in \mathcal{C}$. Then **L** can be turned into an $\mathcal{F}$-coalgebra (where $\mathcal{F}$ is the filter functor as defined in [2]) by defining the coalgebra map precisely in the same way followed by Gumm to turn topological spaces into $\mathcal{F}$-coalgebras.

Then, as it is shown by Gumm, coalgebra morphisms are exactly the continuous and open maps. This does not suit our case, because there are examples of strict morphisms which are not open (for example, if $\mathbf{L} = \langle A, \mathcal{C}\rangle$, $T \in \mathcal{C}$, T is not open, and $\mathbf{L}' = \langle T, \mathcal{C}'\rangle$, where $\mathcal{C}' = \{X \cap T \mid X \in \mathcal{C}\}$, then the embedding map of $\mathbf{L}'$ into $\mathbf{L}$ is a strict morphism which is not open), so not all morphisms which are relevant in the context of abstract logics would give rise to morphisms of the corresponding coalgebras.

4 Abstract logics as dialgebras

Remark 4.1 Let τ be an algebraic similarity type, for every $f \in \tau$ let n_f be the arity of f and let $\mathcal{A} = \langle A, (f^{\mathcal{A}})_{f\in\tau}\rangle$ be a τ-algebra. $\mathcal{A}$ can be turned into an F_τ-algebra $\langle A, \alpha\rangle$, where F_τ is the covariant endofunctor on **Set** defined as follows: $F_\tau(A) = \coprod_{f\in\tau} A^{n_f}$ for every set A, and for every map $g : A \longrightarrow B$

$$\begin{array}{rcl} F_\tau(g) : \coprod_{f\in\tau} A^{n_f} & \longrightarrow & \coprod_{f\in\tau} B^{n_f} \\ \langle \bar{a}, f\rangle & \longmapsto & \langle \overline{g(a)}, f\rangle \end{array}$$

(so if $\tau = \emptyset$ then $F_\tau = Id$), and α is defined as follows:

$$\begin{array}{rcl} \alpha : F_\tau(A) & \longrightarrow & A \\ \langle \bar{a}, f\rangle & \longmapsto & f^{\mathcal{A}}(\bar{a}). \end{array}$$

Remark 4.2 Let $\langle A, \alpha\rangle$ be an F_τ-algebra. For every $f \in \tau$ let us define

$$\begin{array}{rcl} f^{\mathcal{A}} : A^{n_f} & \longrightarrow & A \\ \bar{a} & \longmapsto & \alpha(\langle \bar{a}, f\rangle). \end{array}$$

So $\mathcal{A} = \langle A, (f^{\mathcal{A}})_{f\in\tau}\rangle$ is a τ-algebra, therefore every F_τ-algebra comes from a τ-algebra.

The two correspondences mentioned in the previous remarks can be extended to isomorphisms between the category $\mathbf{Alg}(\tau)$ of τ-algebras and their homomorphisms and the category $\mathbf{Alg}(\mathbf{F}_\tau)$ of F_τ-algebras and F_τ-algebra morphisms.

Definition 4.3 (Dialgebraic signature) (cf. def 3.1 in [3]) A *dialgebraic signature* is a signature of the form

$$\Sigma(X) = \Sigma_1(X) \times \cdots \times \Sigma_n(X),$$

where for every $i = 1, \ldots, n$

$$\Sigma_i(X) = F_i(X) \longrightarrow G_i(X),$$

and F_i, G_i are endofunctors on **Set**. [1]

[1] This definition is more general than the one in [3], where F_i and G_i are polynomial signatures. The reason is that I need to account for the closure system functor which is not

$\Sigma(X)$ is *algebraic* if $G_i(X) = X$ for every i, and it is *coalgebraic* if $F_i(X) = X$ for every i.

Definition 4.4 (Dialgebra) (cf. def 3.2 in [3]) A Σ-*dialgebra* is a pair $\langle A, \sigma\rangle$, where A is a set and $\sigma \in \Sigma(A)$, i.e. $\sigma = \langle \sigma_1, \ldots, \sigma_n\rangle$ and for every $i = 1, \ldots, n$ $\sigma_i \in \Sigma_i(A) = F_i \longrightarrow G_i$.

Notation: Let τ be an algebraic similarity type, and let us denote

$$\Sigma_\tau(X) := (F_\tau(X) \longrightarrow X) \times (X \longrightarrow \mathcal{C}(X)).$$

Remark 4.5 Let $\mathbf{L} = \langle \mathcal{A}, \mathcal{F}\rangle$ be a τ-abstract logic. $\mathbf{L}$ can be turned into a Σ_τ-dialgebra $\langle A, \alpha, \xi\rangle$ by defining the maps $\alpha : F_\tau(A) \longrightarrow A$ and $\xi : A \longrightarrow \mathcal{C}(A)$ as in Remarks 4.1 and 3.4. So τ-abstract logics can be regarded as dialgebras of a very special signature which admits an algebraic part and a coalgebraic part: given a Σ_τ-dialgebra $\langle A, \alpha, \xi\rangle$, $\langle A, \alpha\rangle$ is its *algebraic reduct* and $\langle A, \xi\rangle$ is its *coalgebraic reduct.*

Definition 4.6 A Σ_τ-dialgebra $\langle A, \alpha, \xi\rangle$ *comes from a τ-abstract logic* iff its coalgebraic reduct comes from an abstract logic.

Definition 4.7 (Dialgebra morphism) Let $\langle A, \alpha, \xi\rangle$ and $\langle B, \beta, \sigma\rangle$ be Σ_τ-dialgebras. A map $f : A \longrightarrow B$ is a *dialgebra morphism* iff the following diagram commutes:

$$\begin{array}{ccccc} F_\tau(A) & \xrightarrow{\alpha} & A & \xrightarrow{\xi} & \mathcal{C}(A) \\ F_\tau(f)\big\downarrow & & f\big\downarrow & & \mathcal{C}(f)\big\uparrow \\ F_\tau(B) & \xrightarrow{\beta} & B & \xrightarrow{\sigma} & \mathcal{C}(B) \end{array} \tag{7}$$

Proposition 4.8 *Let $\langle A, \alpha, \xi\rangle$ and $\langle B, \beta, \sigma\rangle$ be Σ_τ-dialgebras which come from τ-abstract logics, let $\mathcal{F} \in \mathcal{C}(A)$, $\mathcal{G} \in \mathcal{C}(B)$ such that for every $a \in A$ $\xi(a) = C_\mathcal{F}(a)$ and for every $b \in B$ $\sigma(b) = \mathcal{C}_\mathcal{G}(b)$, let $\mathcal{F}' = \mathcal{F} \cup \{\emptyset\}$, $\mathcal{G}' = \mathcal{G} \cup \{\emptyset\}$, let $\mathbf{L} = \langle \mathcal{A}, \mathcal{F}'\rangle$, $\mathbf{L}' = \langle \mathcal{B}, \mathcal{G}'\rangle$ be τ-abstract logics and let $f : A \longrightarrow B$ be a map. The following are equivalent:*

(i) *f is a dialgebra morphism.*

(ii) *$f : \mathbf{L} \longrightarrow \mathbf{L}'$ is a strict morphism of τ-abstract logics.*

Definition 4.9 (Dialgebra Bisimulation) Let $\langle A, \alpha, \xi\rangle$ and $\langle B, \beta\sigma\rangle$ be Σ_τ-dialgebras. A relation $R \subseteq A \times B$ is a *bisimulation* between them iff there is

polynomial.

some Σ_τ-dialgebra $\langle R, \rho, \zeta \rangle$ such that the following diagram commutes:

$$
\begin{array}{ccccc}
F_\tau(A) & \xleftarrow{F_\tau(\pi_1)} & F_\tau(R) & \xrightarrow{F_\tau(\pi_2)} & F_\tau(B) \\
\alpha\big\downarrow & & \rho\big\downarrow & & \beta\big\downarrow \\
A & \xleftarrow{\pi_1} & R & \xrightarrow{\pi_2} & B \\
\xi\big\downarrow & & \rho\big\downarrow & & \sigma\big\downarrow \\
\mathcal{C}(A) & \xrightarrow{\mathcal{C}(\pi_1)} & \mathcal{C}(R) & \xleftarrow{\mathcal{C}(\pi_2)} & \mathcal{C}(B)
\end{array}
\tag{8}
$$

where π_1 and π_2 are the canonical projections restricted to R.

Proposition 4.10 *Let $\langle A, \alpha, \xi \rangle$ and $\langle B, \beta, \sigma \rangle$ be Σ_τ-dialgebras which come from τ-abstract logics, let $\langle R, \rho, \zeta \rangle$ be a bisimulation between them. Then $\langle R, \rho, \zeta \rangle$ comes from a τ-abstract logic.*

Proposition 4.11 *Let $\langle A, \alpha, \xi \rangle$ and $\langle B, \beta, \sigma \rangle$ be Σ_τ-dialgebras, let $f : A \longrightarrow B$ be a surjective dialgebra morphism. If $\langle A, \alpha, \xi \rangle$ comes from a τ-abstract logic, so does $\langle B, \beta, \sigma \rangle$.*

Definition 4.12 (Subdialgebra) Let $B \subseteq A$ and $\langle A, \alpha, \xi \rangle$ and $\langle B, \beta, \sigma \rangle$ be Σ_τ-dialgebras. $\langle B, \beta, \sigma \rangle$ is a *subdialgebra* of $\langle A, \alpha, \xi \rangle$ iff the natural inclusion map $i : B \longrightarrow A$ is a dialgebra morphism.

Proposition 4.13 *Let $\langle A, \alpha, \xi \rangle$ and $\langle B, \beta, \sigma \rangle$ be Σ_τ-dialgebras, and let $\langle B, \beta, \sigma \rangle$ be a subdialgebra of $\langle A, \alpha, \xi \rangle$. If $\langle A, \alpha, \xi \rangle$ comes from a τ-abstract logic, so does $\langle B, \beta, \sigma \rangle$.*

As there is no canonical way of defining a structure of τ-algebra on the sum of a family of τ-algebras for an arbitrary algebraic similarity type τ, Proposition 3.21 cannot be extended to Σ_τ-dialgebras.

5 Further developments

About the Tarski congruence. The Tarski congruence plays a central role in the theory of abstract logics because it is used for generalizing the Lindenbaum-Tarski construction to the case of abstract logics. It should be easy to see that if $\langle A, \alpha, \xi \rangle$ comes from an abstract logic $\mathbf{L}$, $\widetilde{\Omega}(\mathbf{L})$ is the greatest bisimulation between $\langle A, \alpha, \xi \rangle$ and itself.

Finitary abstract logics. An abstract logic $\langle A, C \rangle$ is *finitary* iff for every subset $X \subseteq A$

$$C(X) = \bigcup \{C(Y) \mid Y \subseteq X, Y \text{ finite}\}.$$

Finitary abstract logics are an interesting and well behaved subclass of abstract logics. The closure systems associated with finitary abstract logics are exactly the *inductive* ones, i.e. the ones which are closed under unions of *upwards directed* [2] subfamilies (cf. Theorem 1.3.5. of [4]). One might ask if the subclass of co-/dialgebras which come from finitary abstract logics has a nice internal characterization, or if a special co-/dialgebraic signature is suitable to turn finitary abstract logics into co-/dialgebras. More in general, one might study finitary abstract logics from a co-/dialgebraic perspective.

Metalogical properties of closure operators. In the picture presented so far there is no interaction between algebraic operations and closure systems. In the theory of abstract logics this interaction is accounted for by the *metalogical properties* (see section 2.4. of [1]) that can be imposed on closure operators. For example, let $\langle \mathcal{A}, C \rangle$ be a τ-abstract logic such that the binary operation $\vee$ is in τ. The closure operator C *has the weak property of disjunction w.r.t.* $\vee$ iff for every $a, b \in A$,

$$C(\{a \vee b\}) = C(\{a\}) \cap C(\{b\}).$$

This property is intended to capture some of the proof-theoretic behaviour of disjunction, namely, it says that for every $x \in A$, "x follows from $a \vee b$" if and only if "x follows from a" and "x follows from b". Metalogical properties are important not only because they are essential in linking abstract logics with concrete logics, but also because some of them guarantee a good algebraic behaviour of abstract logics that enjoy them, for example some of them imply that the Frege and the Tarski relations coincide. Enriching this picture with metalogical properties would be a natural development of this work.

Properties of the functor $\mathcal{C}$. Another natural development is to investigate properties of the functor $\mathcal{C}$. For example, as $\mathcal{C}$ is a contravariant functor, some of the properties which guarantee that covariant functors have a good behaviour, such as the preservation of pullbacks, do not apply. Are there properties of contravariant functors which would guarantee the same good behaviour that the property of preservation of (weak) pullbacks guarantees for covariant endofunctors? Does $\mathcal{C}$ enjoy them?

The Algebraic Logic perspective. A given logic $\mathcal{S}$ is investigated from the perspective of Algebraic Logic by associating it with a class of algebras "in a canonical way" and studying how tightly $\mathcal{S}$ is connected with its class of algebras. Using abstract logics is one of the possible ways in which this connection is made (see [1]). One might investigate whether it is possible

[2] $\mathcal{F} \subseteq \mathcal{C}$ is *upwards directed* iff for all $X, Y \in \mathcal{F}$ there is $Z \in \mathcal{F}$ such that $X \cup Y = Z$.

to translate some of these procedures and constructions to the context presented in this report, and whether some of them can be generalized to different signatures.

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