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Unified Apostol–Bernoulli, Euler and Genocchi polynomials

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ABSTRACT

In this paper, we present a unified family of polynomials including not only the Apostol–Bernoulli, Euler and Genocchi polynomials, but also a general family of polynomials suggested by Özden et al. [H. Ozden, Y. Simsek, H.M. Srivastava, A unified presentation of the generating functions of the generalized Bernoulli, Euler and Genocchi polynomials. Comput. Math. Appl. 60 (10) (2010) 2779–2787]. We obtain the explicit representation of this unified family, in terms of a Gaussian hypergeometric function. Some symmetry identities and multiplication formula are also given.

1. Introduction

The generalized Apostol–Bernoulli polynomials $\mathcal{B}_n^{(\alpha)}(x;\lambda)$ are defined, by Luo and Srivastava [1–4], through the generating relation

$$\left(\frac{t}{\lambda e^t-1}\right)^{\alpha} e^{xt}=\sum_{n=0}^{\infty}\mathcal{B}_n^{(\alpha)}(x;\lambda)\frac{t^n}{n!}\quad (|t+\log\lambda|<2\pi,\ 1^{\alpha}:=1),$$

where α and λ are the arbitrary real or complex parameters and $x\in\mathbb{R}$. The Apostol–Bernoulli polynomials and the Apostol–Bernoulli numbers can be obtained from the generalized Apostol–Bernoulli polynomials by

$$B_n(x;\lambda)=\mathcal{B}_n^{(1)}(x;\lambda),\qquad B_n(\lambda)=B_n(0;\lambda)\quad (n\in\mathbb{N}_0),$$

respectively. The case $\lambda=1$ in the above relations gives the classical Bernoulli polynomials $B_n(x)$ and the Bernoulli numbers B_n.

Recently, for the arbitrary real or complex parameters α and λ and $x\in\mathbb{R}$, Luo [5] generalized the Apostol–Euler polynomials $E_n^{\alpha}(x;\lambda)$ by the generating relation

$$\left(\frac{2}{\lambda e^t+1}\right)^{\alpha} e^{xt}=\sum_{n=0}^{\infty}\mathcal{E}_n^{\alpha}(x;\lambda)\frac{t^n}{n!}\quad (|t+\log\lambda|<\pi,\ 1^{\alpha}:=1).$$

The Apostol–Euler polynomials and the Apostol–Euler numbers are given by

$$E_n(x;\lambda)=\mathcal{E}_n^{1}(x;\lambda),\qquad E_n(\lambda)=E_n(1;\lambda),$$

respectively. The above relations give the classical Euler polynomials $E_n(x)$ and the Euler numbers E_n when $\lambda=1$.

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Let $x \in \mathbb{R}$. For an arbitrary real or complex parameters α and λ, the Apostol–Genocchi polynomials of order α are defined by (see [6,7])

$$\left(\frac{2t}{\lambda e^t + 1}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} \mathcal{G}_n^{\alpha}(x; \lambda) \frac{t^n}{n!} \quad (|t + \log \lambda| < \pi,\ 1^{\alpha} := 1).$$

The Apostol–Genocchi polynomials and the Apostol–Genocchi numbers are given by

$$G_n(x; \lambda) = \mathcal{G}_n^1(x; \lambda), \qquad G_n(\lambda) = G_n(0; \lambda),$$

respectively. When $\lambda = 1$, the above relations give the classical Genocchi polynomials $G_n(x)$ and the Genocchi numbers G_n.

We should note that the above polynomials have recently been studied in the papers [8–12].

Motivated by the generating relation [13]

$$f_{a,b}(x; t; k, \beta) := \frac{2^{1-k} t^k e^{xt}}{\beta^b e^t - a^b} = \sum_{n=0}^{\infty} y_{n,\beta}(x; k, a, b) \frac{t^n}{n!}; \quad \left(\left| t + b \log\left(\frac{\beta}{a}\right) \right| < 2\pi,\ x \in \mathbb{R} \right)$$
$$(k \in \mathbb{N}_0;\ a, b \in \mathbb{R}^+,\ \beta \in \mathbb{C}),$$

where the associated numbers are given by

$$y_{n,\beta}(0; k, a, b) := y_{n,\beta}(k, a, b),$$

in this paper, we consider the following unified form of the Apostol–Bernoulli, Euler and Genocchi polynomials

$$f_{a,b}^{(\alpha)}(x; t; k, \beta) := \left(\frac{2^{1-k} t^k}{\beta^b e^t - a^b}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} P_{n,\beta}^{(\alpha)}(x; k, a, b) \frac{t^n}{n!}; \quad (k \in \mathbb{N}_0;\ a, b \in \mathbb{R} \setminus \{0\}; \alpha, \beta \in \mathbb{C}). \tag{1}$$

For the convergence of the series involved in (1), we have

(i) If $a^b > 0$ and $k \in \mathbb{N}$, then $|t + b \log(\frac{\beta}{a})| < 2\pi;\ 1^{\alpha} := 1,\ x \in \mathbb{R}, \beta \in \mathbb{C}$.
(ii) If $a^b > 0$ and $k = 0$, then $0 < \mathrm{Im}(t + b \log(\frac{\beta}{a})) < 2\pi;\ 1^{\alpha} := 1,\ x \in \mathbb{R}, \beta \in \mathbb{C}$.
(iii) If $a^b < 0$, then $|t + b \log(\frac{\beta}{a})| < \pi;\ 1^{\alpha} := 1,\ x \in \mathbb{R}, k \in \mathbb{N}_0, \beta \in \mathbb{C}$.

It should be noted that, the family of polynomials $P_{n,\beta}^{(\alpha)}(x; k, a, b)$ includes the above mentioned well known polynomials some of which we list below:

Remark 1.1. Setting $k = a = b = 1$ and $\beta = \lambda$ in (1), we get

$$P_{n,\lambda}^{(\alpha)}(x; 1, 1, 1) = \mathcal{B}_n^{(\alpha)}(x; \lambda),$$

where $\mathcal{B}_n^{(\alpha)}(x; \lambda)$ are the generalized Apostol–Bernoulli polynomials.

Remark 1.2. Choosing $k + 1 = -a = b = 1$ and $\beta = \lambda$ in (1), we get

$$P_{n,\lambda}^{(\alpha)}(x; 0, -1, 1) = \mathcal{E}_n^{(\alpha)}(x; \lambda),$$

where $\mathcal{E}_n^{(\alpha)}(x; \lambda)$ are the generalized Apostol–Euler polynomials.

Remark 1.3. Letting $k = -2a = b = 1$ and $2\beta = \lambda$ in (1), we get

$$P_{n,\frac{\lambda}{2}}^{(\alpha)}\left(x; 1, \frac{-1}{2}, 1\right) = \mathcal{G}_n^{\alpha}(x; \lambda),$$

where $\mathcal{G}_n^{\alpha}(x; \lambda)$ are the generalized Apostol–Genocchi polynomials.

Remark 1.4. Setting $\alpha = 1$ in (1), we get

$$P_{n,\beta}^{(1)}(x; k, a, b) = y_{n,\beta}(x; k, a, b).$$

For the other known polynomials which are related with the generalized family $P_{n,\beta}^{(\alpha)}(x; k, a, b)$, we refer [14–19]. We organize the paper as follows.

In Section 2, we obtain the explicit representation of the unified family $P_{n,\beta}^{(\alpha)}(x; k, a, b)$, in terms of Gaussian hypergeometric function. In Section 3, some symmetry identities for the polynomials $P_{n,\beta}^{(\alpha)}(x; k, a, b)$ are given. In Section 4, multiplication formula is obtained for this unified family.

2. Explicit expression

In this section, we aim to obtain the explicit expression of the polynomials $P_{n,\beta}^{(\alpha)}(x; k, a, b)$, in terms of a Gauss hypergeometric function $F(a, b; c; z)$, which is defined by

$$F(a, b; c; z) = {}_2F_1(a, b; c; z) := \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}$$

where $c \notin \mathbb{Z}_0^- := \{0, -1, -2, \ldots\}$; $|z| < 1$; $z = 1$ and $\mathrm{Re}(c - a - b) > 0$; $z = -1$ and $\mathrm{Re}(c - a - b) > -1$. Our main result in this section is stated in the following theorem.

Theorem 2.1. *For $n, l, k \in \mathbb{N}_0$, $a, b \in \mathbb{R}$, $\beta \neq a$, we have*

$$P_{n,\beta}^{(l)}(x; k, a, b) = 2^{(1-k)l}(kl)! \binom{n}{kl} \sum_{i=0}^{n-kl} \binom{l+i-1}{i} \binom{n-kl}{i} \frac{\beta^{bi}}{(\beta^b - a^b)^{l+i}}$$
$$\times \sum_{m=0}^{i} (-1)^m \binom{i}{m} m^i (x+m)^{n-i-kl} F\left(-n + kl + i, i; 1 + i; \frac{m}{m+x}\right).$$

Proof. Let $D_t = \frac{\mathrm{d}}{\mathrm{d}t}$. We have, by (1), that

$$\begin{aligned} P_{n,\beta}^{(l)}(x; k, a, b) &= D_t^n \left[\left(\frac{2^{1-k} t^k}{\beta^b \mathrm{e}^t - a^b} \right)^l \mathrm{e}^{xt} \right]_{t=0} \\ &= 2^{(1-k)l} \sum_{s=0}^{n} \binom{n}{s} x^{n-s} D_t^s [t^{kl} (\beta^b \mathrm{e}^t - a^b)^{-l}]|_{t=0} \\ &= 2^{(1-k)l} \sum_{s=kl}^{n} \binom{n}{s} x^{n-s} (kl)! \binom{s}{kl} D_t^{s-kl} [(\beta^b \mathrm{e}^t - a^b)^{-l}]_{t=0} \\ &= 2^{(1-k)l} \sum_{s=kl}^{n} \binom{n}{s} x^{n-s} (kl)! \binom{s}{kl} D_t^{s-kl} [(\beta^b - a^b + \beta^b (\mathrm{e}^t - 1))^{-l}]_{t=0}. \end{aligned}$$

Now, since

$$(A + w)^{-l} = \sum_{i=0}^{\infty} \binom{l+i-1}{i} A^{-l-i} (-w)^i \quad (|w| < |A|)$$

and since (see [20, p. 58 (15)]),

$$(\mathrm{e}^t - 1)^i = i! \sum_{j=i}^{\infty} S(j, i) \frac{t^j}{j!},$$

where $S(j, i)$ are the Stirling numbers of the second kind, we get

$$\begin{aligned} P_{n,\beta}^{(l)}(x; k, a, b) &= 2^{(1-k)l} \sum_{s=kl}^{n} \binom{n}{s} x^{n-s} (kl)! \binom{s}{kl} \sum_{i=0}^{\infty} \binom{l+i-1}{i} (\beta^b - a^b)^{-l-i} (-\beta^b)^i D_t^{s-kl} [(\mathrm{e}^t - 1)^i]_{t=0} \\ &= 2^{(1-k)l} \sum_{s=kl}^{n} \binom{n}{s} x^{n-s} (kl)! \binom{s}{kl} \sum_{i=0}^{\infty} \binom{l+i-1}{i} (\beta^b - a^b)^{-l-i} (-\beta^b)^i D_t^{s-kl} \left[i! \sum_{j=i}^{\infty} S(j, i) \frac{t^j}{j!} \right]_{t=0} \\ &= 2^{(1-k)l} \sum_{s=kl}^{n} \binom{n}{s} x^{n-s} (kl)! \binom{s}{kl} \sum_{i=0}^{s-kl} \binom{l+i-1}{i} (\beta^b - a^b)^{-l-i} (-\beta^b)^i i! S(s - kl, i). \end{aligned}$$

Using the formula (see [20, p. 58 (20)]),

$$S(j, i) = \frac{1}{i!} \sum_{m=0}^{i} (-1)^{i-m} \binom{i}{m} m^j,$$

and

$$\binom{n}{s} \binom{s}{kl} = \binom{n}{kl} \binom{n-kl}{n-s},$$

we obtain

$$\begin{aligned}
P_{n,\beta}^{(l)}(x;k,a,b) &= 2^{(1-k)l}\sum_{s=kl}^{n}\binom{n}{s}(kl)!\binom{s}{kl}\sum_{i=0}^{s-kl}\binom{l+i-1}{i}\frac{(-\beta^b)^i x^{n-s}}{(\beta^b-a^b)^{l+i}}\sum_{m=0}^{i}(-1)^{i-m}\binom{i}{m}m^{s-kl}\\
&= 2^{(1-k)l}(kl)!\binom{n}{kl}\sum_{i=0}^{n-kl}\sum_{s=i+kl}^{n}\binom{n-kl}{n-s}\binom{l+i-1}{i}\frac{(-\beta^b)^i x^{n-s}}{(\beta^b-a^b)^{l+i}}\sum_{m=0}^{i}(-1)^{i-m}\binom{i}{m}m^{s-kl}\\
&= 2^{(1-k)l}(kl)!\binom{n}{kl}\sum_{i=0}^{n-kl}\sum_{s=0}^{n-i-kl}\binom{n-kl}{n-s-kl-i}\binom{l+i-1}{i}\frac{(-\beta^b)^i x^{n-s-i-kl}}{(\beta^b-a^b)^{l+i}}\sum_{m=0}^{i}(-1)^{i-m}\binom{i}{m}m^{s+i}.
\end{aligned}$$

Since

$$(n-s-kl-i)! = \frac{(-1)^s(n-kl-i)!}{(-n+kl+i)_s},$$

we get

$$\begin{aligned}
P_{n,\beta}^{(l)}(x;k,a,b) &= 2^{(1-k)l}(kl)!\binom{n}{kl}\sum_{i=0}^{n-kl}\binom{l+i-1}{i}\binom{n-kl}{i}\frac{\beta^{bi}x^{n-i-kl}}{(\beta^b-a^b)^{l+i}}\\
&\quad\times\sum_{m=0}^{i}(-1)^m\binom{i}{m}m^i F\left(-n+kl+i,1;1+i;-\frac{m}{x}\right).
\end{aligned}$$

Applying the Pfaff–Kummer hypergeometric transformation

$$F(a,b;c;z) = (1-z)^{-a}F\left(a,c-b;c;\frac{z}{z-1}\right)\quad (c\notin\mathbb{Z}_0^-;\ |\arg(1-z)|\leq\pi-\varepsilon\ (0<\varepsilon<\pi)),$$

we obtain

$$\begin{aligned}
P_{n,\beta}^{(l)}(x;k,a,b) &= 2^{(1-k)l}(kl)!\binom{n}{kl}\sum_{i=0}^{n-kl}\binom{l+i-1}{i}\binom{n-kl}{i}\frac{\beta^{bi}}{(\beta^b-a^b)^{l+i}}\\
&\quad\times\sum_{m=0}^{i}(-1)^m\binom{i}{m}m^i(x+m)^{n-i-kl}F\left(-n+kl+i,i;1+i;\frac{m}{m+x}\right).
\end{aligned}$$

Whence the result. □

For $l=1$, we get the explicit representation of the polynomials $y_{n,\beta}(x;k,a,b)$.

Corollary 2.2. *For $n,k\in\mathbb{N}_0,\ a,b\in\mathbb{R}^+,\ \beta\neq a$, we have*

$$\begin{aligned}
y_{n,\beta}(x;k,a,b) &= 2^{(1-k)}k!\binom{n}{k}\sum_{i=0}^{n-k}\binom{n-k}{i}\frac{\beta^{bi}}{(\beta^b-a^b)^{1+i}}\\
&\quad\times\sum_{m=0}^{i}(-1)^m\binom{i}{m}m^i(x+m)^{n-i-k}F\left(-n+k+i,i;1+i;\frac{m}{m+x}\right).
\end{aligned}$$

For $k=a=b=1$ and $\beta=\lambda$, we get Theorem 1 of [3].

Corollary 2.3. *For $n,l\in\mathbb{N}_0,\ \lambda\in\mathbb{C}\setminus\{1\}$, we have the following explicit representation of the generalized Apostol–Bernoulli polynomials* [3]:

$$\begin{aligned}
\mathcal{B}_n^{(l)}(x;\lambda) &= l!\binom{n}{l}\sum_{i=0}^{n-l}\binom{l+i-1}{i}\binom{n-l}{i}\frac{\lambda^i}{(\lambda-1)^i}\\
&\quad\times\sum_{m=0}^{i}(-1)^m\binom{i}{m}m^i(x+m)^{n-i-l}F\left(-n+l+i,i;1+i;\frac{m}{m+x}\right).
\end{aligned}$$

For $k+1=-a=b=1$ and $\beta=\lambda$, we get Theorem 1 of [5] when $\alpha=l\in\mathbb{N}_0$.

Corollary 2.4. *For $n, l \in \mathbb{N}_0, \lambda \in \mathbb{C} \setminus \{1\}$, we have the following explicit representation of the generalized Apostol–Euler polynomials* [5]*:*

$$\mathcal{E}_n^{(l)}(x;\lambda) = 2^l \sum_{i=0}^{n} \binom{l+i-1}{i} \binom{n-l}{i} \frac{\lambda^i}{(\lambda-1)^{l+i}} \sum_{m=0}^{i} (-1)^m \binom{i}{m} m^i (x+m)^{n-i} F\left(-n+i, i; 1+i; \frac{m}{m+x}\right).$$

For $k = -2a = b = 1$ and $2\beta = \lambda$, we get Theorem 1 of [21].

Corollary 2.5. *For $n, l \in \mathbb{N}_0,\ \lambda \in \mathbb{C} \setminus \{1\}$, we have the following explicit representation of the generalized Apostol–Genocchi polynomials* [21]*:*

$$\mathcal{G}_n^{(l)}(x;\lambda) = 2^l l! \binom{n}{l} \sum_{i=0}^{n-l} \binom{l+i-1}{i} \binom{n-l}{i} \frac{\lambda^i}{(\lambda-1)^{l+i}} \times \sum_{m=0}^{i} (-1)^m \binom{i}{m} m^i (x+m)^{n-i-l} F\left(-n+l+i, i; 1+i; \frac{m}{m+x}\right).$$

3. Symmetry identities for the unified family

For each $k \in \mathbb{N}_0$, $S_k(n) = \sum_{i=0}^{n} i^k$ is known as the power sum and the following generating relation is straightforward:

$$\sum_{k=0}^{\infty} S_k(n) \frac{t^k}{k!} = 1 + e^t + e^{2t} + \cdots + e^{nt} = \frac{e^{(n+1)t} - 1}{e^t - 1}.$$

For an arbitrary real or complex λ, the generalized sum of integer powers $S_k(n, \lambda)$ is defined, in [22], via the following generating relation:

$$\sum_{k=0}^{\infty} S_k(n, \lambda) \frac{t^k}{k!} = \frac{\lambda e^{(n+1)t} - 1}{\lambda e^t - 1}.$$

Obviously $S_k(n, 1) = S_k(n)$.

For each $k \in \mathbb{N}_0$, $M_k(n) = \sum_{i=0}^{n} (-1)^k i^k$ is known as the alternative integer powers given by the generating relation:

$$\sum_{k=0}^{\infty} M_k(n) \frac{t^k}{k!} = 1 - e^t + e^{2t} - \cdots (-1)^n e^{nt} = \frac{1 - (-e^t)^{(n+1)}}{e^t + 1}.$$

For an arbitrary real or complex λ, the generalized sum of alternative integer powers $M_k(n, \lambda)$ is defined, in [22], by

$$\sum_{k=0}^{\infty} M_k(n, \lambda) \frac{t^k}{k!} = \frac{1 - \lambda(-e^t)^{(n+1)}}{\lambda e^t + 1}.$$

Obviously $M_k(n, 1) = M_k(n)$. On the other hand, if n is even, then

$$S_k(n, -\lambda) = M_k(n, \lambda). \tag{2}$$

We start with obtaining several symmetric identities, which includes the results given in [23–26,22].

Theorem 3.1. *For all $c, d, m \in \mathbb{N},\ n \in \mathbb{N}_0$, we have the following symmetry identity:*

$$\sum_{r=0}^{n} \binom{n}{r} c^{n-r} d^{r+k} P_{n-r,\beta}^{(m)}(dx; k, a, b) \sum_{l=0}^{r} \binom{r}{l} S_l\left(c-1; \left(\frac{\beta}{a}\right)^b\right) P_{r-l,\beta}^{(m-1)}(cy; k, a, b)$$
$$= \sum_{r=0}^{n} \binom{n}{r} d^{n-r} c^{r+k} P_{n-r,\beta}^{(m)}(cx; k, a, b) \sum_{l=0}^{r} \binom{r}{l} S_l\left(d-1; \left(\frac{\beta}{a}\right)^b\right) P_{r-l,\beta}^{(m-1)}(dy; k, a, b).$$

Proof. Let

$$G(t) := \frac{2^{(1-k)(2m-1)} t^{2km-k} e^{cdxt} (\beta^b e^{cdt} - a^b) e^{cdyt}}{(\beta^b e^{ct} - a^b)^m (\beta^b e^{dt} - a^b)^m}.$$

Expanding $G(t)$ into a series, we get

$$\begin{aligned}
G(t) &= \frac{1}{c^{km}d^{k(m-1)}}\left(\frac{2^{1-k}c^kt^k}{\beta^b e^{ct}-a^b}\right)^m e^{cdxt}\left(\frac{\beta^b e^{cdt}-a^b}{\beta^b e^{dt}-a^b}\right)\left(\frac{2^{1-k}d^kt^k}{\beta^b e^{dt}-a^b}\right)^{m-1} e^{cdyt}\\
&= \frac{1}{c^{km}d^{k(m-1)}}\left[\sum_{n=0}^{\infty}P_{n,\beta}^{(m)}(dx;k,a,b)\frac{(ct)^n}{n!}\right]\left[\sum_{n=0}^{\infty}S_n\left(c-1;\left(\frac{\beta}{a}\right)^b\right)\frac{(dt)^n}{n!}\right]\left[\sum_{n=0}^{\infty}P_{n,\beta}^{(m-1)}(cy;k,a,b)\frac{(dt)^n}{n!}\right]\\
&= \frac{1}{c^{km}d^{km}}\sum_{n=0}^{\infty}\left[\sum_{r=0}^{n}\binom{n}{r}c^{n-r}d^{r+k}P_{n-r,\beta}^{(m)}(dx;k,a,b)\sum_{l=0}^{r}\binom{r}{l}S_l\left(c-1;\left(\frac{\beta}{a}\right)^b\right)P_{r-l,\beta}^{(m-1)}(cy;k,a,b)\right]\frac{t^n}{n!}. \qquad (3)
\end{aligned}$$

In a similar manner,

$$\begin{aligned}
G(t) &= \frac{1}{d^{km}c^{k(m-1)}}\left(\frac{2^{1-k}d^kt^k}{\beta^b e^{dt}-a^b}\right)^m e^{cdxt}\left(\frac{\beta^b e^{cdt}-a^b}{\beta^b e^{dt}-a^b}\right)\left(\frac{2^{1-k}c^kt^k}{\beta^b e^{ct}-a^b}\right)^{m-1} e^{cdyt}\\
&= \frac{1}{c^{km}d^{km}}\sum_{n=0}^{\infty}\left[\sum_{r=0}^{n}\binom{n}{r}d^{n-r}c^{r+k}P_{n-r,\beta}^{(m)}(cx;k,a,b)\sum_{l=0}^{r}\binom{r}{l}S_l\left(d-1;\left(\frac{\beta}{a}\right)^b\right)P_{r-l,\beta}^{(m-1)}(dy;k,a,b)\right]\frac{t^n}{n!}. \qquad (4)
\end{aligned}$$

From (3) and (4), we get the result. □

For $k=a=b=1$ and $\beta=\lambda$, we get Theorem 2.1 of [22].

Corollary 3.2. *For all $c,d,m\in\mathbb{N}$, $n\in\mathbb{N}_0,\lambda\in\mathbb{C}$, we have the following symmetry identity for the generalized Apostol–Bernoulli polynomials* [22]*:*

$$\begin{aligned}
&\sum_{r=0}^{n}\binom{n}{r}c^{n-r}d^{r+1}\mathcal{B}_{n-r}^{(m)}(dx,\lambda)\sum_{l=0}^{r}\binom{r}{l}S_l(c-1;\lambda)\mathcal{B}_{r-l}^{(m-1)}(cy,\lambda)\\
&\quad=\sum_{r=0}^{n}\binom{n}{r}d^{n-r}c^{r+1}\mathcal{B}_{n-r}^{(m)}(cx,\lambda)\sum_{l=0}^{r}\binom{r}{l}S_l(d-1;\lambda)\mathcal{B}_{r-l}^{(m-1)}(dy,\lambda).
\end{aligned}$$

For $k+1=-a=b=1$ and $\beta=\lambda$, we get the following corollary, by taking into account (2).

Corollary 3.3. *For all $m\in\mathbb{N}$, $n\in\mathbb{N}_0,\lambda\in\mathbb{C}$, we have for each pair of positive even integers c and d, or for each pair of positive odd integers c and d, that* [22]*:*

$$\begin{aligned}
&\sum_{r=0}^{n}\binom{n}{r}c^{n-r}d^{r+1}\mathcal{E}_{n-r}^{(m)}(dx,\lambda)\sum_{l=0}^{r}\binom{r}{l}M_l(c-1;\lambda)\mathcal{E}_{r-l}^{(m-1)}(cy,\lambda)\\
&\quad=\sum_{r=0}^{n}\binom{n}{r}d^{n-r}c^{r+1}\mathcal{E}_{n-r}^{(m)}(cx,\lambda)\sum_{l=0}^{r}\binom{r}{l}M_l(d-1;\lambda)\mathcal{E}_{r-l}^{(m-1)}(dy,\lambda).
\end{aligned}$$

For $k=-2a=b=1$ and $2\beta=\lambda$ we get the following corollary, by taking into account (2).

Corollary 3.4. *For all $m\in\mathbb{N}$, $n\in\mathbb{N}_0,\lambda\in\mathbb{C}$, we have for each pair of positive even integers c and d, or for each pair of positive odd integers c and d, that*

$$\begin{aligned}
&\sum_{r=0}^{n}\binom{n}{r}c^{n-r}d^{r+1}\mathcal{G}_{n-r}^{(m)}(dx,\lambda)\sum_{l=0}^{r}\binom{r}{l}M_l(c-1;\lambda)\mathcal{G}_{r-l}^{(m-1)}(cy,\lambda)\\
&\quad=\sum_{r=0}^{n}\binom{n}{r}d^{n-r}c^{r+1}\mathcal{G}_{n-r}^{(m)}(cx,\lambda)\sum_{l=0}^{r}\binom{r}{l}M_l(d-1;\lambda)\mathcal{G}_{r-l}^{(m-1)}(dy,\lambda).
\end{aligned}$$

Theorem 3.5. *For all $c,d,m\in\mathbb{N}$, $n\in\mathbb{N}_0,\beta\in\mathbb{C}$ we have the following identity:*

$$\begin{aligned}
&\sum_{r=0}^{n}\binom{n}{r}\sum_{i=0}^{c-1}\sum_{j=0}^{d-1}\left(\frac{\beta}{a}\right)^{b(i+j)}c^rd^{n-r}P_{r,\beta}^{(m)}\left(dx+\frac{d}{c}i;k,a,b\right)P_{n-r,\beta}^{(m)}\left(cy+\frac{c}{d}j;k,a,b\right)\\
&\quad=\sum_{r=0}^{n}\binom{n}{r}\sum_{i=0}^{d-1}\sum_{j=0}^{c-1}\left(\frac{\beta}{a}\right)^{b(i+j)}d^rc^{n-r}P_{r,\beta}^{(m)}\left(cx+\frac{c}{d}i;k,a,b\right)P_{n-r,\beta}^{(m)}\left(dy+\frac{d}{c}j;k,a,b\right).
\end{aligned}$$

Proof. Let

$$H(t) := \frac{2^{2m(1-k)}t^{2km}e^{cdxt}(\beta^{bc}e^{cdt}-a^{bc})(\beta^{bd}e^{cdt}-a^{bd})e^{cdyt}}{(\beta^b e^{ct}-a^b)^{m+1}(\beta^b e^{dt}-a^b)^{m+1}}.$$

Expanding $H(t)$ into a series, we get

$$\begin{aligned}
H(t) &= \frac{1}{(cd)^{km}}\left(\frac{2^{1-k}c^k t^k}{\beta^b e^{ct}-a^b}\right)^m e^{cdxt}a^{b(c-1)}\left(\frac{\left(\frac{\beta}{a}\right)^{bc}e^{cdt}-1}{\left(\frac{\beta}{a}\right)^{b}e^{dt}-1}\right)\left(\frac{2^{1-k}d^k t^k}{\beta^b e^{dt}-a^b}\right)^m e^{cdyt}a^{b(d-1)}\left(\frac{\left(\frac{\beta}{a}\right)^{bd}e^{cdt}-1}{\left(\frac{\beta}{a}\right)^{b}e^{ct}-1}\right)\\
&= \frac{a^{b(c-1)}a^{b(d-1)}}{(cd)^{km}}\sum_{i=0}^{c-1}\left(\frac{\beta}{a}\right)^{bi}\left(\frac{2^{1-k}c^k t^k}{\beta^b e^{ct}-a^b}\right)^m e^{(dx+\frac{d}{c}i)ct}\sum_{j=0}^{d-1}\left(\frac{\beta}{a}\right)^{bj}\left(\frac{2^{1-k}d^k t^k}{\beta^b e^{dt}-a^b}\right)^m e^{(cy+\frac{c}{d}j)dt}\\
&= \frac{a^{b(c-1)}a^{b(d-1)}}{(cd)^{km}}\left[\sum_{i=0}^{c-1}\left(\frac{\beta}{a}\right)^{bi}\sum_{n=0}^{\infty}P_{n,\beta}^{(m)}\left(dx+\frac{d}{c}i;k,a,b\right)\frac{(ct)^n}{n!}\right]\\
&\quad\times\left[\sum_{j=0}^{d-1}\left(\frac{\beta}{a}\right)^{bj}\sum_{n=0}^{\infty}P_{n,\beta}^{(m)}\left(cy+\frac{c}{d}j;k,a,b\right)\frac{(dt)^n}{n!}\right]\\
&= \frac{a^{b(c-1)}a^{b(d-1)}}{(cd)^{km}}\sum_{n=0}^{\infty}\left[\sum_{r=0}^{n}\binom{n}{r}\sum_{i=0}^{c-1}\sum_{j=0}^{d-1}\left(\frac{\beta}{a}\right)^{b(i+j)}c^r d^{n-r}P_{r,\beta}^{(m)}\left(dx+\frac{d}{c}i;k,a,b\right)P_{n-r,\beta}^{(m)}\left(cy+\frac{c}{d}j;k,a,b\right)\right]\frac{t^n}{n!}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
H(t) &= \frac{1}{(cd)^{km}}\left(\frac{2^{1-k}d^k t^k}{\beta^b e^{dt}-a^b}\right)^m e^{cdxt}a^{b(d-1)}\left(\frac{\left(\frac{\beta}{a}\right)^{bd}e^{cdt}-1}{\left(\frac{\beta}{a}\right)^{b}e^{ct}-1}\right)\left(\frac{2^{1-k}c^k t^k}{\beta^b e^{ct}-a^b}\right)^m e^{cdyt}a^{b(c-1)}\left(\frac{\left(\frac{\beta}{a}\right)^{bc}e^{cdt}-1}{\left(\frac{\beta}{a}\right)^{b}e^{dt}-1}\right)\\
&= \frac{a^{b(c-1)}a^{b(d-1)}}{(cd)^{km}}\sum_{n=0}^{\infty}\left[\sum_{r=0}^{n}\binom{n}{r}\sum_{i=0}^{d-1}\sum_{j=0}^{c-1}\left(\frac{\beta}{a}\right)^{b(i+j)}d^r c^{n-r}\right.\\
&\quad\times\left.P_{r,\beta}^{(m)}\left(cx+\frac{c}{d}i;k,a,b\right)P_{n-r,\beta}^{(m)}\left(dy+\frac{d}{c}j;k,a,b\right)\right]\frac{t^n}{n!}.
\end{aligned}$$

Whence the result. □

For $k=a=b=1$ and $\beta=\lambda$, we get Theorem 2.7 of [22].

Corollary 3.6. *For all $c,d,m\in\mathbb{N},\ n\in\mathbb{N}_0,\lambda\in\mathbb{C}$, we have the following symmetry identity for the generalized Apostol–Bernoulli polynomials* [22]*:*

$$\begin{aligned}
&\sum_{r=0}^{n}\binom{n}{r}\sum_{i=0}^{c-1}\sum_{j=0}^{d-1}\lambda^{i+j}c^r d^{n-r}\mathcal{B}_r^{(m)}\left(dx+\frac{d}{c}i,\lambda\right)\mathcal{B}_{n-r}^{(m)}\left(cy+\frac{c}{d}j,\lambda\right)\\
&\quad=\sum_{r=0}^{n}\binom{n}{r}\sum_{i=0}^{d-1}\sum_{j=0}^{c-1}\lambda^{i+j}d^r c^{n-r}\mathcal{B}_r^{(m)}\left(cx+\frac{c}{d}i,\lambda\right)\mathcal{B}_{n-r}^{(m)}\left(dy+\frac{d}{c}j,\lambda\right).
\end{aligned}$$

For $k+1=-a=b=1$ and $\beta=\lambda$, we get the following corollary.

Corollary 3.7. *For all $c,d,m\in\mathbb{N},\ n\in\mathbb{N}_0,\lambda\in\mathbb{C}$, we have the following symmetry identity for the generalized Apostol–Euler polynomials:*

$$\begin{aligned}
&\sum_{r=0}^{n}\binom{n}{r}\sum_{i=0}^{c-1}\sum_{j=0}^{d-1}(-\lambda)^{i+j}c^r d^{n-r}\mathcal{E}_r^{(m)}\left(dx+\frac{d}{c}i,\lambda\right)\mathcal{E}_{n-r}^{(m)}\left(cy+\frac{c}{d}j,\lambda\right)\\
&\quad=\sum_{r=0}^{n}\binom{n}{r}\sum_{i=0}^{d-1}\sum_{j=0}^{c-1}(-\lambda)^{i+j}d^r c^{n-r}\mathcal{E}_r^{(m)}\left(cx+\frac{c}{d}i,\lambda\right)\mathcal{E}_{n-r}^{(m)}\left(dy+\frac{d}{c}j,\lambda\right).
\end{aligned}$$

For $k=-2a=b=1$ and $2\beta=\lambda$, we get Theorem 2.5 of [27].

Corollary 3.8. *For all $c, d, m \in \mathbb{N}$, $n \in \mathbb{N}_0, \lambda \in \mathbb{C}$, we have the following symmetry identity for the generalized Apostol–Genocchi polynomials* [27]*:*

$$\sum_{r=0}^{n}\binom{n}{r}\sum_{i=0}^{c-1}\sum_{j=0}^{d-1}(-\lambda)^{i+j}c^r d^{n-r}\mathcal{G}_r^{(m)}\left(dx+\frac{d}{c}i,\lambda\right)\mathcal{G}_{n-r}^{(m)}\left(cy+\frac{c}{d}j,\lambda\right)$$
$$=\sum_{r=0}^{n}\binom{n}{r}\sum_{i=0}^{d-1}\sum_{j=0}^{c-1}(-\lambda)^{i+j}d^r c^{n-r}\mathcal{G}_r^{(m)}\left(cx+\frac{c}{d}i,\lambda\right)\mathcal{G}_{n-r}^{(m)}\left(dy+\frac{d}{c}j,\lambda\right).$$

Theorem 3.9. *For all $c, d, m \in \mathbb{N}$, $n \in \mathbb{N}_0$, $\beta \in \mathbb{C}$, we have the following identity:*

$$\sum_{r=0}^{n}\binom{n}{r}\sum_{i=0}^{c-1}\sum_{j=0}^{d-1}\left(\frac{\beta}{a}\right)^{b(i+j)}c^r d^{n-r}P_{r,\beta}^{(m)}\left(dx+\frac{d}{c}i+j;k,a,b\right)P_{n-r,\beta}^{(m)}(cy;k,a,b)$$
$$=\sum_{r=0}^{n}\binom{n}{r}\sum_{i=0}^{d-1}\sum_{j=0}^{c-1}\left(\frac{\beta}{a}\right)^{b(i+j)}d^r c^{n-r}P_{r,\beta}^{(m)}\left(cx+\frac{c}{d}i+j;k,a,b\right)P_{n-r,\beta}^{(m)}(dy;k,a,b).$$

Proof. Let

$$L(t):=\frac{2^{2m(1-k)}t^{2km}\mathrm{e}^{cdxt}(\beta^{bc}\mathrm{e}^{cdt}-a^{bc})(\beta^{bd}\mathrm{e}^{cdt}-a^{bd})\mathrm{e}^{cdyt}}{(\beta^b\mathrm{e}^{ct}-a^b)^{m+1}(\beta^b\mathrm{e}^{dt}-a^b)^{m+1}}.$$

Expanding $H(t)$ into a series, we get

$$\begin{aligned}
L(t) &= \frac{1}{(cd)^{km}}\left(\frac{2^{1-k}c^kt^k}{\beta^b\mathrm{e}^{ct}-a^b}\right)^m\mathrm{e}^{cdxt}a^{b(c-1)}\left(\frac{\left(\frac{\beta}{a}\right)^{bc}\mathrm{e}^{cdt}-1}{\left(\frac{\beta}{a}\right)^{b}\mathrm{e}^{dt}-1}\right)a^{b(d-1)}\left(\frac{\left(\frac{\beta}{a}\right)^{bd}\mathrm{e}^{cdt}-1}{\left(\frac{\beta}{a}\right)^{b}\mathrm{e}^{ct}-1}\right)\left(\frac{2^{1-k}d^kt^k}{\beta^b\mathrm{e}^{dt}-a^b}\right)^m\mathrm{e}^{cdyt}\\
&= \frac{a^{b(c-1)}a^{b(d-1)}}{(cd)^{km}}\sum_{i=0}^{c-1}\sum_{j=0}^{d-1}\left(\frac{\beta}{a}\right)^{b(i+j)}\left(\frac{2^{1-k}c^kt^k}{\beta^b\mathrm{e}^{ct}-a^b}\right)^m\mathrm{e}^{(dx+\frac{d}{c}i+j)ct}\left(\frac{2^{1-k}d^kt^k}{\beta^b\mathrm{e}^{dt}-a^b}\right)^m\mathrm{e}^{cdyt}\\
&= \frac{a^{b(c-1)}a^{b(d-1)}}{(cd)^{km}}\left[\sum_{i=0}^{c-1}\sum_{j=0}^{d-1}\left(\frac{\beta}{a}\right)^{b(i+j)}\sum_{n=0}^{\infty}P_{n,\beta}^{(m)}\left(dx+\frac{d}{c}i+j;k,a,b\right)\frac{(ct)^n}{n!}\right]\left[\sum_{n=0}^{\infty}P_{n,\beta}^{(m)}(cy;k,a,b)\frac{(\mathrm{d}t)^n}{n!}\right]\\
&= \frac{a^{b(c-1)}a^{b(d-1)}}{(cd)^{km}}\sum_{n=0}^{\infty}\left[\sum_{r=0}^{n}\binom{n}{r}\sum_{i=0}^{c-1}\sum_{j=0}^{d-1}\left(\frac{\beta}{a}\right)^{b(i+j)}c^rd^{n-r}P_{r,\beta}^{(m)}\left(dx+\frac{d}{c}i+j;k,a,b\right)P_{n-r,\beta}^{(m)}(cy;k,a,b)\right]\frac{t^n}{n!}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
L(t) &= \frac{1}{(cd)^{km}}\left(\frac{2^{1-k}c^kt^k}{\beta^b\mathrm{e}^{ct}-a^b}\right)^m\mathrm{e}^{cdxt}a^{b(d-1)}\left(\frac{\left(\frac{\beta}{a}\right)^{bd}\mathrm{e}^{cdt}-1}{\left(\frac{\beta}{a}\right)^{b}\mathrm{e}^{ct}-1}\right)a^{b(c-1)}\left(\frac{\left(\frac{\beta}{a}\right)^{bc}\mathrm{e}^{cdt}-1}{\left(\frac{\beta}{a}\right)^{b}\mathrm{e}^{dt}-1}\right)\left(\frac{2^{1-k}d^kt^k}{\beta^b\mathrm{e}^{dt}-a^b}\right)^m\mathrm{e}^{cdyt}\\
&= \frac{a^{b(c-1)}a^{b(d-1)}}{(cd)^{km}}\sum_{n=0}^{\infty}\left[\sum_{r=0}^{n}\binom{n}{r}\sum_{i=0}^{d-1}\sum_{j=0}^{c-1}\left(\frac{\beta}{a}\right)^{b(i+j)}d^rc^{n-r}P_{r,\beta}^{(m)}\left(cx+\frac{c}{d}i+j;k,a,b\right)P_{n-r,\beta}^{(m)}(dy;k,a,b)\right]\frac{t^n}{n!}.
\end{aligned}$$

Whence the result. □

For $k=a=b=1$ and $\beta=\lambda$, we get Theorem 2.10 of [22].

Corollary 3.10. *For all $c, d, m \in \mathbb{N}$, $n \in \mathbb{N}_0, \lambda \in \mathbb{C}$, we have the following symmetry identity for the generalized Apostol–Bernoulli polynomials* [22]*:*

$$\sum_{r=0}^{n}\binom{n}{r}\sum_{i=0}^{c-1}\sum_{j=0}^{d-1}\lambda^{i+j}c^r d^{n-r}\mathcal{B}_r^{(m)}\left(dx+\frac{d}{c}i+j,\lambda\right)\mathcal{B}_{n-r}^{(m)}(cy,\lambda)$$
$$=\sum_{r=0}^{n}\binom{n}{r}\sum_{i=0}^{d-1}\sum_{j=0}^{c-1}\lambda^{i+j}d^r c^{n-r}\mathcal{B}_r^{(m)}\left(cx+\frac{c}{d}i+j;k,a,b\right)\mathcal{B}_{n-r}^{(m)}(dy;k,a,b).$$

For $k+1=-a=b=1$ and $\beta=\lambda$, we get the following corollary.

Corollary 3.11. *For all $c, d, m \in \mathbb{N},\ n \in \mathbb{N}_0, \lambda \in \mathbb{C}$, we have the following symmetry identity for the generalized Apostol–Euler polynomials:*

$$\sum_{r=0}^{n}\binom{n}{r}\sum_{i=0}^{c-1}\sum_{j=0}^{d-1}(-\lambda)^{i+j}c^r d^{n-r}\mathcal{E}_r^{(m)}\left(dx+\frac{d}{c}i+j,\lambda\right)\mathcal{E}_{n-r}^{(m)}(cy,\lambda)$$
$$=\sum_{r=0}^{n}\binom{n}{r}\sum_{i=0}^{d-1}\sum_{j=0}^{c-1}(-\lambda)^{i+j}d^r c^{n-r}\mathcal{E}_r^{(m)}\left(cx+\frac{c}{d}i+j;k,a,b\right)\mathcal{E}_{n-r}^{(m)}(dy;k,a,b).$$

For $k=-2a=b=1$ and $2\beta=\lambda$, we get the following corollary.

Corollary 3.12. *For all $c, d, m \in \mathbb{N},\ n \in \mathbb{N}_0, \lambda \in \mathbb{C}$, we have the following symmetry identity for the generalized Apostol–Genocchi polynomials:*

$$\sum_{r=0}^{n}\binom{n}{r}\sum_{i=0}^{c-1}\sum_{j=0}^{d-1}(-\lambda)^{i+j}c^r d^{n-r}\mathcal{G}_r^{(m)}\left(dx+\frac{d}{c}i+j,\lambda\right)\mathcal{G}_{n-r}^{(m)}(cy,\lambda)$$
$$=\sum_{r=0}^{n}\binom{n}{r}\sum_{i=0}^{d-1}\sum_{j=0}^{c-1}(-\lambda)^{i+j}d^r c^{n-r}\mathcal{G}_r^{(m)}\left(cx+\frac{c}{d}i+j;k,a,b\right)\mathcal{G}_{n-r}^{(m)}(dy;k,a,b).$$

4. Multiplication formula for the unified family

Now let us recall some basic identities which are needed to prove the multiplication formula for the unified family.

Lemma 4.1 (*Multinomial Identity*). *If $x_1, x_2, \ldots, x_r$ are commuting elements of a ring, then for all $n \in \mathbb{N}_0$, we have*

$$(x_1+x_2+\cdots+x_r)^n=\sum_{\substack{n_1,n_2,\ldots,n_r\geq 0\\ n_1+n_2+\cdots+n_r=n}}\binom{n}{n_1,n_2,\ldots,n_r}x_1^{n_1}x_2^{n_2}\cdots x_r^{n_r}.$$

Lemma 4.2 (*Generalized Multinomial Identity*). *If $x_1, x_2, \ldots, x_r$ are commuting elements of a ring, then for all $n \in \mathbb{N}_0$ and $\alpha \in \mathbb{C}$, we have*

$$(1+x_1+x_2+\cdots+x_r)^\alpha=\sum_{n_1,n_2,\ldots,n_r\geq 0}\binom{\alpha}{n_1,n_2,\ldots,n_r}x_1^{n_1}x_2^{n_2}\cdots x_r^{n_r},$$

where

$$\binom{\alpha}{n_1,n_2,\ldots,n_r}=\frac{\alpha(\alpha-1)\cdots(\alpha-n_1-n_2-\cdots-n_r+1)}{n!n_2!\cdots n_r!}.$$

Theorem 4.3. *For $r \in \mathbb{N}, k, n \in \mathbb{N}_0$ and $a, b \in \mathbb{R}; \alpha, \beta \in \mathbb{C}$, we have the following multiplication formula:*

$$P_{n,\beta}^{(\alpha)}(rx;k,a,b)=r^{n-k\alpha}\sum_{n_1,n_2,\ldots,n_{r-1}\geq 0}\binom{\alpha}{n_1,n_2,\ldots,n_{r-1}}\left(\frac{\beta}{a}\right)^{mb}P^{(\alpha)}_{n,\frac{\beta^r}{a^{r-1}}}\left(x+\frac{m}{r};k,a,b\right)$$

where $m=n_1+2n_2+\cdots+(r-1)n_{r-1}$.

Proof. From (1), we have

$$\sum_{n=0}^{\infty}P_{n,\beta}^{(\alpha)}(rx;k,a,b)\frac{t^n}{n!}=\left(\frac{2^{1-k}t^k}{\beta^b e^t-a^{rb}}\right)^\alpha e^{rxt}=\frac{1}{a^{b\alpha}}\left(\frac{2^{1-k}t^k}{\left(\frac{\beta}{a}\right)^b e^t-1}\right)^\alpha e^{rxt}.$$

Since

$$\frac{1}{\left(\frac{\beta}{a}\right)^b e^t-1}=\frac{1+\left(\frac{\beta}{a}\right)^b e^t+\left(\frac{\beta}{a}\right)^{2b}e^{2t}+\cdots+\left(\frac{\beta}{a}\right)^{(r-1)b}e^{(r-1)t}}{\left(\frac{\beta}{a}\right)^{rb}e^{rt}-1}$$

we get

$$\sum_{n=0}^{\infty} P_{n,\beta}^{(\alpha)}(rx;k,a,b)\frac{t^n}{n!} = \frac{1}{r^{k\alpha}a^{b\alpha}}\left(\frac{2^{1-k}(rt)^k}{\left(\frac{\beta}{a}\right)^{rb}e^{rt}-1}\right)^{\alpha}\left(\frac{\left(\frac{\beta}{a}\right)^{rb}e^{rt}-1}{\left(\frac{\beta}{a}\right)^{b}e^{t}-1}\right)^{\alpha}e^{rxt}$$
$$= \frac{1}{r^{k\alpha}a^{b\alpha}}\left(\frac{2^{1-k}(rt)^k}{\left(\frac{\beta}{a}\right)^{rb}e^{rt}-1}\right)^{\alpha}\left(\sum_{j=0}^{r-1}\left(\frac{\beta}{a}\right)^{jb}e^{jt}\right)^{\alpha}e^{rxt}.$$

By generalized multinomial identity, we have

$$\sum_{n=0}^{\infty} P_{n,\beta}^{(\alpha)}(rx;k,a,b)\frac{t^n}{n!} = \frac{1}{r^{k\alpha}a^{b\alpha}}\sum_{n_1,n_2,\ldots,n_{r-1}\geq 0}\binom{\alpha}{n_1,n_2,\ldots,n_{r-1}}\left(\frac{\beta}{a}\right)^{mb}\left(\frac{2^{1-k}\,(rt)^k}{\left(\frac{\beta}{a}\right)^{rb}e^{rt}-1}\right)^{\alpha}e^{(x+\frac{m}{r})rt}$$
$$= \left[r^{n-k\alpha}\sum_{n_1,n_2,\ldots,n_{-1}\geq 0}\binom{\alpha}{n_1,n_2,\ldots,n_{r-1}}\left(\frac{\beta}{a}\right)^{mb}P_{n,\frac{\beta^r}{a^{r-1}}}^{(\alpha)}\left(x+\frac{m}{r};k,a,b\right)\right]\frac{t^n}{n!}.$$

Whence the result. □

For $k = a = b = 1$ and $\beta = \lambda$, we get Theorem 2.1 of [28].

Corollary 4.4. *For $r \in \mathbb{N}$, $n \in \mathbb{N}_0$, $\alpha, \beta \in \mathbb{C}$, we have the following multiplication formula for the generalized Apostol–Bernoulli polynomials* [28]*:*

$$\mathcal{B}_n^{(\alpha)}(rx,\lambda) = r^{n-\alpha}\sum_{n_1,n_2,\ldots,n_{r-1}\geq 0}\binom{\alpha}{n_1,n_2,\ldots,n_{r-1}}\lambda^m\mathcal{B}_n^{(\alpha)}\left(x+\frac{m}{r},\lambda^r\right).$$

Now observe from (1) that, for $k, l \in \mathbb{N}$, $a \in \mathbb{R}^+$; $\beta \in \mathbb{C}$, we have

$$P_{n,-\beta}^{(l)}(x;k-1,-a,1) = \frac{(-2)^l}{(n+1)_l}P_{n+l,\beta}^{(l)}(x;k,a,1) \tag{5}$$

and

$$P_{n,\frac{-\beta}{2}}^{(l)}\left(x;k,\frac{-a}{2},1\right) = (-2)^l P_{n,\beta}^{(l)}(x;k,a,1) \tag{6}$$

where $(\gamma)_\nu$, $(\gamma)_0 \equiv 1$ denotes the Pochhammer symbol defined by

$$(\gamma)_\nu := \frac{\Gamma(\gamma+\nu)}{\Gamma(\gamma)}$$

by means of familiar Gamma functions. By (5) and (6), we have

$$\mathcal{E}_n^{(l)}(x,\lambda) = \frac{(-2)^l}{(n+1)_l}\mathcal{B}_{n+l}^{(l)}(x,\lambda) \tag{7}$$

and

$$\mathcal{G}_n^{(l)}(x,\lambda) = (-2)^l\mathcal{B}_n^{(l)}(x,\lambda), \tag{8}$$

respectively.

For $k+1 = -a = b = 1$ and $\beta = \lambda$, taking into account (7), we get Theorem 3.1 of [28].

Corollary 4.5. *For $r \in \mathbb{N}$, $n, l \in \mathbb{N}_0$, $\alpha, \beta \in \mathbb{C}$, we have the following multiplication formula for the generalized Apostol–Euler polynomials:*

When r is odd, we have

$$\mathcal{E}_n^{(\alpha)}(rx,\lambda) = r^n\sum_{n_1,n_2,\ldots,n_{r-1}\geq 0}\binom{\alpha}{n_1,n_2,\ldots,n_{r-1}}(-\lambda)^m\mathcal{E}_n^{(\alpha)}\left(x+\frac{m}{r},\lambda^r\right)$$

and when r is even, we have

$$\mathcal{E}_n^{(l)}(rx,\lambda)=\frac{(-2)^l r^n}{(n+1)_l}\sum_{\substack{0\le n_1,n_2,\ldots,n_{r-1}\le l\\ n_1+\cdots+n_{r-1}=l}}\binom{l}{n_1,n_2,\ldots,n_{r-1}}(-\lambda)^m\mathcal{B}_{n+l}^{(l)}\left(x+\frac{m}{r},\lambda^r\right).$$

For $k=-2a=b=1$ and $2\beta=\lambda$, taking into account (8), we get the following corollary.

Corollary 4.6. *For $r\in\mathbb{N}, n,l\in\mathbb{N}_0$, $\alpha,\beta\in\mathbb{C}$, we have the following multiplication formula for the generalized Apostol–Genocchi polynomials.*

When r is odd, we have

$$\mathcal{G}_n^{(\alpha)}(rx,\lambda)=r^{n-\alpha}\sum_{n_1,n_2,\ldots,n_{r-1}\ge 0}\binom{\alpha}{n_1,n_2,\ldots,n_{r-1}}(-\beta)^m\mathcal{G}_n^{(\alpha)}\left(x+\frac{m}{r},\beta^r\right)$$

and when r is even, we have

$$\mathcal{G}_n^{(l)}(rx,\lambda)=(-2)^l r^{n-l}\sum_{\substack{0\le n_1,n_2,\ldots,n_{r-1}\le l\\ n_1+\cdots+n_{r-1}=l}}\binom{l}{n_1,n_2,\ldots,n_{r-1}}(-\lambda)^m\mathcal{B}_n^{(l)}\left(x+\frac{m}{r},\lambda^r\right).$$

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