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Equilibrium semantics of languages of imperfect information

Merlijn Sevenster [a,*], Gabriel Sandu [b]

[a] Philips Research, Prof. Holstlaan 4, 5656AA, Eindhoven, The Netherlands

[b] Department of Philosophy, University of Helsinki, Unioninkatu 40 A, PB 24 00014, Helsinki, Finland

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ABSTRACT

In this paper, we introduce a new approach to independent quantifiers, as originally introduced in *Informational independence as a semantic phenomenon* by Hintikka and Sandu (1989) [9] under the header of independence-friendly (IF) languages. Unlike other approaches, which rely heavily on compositional methods, we shall analyze independent quantifiers *via* equilibriums in strategic games. In this approach, coined *equilibrium semantics*, the value of an IF sentence on a particular structure is determined by the expected utility of the existential player in any of the game's equilibriums. This approach was suggested in *Henkin quantifiers and complete problems* by Blass and Gurevich (1986) [2] but has not been taken up before. We prove that each rational number can be realized by an IF sentence. We also give a lower and upper bound on the expressive power of IF logic under equilibrium semantics.

1. Introduction

Independence-friendly logic (IF logic), the logic with the slashed quantifiers $(\exists x/W)$ and $(\forall x/W)$ was primarily [9] given a game-theoretic semantics in terms of games of imperfect information. Let us introduce it by way of an example. Consider the following simple game played by two players, Abelard (the universal player) and Eloise (the existential player): Abelard hides 1 euro in his left or right hand, without Eloise's seeing it. Eloise has to guess where it is. If she does, she wins and Abelard loses; otherwise Abelard wins and Eloise loses.

We model this game by fixing a structure $\mathfrak{M}$ with universe $M = \{l, r\}$ and a sentence of IF logic φ given by $\forall x(\exists y/\{x\})x = y$. φ is interpreted by a semantic game $G(M, \varphi)$, which has two moves: Abelard chooses an individual $a \in M$, after which Eloise chooses $b \in M$, without "seeing" the individual chosen earlier by her opponent. The game stops: if $a = b$, Eloise wins and Abelard loses; otherwise Abelard wins and Eloise loses. Notice how the syntax of the sentence indicates the patterns of knowledge of the players in the semantic game.

What matters in a game is not winning a particular play but having a systematic method, a strategy, which gets a player a win no matter what his or her opponent does. More exactly, a strategy for a player is represented by a set of functions, one for each of his or her moves, defined on all the possible earlier sequences of "known" or "seen" elements at that move. In our particular game, a strategy for Abelard is any individual in M and so is a strategy for Eloise. Were Eloise to know the earlier move of her opponent, the sentence of the game would have been the ordinary first-order sentence $\forall x\exists y(x = y)$ and the game itself would have been one of perfect information: in that case, a strategy for Abelard would be any individual in M, while a strategy for Eloise would be any function $f : M \to M$.

* Corresponding author.
E-mail address: merlijn.sevenster@philips.com (M. Sevenster).

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It is well known that imperfect information introduces indeterminacy in the game. In fact, the game in our example is the simplest indeterminate game: Neither Abelard nor Eloise has a winning strategy in it. Abelard has one if and only if $\exists a\forall b(a \neq b)$, which is impossible; and Eloise has one if and only if $\exists b\forall a(a = b)$, which is also impossible.

The semantic game described here is a game in extensive form. It provides a game-theoretic semantics for IF languages. In [10], Hodges replaces game-theoretic semantics with a compositional interpretation ("trump semantics") which is a generalization of the Tarski-type semantics. His work has stimulated a host of similar interpretations, all of which sacrifice the game-theoretic interpretation for compositional methods. Recent work by Väänänen [18], also inspired by Hodges' trump semantics, abandons slash quantifiers altogether in favor of basic dependence relations between variables (cf. below).

For us, the replacement of the original game-theoretic interpretation with compositional methods is a symptom of the fact that the former was not sufficiently exploited. In the present paper, we shall stick to the game-theoretic paradigm and push it much further than was done before, by studying games for IF logic in strategic form. This approach was suggested in [2], but has not been taken up since. The central concept of our approach is that of strategic equilibria (instead of winning strategies). One of its interesting aspects is the emergence of a multi-valued semantics with truth values in the interval [0,1].

In Section 2, we introduce the syntax of IF logic, together with its Skolem semantics, which is supposed to be the counterpart of the game-theoretic semantics sketched above. Section 3 gives a brief account of Väänänen's dependence logic together with few observations concerning the comparison between the two frameworks. Section 4 introduces IF games in strategic form, together with the appropriate notion of equilibrium that will be used to define the truth value of an IF sentence on a given structure. An easy example shows that there is an IF sentence that has the value $1/n$, where n is the size of the structure on which the given sentence is evaluated. The resulting concept of truth induces a new semantic relation $\models_\varepsilon$ coined "equilibrium semantics". In Section 5, we give a more elaborate example which shows how to establish the truth value of an IF sentence under the equilibrium semantics by the technique of eliminating weakly dominated strategies. In Section 6, we show that, for every rational $q \in [0, 1]$, there is an IF sentence that has truth value q regardless of the structure. We use this result to derive some interesting properties of the equilibrium semantics. Section 7 gives a lower and upper bound on the expressive power of IF logic under equilibrium semantics by drawing on insights from descriptive complexity theory and combinatorial game theory. Section 8 concludes the paper.

2. Syntax and semantics: IF logic

In this section, we give a short introduction to the syntax and semantics of IF logic. It has been designed to represent patterns of dependent and independent quantifiers that extend far beyond those possible in the Frege–Tarski linear tradition.

We fix a vocabulary $L = \{c_1, c_2, \ldots, R_1, R_2, \ldots\}$, that is, a collection of individual constants c_i and relation symbols R_i. An L-term is a first-order variable (usually denoted by x_i, y_i, z_i, etc.) or a constant c_i. The main novelty in the syntax is the presence of slashed quantifiers (Qx/W). The first definition introduces the notion of "pseudo-formula", that is, an IF formula in which the slashed quantifiers are quite arbitrary.

Definition 1. The *pseudo-formulas of IF logic in the vocabulary L* (L-pseudo-formulas) are generated by the following rules:

$$\varphi \ ::= \ R(t_1, \ldots, t_n) \mid \neg R(t_1, \ldots, t_n) \mid \varphi \vee \varphi \mid \varphi \wedge \varphi \mid (\exists x/W)\varphi \mid (\forall x/W)\varphi,$$

where $R \in L$, $t_1, \ldots, t_n$ are L-terms, and W is a finite set of variables.

W is called a "slash-set". The intended interpretation of $(\exists x/W)$ is: there exists an x independent of the quantifiers which binds the variables in W. The intended meaning of $(\forall x/W)$ is similar.

The first-order notions of subformula and scope (i.e. a quantifier being superordinate to a subformula) are readily generalized to pseudo-formulas of IF logic. Let x be a variable in the pseudo-formula φ that occurs either in $\psi = R(t_1, \ldots, t_n)$ or in the set W (where $\psi = (Qy/W)\theta$). Let (Qz/U) be a quantifier in φ. We say that x is *bound* by (Qz/U) if (Qz/U) is superordinate to ψ in φ and $z = x$. We say that x is bound if x is bound by one or more quantifiers. A *free variable* in φ is a variable that is not bound in φ.

Definition 2. Let φ be an L-pseudo-formula of IF logic. Then φ is a *formula of IF logic in the vocabulary L* (*L-IF formula* or simply *IF formula*) if it meets the following conditions:

- each variable x that appears in $R(t_1, \ldots, t_n)$ in φ is bound by at most one quantifier;
- each variable x that appears in a slash-set W in φ is bound by one quantifier;
- each quantifier (Qx/W) in φ is such that $x \notin W$.

Every pseudo-formula is an IF formula, but the converse is not true. An IF *sentence* is an IF formula with no free variables. Thus $(\exists y/\{x\})R(x, y, z)$ is a pseudo-formula; $\forall x(\exists y/\{x\})R(x, y, z)$ is an IF formula; and $\forall x(\exists y/\{x\})\forall zR(x, y, z)$ is an IF sentence.

Truth is defined only for IF formulas φ and is given via skolemization, that is, a translation procedure that associates with each subformula ψ of φ a formula $Sk(\psi)$ in the language L^*: the extension of L with new function symbols.

Definition 3. Let φ be an L-formula of IF logic. The *skolemized form* or *skolemization* of φ is given by the following clauses:

$$Sk(\psi) = \psi, \text{ for } \psi \text{ an atomic subformula or its negation}$$
$$Sk(\psi \circ \theta) = Sk(\psi) \circ Sk(\theta), \text{ for } \circ \in \{\vee, \wedge\}$$
$$Sk((\forall x/W)\psi) = \forall x Sk(\psi)$$
$$Sk((\exists x/W)\psi) = Subst(Sk(\psi), x, f_x(z_{k+1}, \ldots, z_l)),$$

where $W = \{z_1, \ldots, z_k\}$; $(Q_1 z_1/W_1), \ldots, (Q_k z_k/W_k), (Q_{k+1} z_{k+1}/W_{k+1}), \ldots, (Q_l z_l/W_l)$ are all the quantifiers in the scope of which $(\exists x/\{z_1, \ldots, z_k\})$ occurs, and f_x is a new function symbol not contained in L. *Subst* denotes the usual substitution operation.

A note on syntax: One of the reviewers remarks that the slash-sets in universal quantifiers are superfluous, since they do not affect the truth conditions of a formula (cf. the skolemization clause for formulas $(\forall x/W)\psi$). He or she suggests that we exclude them from the syntax of IF logic. Here we remark that, even though removing the universal slash-sets from an IF formula does not affect its truth value (true or untrue) under Skolem semantics, it may affect its truth value (a value in $[0, 1]$) in the strategic framework developed below.

The functions $f_x, \ldots$ in L^* are called *Skolem functions*.

To take an example, consider the IF sentence φ:

$$\forall x \forall y (\exists u/\{y\})(\exists v/\{x\})\psi(x, y, u, v),$$

where $\psi(x, y, u, v)$ is a quantifier-free first-order formula. Its skolemized form is obtained through the following steps:

$$Sk((\exists v/x)\psi(x, y, u, v)) = \psi(x, y, u, f_v(y, u))$$
$$Sk((\exists u/\{y\})(\exists v/\{x\})\psi(x, y, u, v)) = \psi(x, y, f_u(x), f_v(y, f_u(x)))$$
$$Sk(\varphi) = \forall x \forall y \psi(x, y, f_u(x), f_v(y, f_u(x))).$$

The vocabulary $L = \{c_1, c_2, \ldots, R_1, R_2, \ldots\}$ receives an interpretation through an L-structure $\mathfrak{M} = (M, c_1^{\mathfrak{M}}, c_2^{\mathfrak{M}}, \ldots, R_1^{\mathfrak{M}}, R_2^{\mathfrak{M}}, \ldots)$ in the usual way. We are now ready for the truth definition.

Definition 4. Let φ be an L-sentence of IF logic and $\mathfrak{M}$ an L-structure. Then we stipulate that φ is true in $\mathfrak{M}$ if and only if there exist functions $f_{x_1}^{\mathfrak{M}}, \ldots, f_{x_n}^{\mathfrak{M}}$ such that $\mathfrak{M}, f_{x_1}^{\mathfrak{M}}, \ldots, f_{x_n}^{\mathfrak{M}} \models Sk(\varphi)$, where $f_{x_1}, \ldots, f_{x_n}$ are the function symbols in $Sk(\varphi)$. If φ is true in $\mathfrak{M}$, we write $\mathfrak{M} \models^+ \varphi$.

The more general notion of satisfaction of IF formulas is defined relative to an assignment in the usual way, but that notion is not of interest to this paper.

Obviously for every IF sentence φ and structure $\mathfrak{M}$ we have:

$$\mathfrak{M} \models^+ \varphi \quad \text{iff} \quad \mathfrak{M} \models \exists f_{x_1} \ldots \exists f_{x_n} Sk(\varphi),$$

where $f_{x_1}, \ldots, f_{x_n}$ are the new function symbols in $Sk(\varphi)$.

Thus every IF sentence is equivalent to an existential second-order (Σ_1^1) sentence. The converse follows from the theory of Henkin quantifiers [20]: every Σ_1^1 sentence is equivalent to an IF sentence. Thereby, IF logic inherits automatically the model-theoretic properties of the Σ_1^1 logic: the compactness theorem and the Löwenheim–Skolem theorem. IF logic also defines its own truth-predicate, as shown by the second author in the Appendix of [8].

In order to deal with falsity, we can define another translation procedure, *Kr* (from *Kreisel counter-examples*), on IF formulas that first replaces all occurrences of $R(t_1, \ldots, t_n)$ by $\neg R(t_1, \ldots, t_n)$; all disjunctions with conjunctions and vice versa; all existential quantifier symbols $\exists$ with universal quantifier symbols $\forall$ and vice versa; and, then applies $Sk(\cdot)$ to the result. The reader may check that $Kr(\forall x(\exists y/\{x\})x = y) = Sk(\exists x(\forall y/\{x\})x \neq y) = \forall y(f_x \neq y)$, where f_x is a 0-ary function symbol.

By analogy with truth, we stipulate that an IF sentence φ is false in a structure $\mathfrak{M}$ if and only if there exist functions $g_{y_1}^{\mathfrak{M}}, \ldots, g_{y_m}^{\mathfrak{M}}$ such that $\mathfrak{M}, g_{y_1}^{\mathfrak{M}}, \ldots, g_{y_m}^{\mathfrak{M}} \models Kr(\varphi)$, where $g_{y_1}^{\mathfrak{M}}, \ldots, g_{y_m}^{\mathfrak{M}}$ are the function symbols in $Kr(\varphi)$. If φ is false in $\mathfrak{M}$ we shall write $M \models^- \varphi$. Thus,

$$\mathfrak{M} \models^- \varphi \quad \text{iff} \quad \mathfrak{M} \models \exists g_{y_1} \ldots \exists g_{y_n} Kr(\varphi).$$

Let us return to our earlier $\varphi = \forall x(\exists y/\{x\})x = y$. We have

$$\mathfrak{M} \models^+ \varphi \quad \text{iff} \quad \mathfrak{M} \models \exists y \forall x(x = y)$$
$$\mathfrak{M} \models^- \varphi \quad \text{iff} \quad \mathfrak{M} \models \exists x \forall y(x \neq y)$$

and thereby neither $\mathfrak{M} \models^+ \varphi$ nor $\mathfrak{M} \models^- \varphi$ in every structure $\mathfrak{M}$ which has more than one element.

Since both the truth and the falsity of an IF first-order sentence are definable by Σ_1^1 sentences, IF logic also inherits from Σ_1^1 logic a Separation Theorem: If K_1 and K_2 are two disjoint classes of structures, each definable by an IF sentence, then there is an an elementary class (i.e. definable by a first-order sentence) which separates them. Burgess [3] shows how in certain conditions, the separating first-order sentence can be effectively found.

We can easily generalize the game-theoretic semantics (games in extensive form) from the first section to provide an alternative semantic interpretation for IF sentences, as specified in the present section: (slashed) existential quantifiers and disjunctions prompt moves by Eloise; (slashed) universal quantifiers and conjunctions prompt moves by Abelard. Truth (falsity) in a given structure is defined as the existence of a winning strategy for Eloise (Abelard). The reader should convince herself that the existence of the appropriate Skolem functions in Skolem semantics yields a winning strategy for Eloise in the corresponding semantic game of imperfect information, and vice versa. And similarly for the Kreisel counter-examples: they exist if and only if Abelard has a winning strategy in the corresponding semantical game. In other words, game-theoretic semantics and the Skolem semantics given in terms of Skolem functions and Kreisel counter-examples coincide.

3. Alternative frameworks

Väänänen [18], Väänänen and Hodges [19] and Abramsky and Väänänen [1] replace the dependence and independence of quantifiers with dependence of terms. In this notation, the sentence $\exists w\forall x\exists y(\exists z/\{x\})(x = z \wedge w \neq y)$ is rendered as $\exists w\forall x\exists y\exists z(=(w, y, z) \wedge x = z \wedge w \neq y)$. The new element is the atomic formula "$=(w, y, z)$" called a *dependence formula*. More generally, the new syntax contains an infinite number of atomic formulas of the form

$$=(x_1, \ldots, x_{n-1}, x_n),$$

which, for $n \geq 2$, have the intended interpretation

x_n is functionally determined by $x_1, \ldots, x_{n-1}$.

The resulting logic (i.e. first-order logic plus the dependence formulas) is called *dependence logic* (D logic). The semantics follows the compositional semantics in [10]. It generalizes Tarski's semantics by interpreting D formulas on sets of assignments (trump semantics). We shall not enter into details here. Suffice it to give the semantic clause for the dependence formulas. The set X of assignments satisfies the dependence formula $=(x_1, \ldots, x_{n-1}, x_n)$ if and only if, for every two assignments $s, s' \in X$,

if s and s' agree on $x_1, \ldots, x_{n-1}$, then $s(x_n) = s'(x_n)$.

For sentences, IF logic and D logic are intertranslatable into each other. Perhaps the simplest way to see this is the following. D logic has a prenex normal form theorem. It is shown in [18] that every D sentence is equivalent (up to truth) to a sentence of the form

$$\forall x_1 \ldots \forall x_n \exists y_1 \ldots \exists y_m \theta,$$

where θ is a conjunction of dependence formulas and of a quantifier-free first-order formula. Analogously, every IF sentence is logically equivalent (up to truth) to a sentence of the form

$$\forall x_1 \ldots \forall x_k (\exists x_{k+1}/W_{k+1}) \ldots (\exists x_{k+n}/W_{k+n})\psi$$

where ψ is a quantifier-free first-order formula (see, e.g. [4]).

But now it is straightforward to see how the translation operates back and forth between IF and D sentences: Erase the sets $W_{k+1}, \ldots, W_{k+n}$ and add the appropriate dependence formulas to ψ. In the other direction, delete the dependence formulas and express the dependences as slashed variables of the appropriate existential quantifiers.

The equivalence of expressive power notwithstanding, the two logical systems are designed to serve different purposes. Dependence logic fits much better with the ideology of compositional semantics and recursive definitions. It is meant to capture Armstrong's notion of functional dependence in databases. IF logic and its game-theoretic interpretation, on the other side, reflect much better the phenomena of signaling, which typically occur in games of imperfect information.

The classics of game theory observe that signaling is any convention of play whereby one partner informs the others of his holdings or desires. Von Neumann and Morgenstern observe that it occurs in Whist and Bridge. Hodges remarks: "In these games there are two players and each of the players consists of two partners. . . . In general a pair of partners knows different things about the state of the play but one of them can use his public moves to signal information about the other partner." ([10], p. 549). In [10] Hodges extensively connects IF logic with signaling in game theory. One example will suffice: compare our earlier IF sentence $\forall x(\exists y/\{x\})x = y$ with the IF sentence ψ that is given by $\forall x\exists z(\exists y/\{x\})x = y$. Unlike the former, which is neither true nor false in all structures with at least two elements, the latter is a logical truth. In this case, we can think of the existential player as consisting of two partners: a Sender and a Receiver. The Sender sees the choices of the universal player and "signals" them to his partner; the Receiver does not see the choices of the universal player but "copies" the moves of her partner. In terms of Skolem functions, the truth of ψ is witnessed by the choice functions $f_{\text{sender}} = f_{\text{receiver}} = \text{Id}^{\mathfrak{M}}$ where $\text{Id}^{\mathfrak{M}}$ is the identity function on the structure $\mathfrak{M}$ at hand. This sort of "signalling games" are well known in the philosophical literature and have been used by Lewis to give an account of conventional meaning.

In this paper, we pursue the game-theoretic interpretation of IF languages, but we shall push it in a completely different direction than that existing in the literature. The strategies of Eloise and Abelard in semantic (extensive) games of imperfect information are only the starting point. The notions of truth and falsity they define in the relevant semantic games, will turn out to be limit cases of semantic values which arise out of solution concepts in strategic game theory (Nash equilibria).

4. Equilibrium semantics

In this section, we formalize the framework that is suggested in [2] to handle indeterminateness of formulas with Henkin quantifiers. The semantics of a particular class of generalized quantifiers was studied in terms of strategic games in [17]. We learned, after submission, that Galliani [7] has proposed a strategic approach to IF logic (see also our discussion in Section 8), which is close but quite different from the system alluded to in the following quote and the system developed in this paper.

> "... a formula might neither hold nor fail; that is, there might be no winning strategy for either player.
> The simplest example of this phenomenon is given by
>
> $$\begin{pmatrix} \forall x \\ \exists y \end{pmatrix} x = y$$
>
> in any structure with at least two elements. Although unpleasant, this lack of determinacy should not be viewed as pathological; it is the usual situation for games of imperfect information.
> Miklos Ajtai has suggested applying the von Neumann minimax theorem to these games. ... [In this approach, f]ormulas which neither hold nor fail have intermediate truth values; the example
>
> $$\begin{pmatrix} \forall x \\ \exists y \end{pmatrix} x = y$$
>
> has truth value $1/n$ in structures of cardinality n." [2]

We start out with some preliminary definitions from game theory.

Definition 5. Let $N = \{0, 1\}$ be a set of two players. For $i \in N$, let S_i be the set of *strategies* of player i. For $i \in N$, let $u_i : S_0 \times S_1 \to \mathbb{R}$ be a *utility function* for player i. Then, $\Gamma = ((S_i)_{i\in N}, (u_i)_{i\in N})$ is called a *strategic game*.
The size of Γ is given by $n_0 \cdot n_1$, where $n_i = |S_i|$. Γ is finite, if it has finite size.

We shall be interested in the strategic games that are defined by IF sentences.

Definition 6. Let $\mathfrak{M}$ be a structure and let φ be an IF sentence. Let $G(\mathfrak{M}, \varphi)$ be the extensive game determined by $\mathfrak{M}$ and φ. Then, $\Gamma(\mathfrak{M}, \varphi) = ((S_i)_{i\in N}, (u_i)_{i\in N})$ is the *strategic IF game* of $\mathfrak{M}$ and φ, where

- $N = \{\exists, \forall\}$ is the set of players;
- S_i is the set of strategies of player i in $G(\mathfrak{M}, \varphi)$;
- u_i is the utility function of player i, s.t. $u_i(s, t) = 1$ if playing s against t in $G(\mathfrak{M}, \varphi)$ yields a win for player i and $u_i(s, t) = 0$ otherwise.

Strategic IF games have the following two properties:

- *Win–loss*—Every game has only two outcomes, namely 0 and 1.
- *Constant-sum*—There is a constant c, s.t. for every $s \in S_\exists$ and $t \in S_\forall$, $u_\exists(s, t) + u_\forall(s, t) = c$. For strategic IF games, c equals 1.[1]

Since strategic IF games Γ are constant-sum, the utility of Abelard is determined by Eloise's utility. For succinctness we shall therefore write Γ as $((S_i)_{i\in N}, u)$, where u refers to Eloise's utility function $u_\exists$. Sometimes we may write (S, T, u) instead of $(S_\exists, S_\forall, u)$.

The value of a strategic IF game $((S_i)_{i\in N}, u)$ will be defined in terms of its equilibrium pairs of mixed strategies. A *mixed strategy* σ_i for player i is a probability distribution over S_i. That is, σ_i is a function $S_i \to [0, 1]$ such that $\sum_{s\in S_i} \sigma_i(s) = 1$. A mixed strategy $\sigma_i : S_i \to \{0, 1\}$ is *uniform* over $S_i' \subseteq S_i$ if it assigns equal probability to all strategies in S_i' and zero probability to the strategies in $S_i - S_i'$. Given a mixed strategy σ for Eloise and a mixed strategy τ for Abelard, the *expected utility* for player i is given by

$$U_i(\sigma, \tau) = \sum_{s\in S_\exists} \sum_{t\in S_\forall} \sigma(s)\tau(t)u_i(s, t).$$

In line with our earlier convention, we write U for $U_\exists$.

Definition 7. Let $N = \{0, 1\}$ and $\Gamma = ((S_i)_{i\in N}, u)$ be a constant-sum, strategic game. Let (σ_0, σ_1) be a pair of mixed strategies in Γ. Then, (σ_0, σ_1) is an *equilibrium* in Γ, if

- for every mixed strategy σ_0', $U_0(\sigma_0, \sigma_1) \geq U_0(\sigma_0', \sigma_1)$; and
- for every mixed strategy σ_1', $U_1(\sigma_0, \sigma_1) \geq U_1(\sigma_0, \sigma_1')$.

[1] One of the reviewers suggests to consider utility functions u_i with range -1 and 1. If we did so, IF games would be "zero-sum" games. We prefer the present formulation, since falsehood of a formula is typically associated with the value 0.

The notion of equilibrium defines a relation V between strategic IF games Γ and values v in the interval $[0, 1]$: $(\Gamma, v) \in V$ iff there is an equilibrium (σ_0, σ_1) in Γ s.t. $U(\sigma_0, \sigma_1) = v$. To show that V is in fact functional for finite Γ, first observe that every finite, constant-sum, two-player game has an equilibrium in mixed strategies. This is von Neumann's well-known Minimax Theorem (cf. Theorem 28). Moreover, it is well known [14, p. 22] that every two equilibria in constant-sum, two-player games have the same expected utility. It follows that V is functional. This leads us to the next definition.

Definition 8. Let φ be an IF sentence in the vocabulary L and let $\mathfrak{M}$ be a finite L-structure. Then, $V(\Gamma)$ is the *truth value* of φ on $\mathfrak{M}$, where $\Gamma = \Gamma(\mathfrak{M}, \varphi)$.

It is important to notice that the framework set up in this paper concerns only *finite* structures. It is not trivial to generalize the framework to infinite structures, since infinite games may not have an equilibrium.

Consider (the IF version of) Ajtai's sentence $\varphi = \forall x(\exists y/x)x = y$ and some structure $\mathfrak{M}$ with, say, five objects $a_1, \ldots, a_5$. Both Abelard and Eloise have five strategies in $\Gamma(\mathfrak{M}, \varphi) = (S, T, u)$: $S = \{s_a : a \in M\}$ and $T = \{t_a : a \in M\}$. The utility function u is conveniently depicted by the following matrix:

	t_{a_1}	t_{a_2}	t_{a_3}	t_{a_4}	t_{a_5}
s_{a_1}	1	0	0	0	0
s_{a_2}	0	1	0	0	0
s_{a_3}	0	0	1	0	0
s_{a_4}	0	0	0	1	0
s_{a_5}	0	0	0	0	1

Let σ (resp. τ) be the uniform strategy over S (resp. T). It is easy to show that (σ, τ) is an equilibrium with value 1/5.

So, interestingly, as the size of $\mathfrak{M}$ increases, the truth value of φ on $\mathfrak{M}$ asymptotically approaches falsehood.

Definition 9. Let $0 \leq \varepsilon \leq 1$. Let φ be an IF sentence in the vocabulary L and let $\mathfrak{M}$ be a finite L-structure. We define the satisfaction relation $\models_\varepsilon$ by $\mathfrak{M} \models_\varepsilon \varphi$ iff $V(\Gamma) \geq \varepsilon$, where $\Gamma = \Gamma(\mathfrak{M}, \varphi)$. We call the semantics defined by $\models_\varepsilon$ the "equilibrium semantics" for IF logic.

The reader may have noted the asymmetry in the definition of $\models_\varepsilon$ between truth (greater than *or equal to* ε) and falsity (smaller than ε). We shall see in Corollary 20 that this asymmetry evaporates in the face of nonrational ε. A convenient property of the "inclusive" formulation of equilibrium semantics is that it is a "conservative extension" of Skolem semantics and GTS.

Proposition 10. *Let $\mathfrak{M}$ be a finite structure and let φ be an IF sentence. Then, for $\varepsilon = 1$, $\mathfrak{M} \models_\varepsilon \varphi$ iff $\mathfrak{M} \models^+ \varphi$.*

To prove Proposition 10 one can exploit the fact that for $\varepsilon = 1$, finding an equilibrium coincides with finding a winning strategy.

5. Evaluating a sentence by eliminating strategies

Several notions and techniques have been laid down to analyze strategic games. In this section, we will compute the truth value of a particular IF sentence by removing "weakly dominated strategies" from the game. This procedure is well known from the game-theoretic literature.

We will consider an IF sentence φ_{even} on finite circular graphs. A directed graph $\mathfrak{C} = (C, R^{\mathfrak{C}})$ is *circular* if there is an ordering $a_0, \ldots, a_{n-1}$ of the elements in C such that $R^{\mathfrak{C}}$ is given by $\{(a_0, a_1), \ldots, (a_{n-2}, a_{n-1}), (a_{n-1}, a_0)\}$. In circular graphs, the edge relation is functional. We shall thus sometimes write $R^{\mathfrak{C}}(a)$ for the R-successor of the object a. Assume that $\mathfrak{C}$ contains at least three elements.

A *coloring* of a graph $\mathfrak{C}$ is a function f that sends the objects in C to one of two *color objects* $c_0, c_1 \in C$. The color objects c_0 and c_1 reside within the graph and f also sends the color objects c_i to color object $f(c_i)$ that need not be c_i itself.

Clearly, a circular graph $\mathfrak{C}$ has even cardinality iff $\mathfrak{C}$ has a coloring f such that for each $a \in C, f(a) \neq f(R^{\mathfrak{C}}(a))$. It is evident that this property is expressed by the Σ^1_1 sentence $\exists f \forall x \forall y(R(x, y) \to f(x) \neq f(y))$. To mimick the second-order quantifier $\exists f$ in IF logic we need two existential quantifiers. This is accomplished by the IF sentence φ_{even} that is given by[2]

$$\forall x(\forall y/\{x\})(\exists z_0/\{x, y\})(\exists z_1/\{x, y, z_1\})(\exists u/\{y, z_0, z_1\})(\exists v/\{x, z_0, z_1\})\psi,$$

where ψ consists of three conjuncts:

$$(u = z_0 \vee u = z_1) \wedge (x = y \to u = v) \wedge (R(x, y) \to u \neq v).$$

2 This sentence is truth equivalent to

$$\varphi'_{\text{even}} = \exists z_0 \exists z_1 \forall x \forall y(\exists u/\{y\})(\exists v/\{x\})\psi$$

and we are confident that the truth value of this sentence is equal to the truth value of φ_{even}. However, for any given structure with size two or greater, each player has many more strategies in the corresponding strategic game of φ'_{even} than in that of φ_{even}. It will make the analysis more transparent if we consider φ_{even}.

The skolemization of φ_{even} reads[3] $\exists f \exists g \exists z_0 \exists z_1 \forall x \forall y Sk(\psi)$, where $Sk(\psi)$ is

$$(f(x) = z_0 \vee f(x) = z_1) \wedge (x = y \to f(x) = g(y)) \wedge (R(x, y) \to f(x) \neq g(y)). \tag{1}$$

The conjunct $(x = y \to f(x) = g(y))$ enforces that f and g be given the same interpretation. We leave it as an exercise to the reader to check that φ_{even} defines evenness on the class of circular graphs.

Write $\Gamma = (S, T, u)$ for the strategic game of φ_{even} on the circular graph $\mathfrak{C}$ with odd size n. In this section we will produce a game-theoretic argument showing that the value of Γ is $(1 - \frac{1}{2n})$. The argument builds on the insight that we can simplify the game at hand by iteratively removing "weakly dominated" and "payoff equivalent" strategies.

Definition 11. Let $\Gamma = (S_0, S_1, u_0, u_1)$ be a strategic game. We say that the strategy $s_0 \in S_0$ is *weakly dominated* by the strategy $s_0' \in S_0$, if

- $u_0(s_0, s_1) \leq u_0(s_0', s_1)$, for every $s_1 \in S_1$;
- $u_0(s_0, s_1) < u_0(s_0', s_1)$, for some $s_1 \in S_1$;

and likewise for strategies $s_1' \in S_1$ of player 1. The strategy $s_i \in S_i$ is *weakly dominated* in Γ if there is a strategy $s_i' \in S_i$ such that s_i is weakly dominated by s_i'.

Definition 12. Let $\Gamma = (S_0, S_1, u)$ be a constant-sum game and let $s_0, s_0' \in S_0$. We say that s_0 and s_0' are *payoff equivalent* in Γ if $u(s_0, s_1) = u(s_0', s_1)$, for each $s_1 \in S_1$; likewise for the strategies of player 1.

Proposition 13. *Let $\Gamma = (S_0, S_1, u)$ be a finite, constant-sum game. Then, Γ has a mixed strategy equilibrium (σ_0, σ_1) such that for each player i and each pure strategy $s \in S_i$ of player i, if $\sigma_i(s) > 0$ then*

- *s is not weakly dominated in Γ and*
- *s does not have a payoff equivalent strategy in Γ.*

Proof. For the first item see [13] (this proof requires finiteness of the game). The second item is easy. □

Proposition 13 grants us that we can remove the discussed strategies from the game without affecting the game's value. For instance, let $s \in S_0$ be a strategy that is weakly dominated or that has a payoff equivalent in the finite constant-sum game (S_0, S_1, u). Then, by Proposition 13 there is an equilibrium (σ_0, σ_1) for which σ_0 assigns zero probability to s. But then, $(S_0 - \{s\}, S_1, u)$ has (σ_0, σ_1) as an equilibrium with value equal to the value of the initial game. Note that, formally speaking, we should consider, as utility function in $(S_0 - \{s\}, S_1, u)$, the restriction of u to the remaining strategy pairs in this game, but we shall be sloppy in this respect and simply write u.

The strategic IF game $\Gamma = \Gamma(\mathfrak{C}, \varphi_{\text{even}})$ is quite a large beast: Abelard has quadratically many strategies in n and Eloise has an exponential number of strategies in n, where n is the size of $\mathfrak{C}$. The remainder of this section is dedicated to removing weakly dominated and payoff equivalent strategies from Γ until we hit a subgame in which each player has $2n$ strategies.

We define two subsets S_1 and S_2 in S. Fact 16 below shows that every strategy in S that does not sit in $S_1 \cup S_2$ is weakly dominated by a strategy in $S_1 \cup S_2$.

Notation 14. *A strategy s for Eloise (resp. Abelard) in an extensive or strategic IF game can be considered as a series of choice functions s_ψ, where ψ ranges over the subformulas owned by Eloise (resp. Abelard). Let $\psi = (Qx/W)\chi$. When proving that Skolem semantics coincides with GTS, one exploits the fact that there is a one-to-one correspondence between choice functions s_ψ and Skolem functions g_ψ that interpret the function symbol f_ψ: each choice function s_ψ defines a Skolem function g_ψ and vice versa.*

Throughout this section, we write f_ψ^s to denote the Skolem function g_ψ that is defined by the choice function s_ψ. We shall also use the shorthand notation $(\mathfrak{M}, s)$ for $(\mathfrak{M}, (f_\psi^s)_\psi)$, where ψ ranges over the subformulas owned by Eloise (Abelard), to indicate that we "extend" the structure $\mathfrak{M}$ by the strategy s.

Fix two color objects $c_0 \neq c_1$ from C and let $\vec{c} = (c_0, c_1)$. For objects a and $b = R^{\mathfrak{C}}(a)$, let $h_{ab,\vec{c}}$ be the coloring that can be constructed as follows: first set $h_{ab,\vec{c}}(a) = h_{ab,\vec{c}}(b) = z_0$. Then, for $j = 0 \ldots n-3$, starting from b, move from the current object to its R-successor b', increase j by one, and set $h_{ab,\vec{c}}(b') = c_0$ if j is even and $h_{ab,\vec{c}}(b') = z_1$ if j is odd. Notice that

$$h_{ab,\vec{c}} \text{ sends only two adjacent objects to the same color, namely } a \text{ and } b. \tag{2}$$

Let $H_1^{\vec{c}}$ be the set $\{h_{ab,\vec{c}} : (a, b) \in R^{\mathfrak{C}}\}$. Let S_1 be the set of all strategies $s = (z_0^s, z_1^s, f^s, g^s)$ such that $z_0^s \neq z_1^s, f^s = g^s$ and $f^s \in H_1^{\vec{z}}$, where $\vec{z} = (z_0^s, z_1^s)$.

For objects a and $b = R^{\mathfrak{C}}(a)$, let $h_{b,\vec{c}}$ be the coloring that sends b to c_1 and that is otherwise like $h_{ab,\vec{c}}$. Let $H_2^{\vec{c}}$ be the set $\{h_{b,\vec{c}} : b \in C\}$. Let S_2 be the set of all strategies $s = (z_0^s, z_1^s, f^s, g^s)$ such that $z_0^s \neq z_1^s, f^s = h_{ab,\vec{z}} \in H_1^{\vec{z}}$ and $g^s = h_{b,\vec{z}} \in H_2^{\vec{z}}$, where $\vec{z} = (z_0^s, z_1^s)$.

For an illustration of the functions in $H_1^{\vec{c}}$ and $H_2^{\vec{c}}$, see Fig. 1.

The following fact shows that the strategies in S_1 and S_2 lose against precisely one strategy in T in such a way that if $s \in S_1 \cup S_2$ loses against $t \in T$, then there is no other strategy in $S_1 \cup S_2$ that loses against t.

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[3] We use the symbols z_0, z_1 also as nullary function symbols.

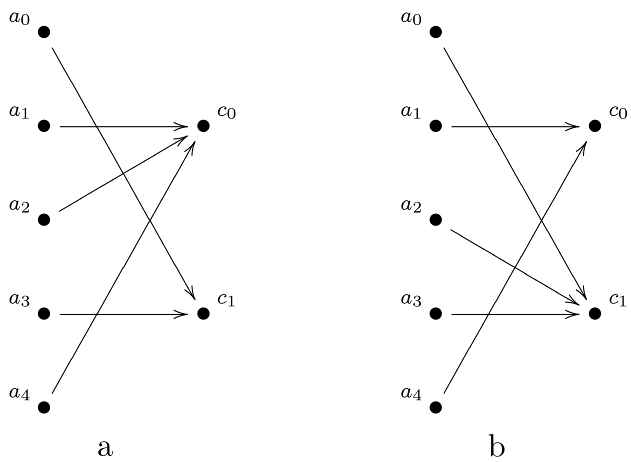

Fig. 1. Depicted are the objects of a circular graph $\mathfrak{C}$ with five objects: $a_0, \ldots, a_4$. We have omitted the binary relation $R^{\mathfrak{C}} = \{(a_0, a_1), \ldots, (a_3, a_4), (a_4, a_0)\}$. The two dedicated color objects c_0 and c_1 are depicted as if they existed on their own, even though $c_i \in \{a_0, \ldots, a_4\}$. Figure (a) shows function $h_{ab,\vec{c}}$ for $a = a_1$ and $b = a_2$; Figure (b) shows function $h_{b,\vec{c}}$ for $b = a_2$.

Fact 15.
- *Let $s \in S_1$, let $s = (z_0^s, z_1^s, f^s, g^s)$ and let a and b be the objects for which $f^s = g^s = h_{ab,\vec{z}}$. Then, for every $t = (x^t, y^t) \in T$, $u(s, t) = 0$ iff $x^t = a$ and $y^t = b$.*
- *Let $s \in S_2$, let $s = (z_0^s, z_1^s, f^s, g^s)$ and let a and b be the objects for which $f^s = h_{ab,\vec{z}}$ and $g^s = h_{b,\vec{z}}$. Then, for every $t = (x^t, y^t) \in T$, $u(s, t) = 0$ iff $x^t = y^t = b$.*

Proof. Let $s \in S_i$ and $t \in T$. We have $u(s, t) = 1$ iff $(\mathfrak{C}, s, t) \models Sk(\psi)$, where $Sk(\psi)$ as in (1). Obviously, the first conjunct in $Sk(\psi)$ is satisfied since f^s is a coloring by construction. The proofs of the two respective items continue as follows:

- Case $s \in S_1$. Since $f^s = g^s$, the second conjunct is true regardless x^t and y^t. Therefore, $u(s, t) = 0$ iff the third conjunct in $Sk(\psi)$ fails, that is,

$$(\mathfrak{C}, f^s, x^t, y^t) \not\models R(x, y) \to f(x) \neq f(y).$$

  We saw in (2) that a function $h_{ab,\vec{z}}$ from $H_1^{\vec{z}}$ only sends a and b to the same color. So $u(s, t) = 0$ iff $x^t = a$ and $y^t = b$. Note that, since $\mathfrak{C}$ is a directed graph in which $(a, b) \in R^{\mathfrak{C}}$ implies not $(b, a) \in R^{\mathfrak{C}}$, playing s against the strategy t' with $x^{t'} = b$ and $y^{t'} = a$ yields a win for Eloise.
- Case $s \in S_2$. Consider three mutually exclusive cases:
  Case $x^t \neq y^t$ and $(x^t, y^t) \notin R^{\mathfrak{C}}$: $u(s, t) = 1$, because the second and third conjunct are automatically satisfied.
  Case $x^t = y^t$: The third conjunct is automatically satisfied. It suffices to show that $u(s, t) = 1$ iff $f^s(x^t) = g^s(x^t)$. By construction, $f^s = h_{ab,\vec{z}}$ and $g^s = h_{b,\vec{z}}$ disagree only on b. Hence, $f^s(x^t) = g^s(x^t)$ iff $x^t \neq b$.
  Case $(x^t, y^t) \in R^{\mathfrak{C}}$: We show that $u(s, t) = 1$. Since the second conjunct is automatically satisfied, it suffices to show that $f^s(x^t) \neq g^s(y^t)$. Recall that $f^s = h_{ab,\vec{z}}$ and $g^s = h_{b,\vec{z}}$. We distinguish three subcases. (i) Suppose $x^t = a$ and $y^t = b$. By construction, $h_{ab,\vec{z}}(a) = z_0^s \neq z_1^s = h_{b,\vec{z}}(b)$. (ii) Suppose $x^t = b$. By construction, $h_{ab,\vec{z}}(b) = z_0^s$. The object y^t is the R-successor of x^t, that is, the first object that is sent to z_1^s when we construct the function $h_{ab,\vec{z}}$. The function $h_{b,\vec{z}}$ disagrees with $h_{ab,\vec{z}}$ only on b. Therefore, $h_{b,\vec{z}}(y^t) = h_{ab,\vec{z}}(y^t) = z_1^s$ and we are done. (iii) Suppose $x^t \neq a$ and $x^t \neq b$. It follows that also $y^t \neq b$. As before, the function $h_{b,\vec{z}}$ disagrees with $h_{ab,\vec{z}}$ only on b. Thus, $h_{b,\vec{z}}(y^t) = h_{ab,\vec{z}}(y^t)$. From (2) we derive that $h_{ab,\vec{z}}(x^t) \neq h_{ab,\vec{z}}(y^t)$. □

The following result allows us to remove all strategies that are not in $S_1 \cup S_2$.

Fact 16. *The value of $\Gamma = (S, T, u)$ is equal to the value of $(S_1 \cup S_2, T, u)$.*

Proof. We show that every strategy that is not in $S_1 \cup S_2$ is weakly dominated by or payoff equivalent to a strategy in $S_1 \cup S_2$. Let $s \in (S - (S_1 \cup S_2))$. Define $L_s = \{t \in T : u(s, t) = 0\}$ and define $N_s = |L_s|$. We split the proof in three cases:

Case $N_s = 0$: It follows that s is a winning strategy and therefore φ_{even} is true on $\mathfrak{C}$. Contradiction.

Before we proceed, observe that:

$$\text{If there is an } a \text{ such that } f^s(a) \notin \{z_0^s, z_1^s\}, \text{ then } \{t \in T : x^t = a\} \subseteq L_s. \tag{3}$$

Case $N_s = 1$: Let t be the single strategy from L_s. Since $u(s, t) = 0$, at least one of $Sk(\psi)$'s conjuncts fails on $(\mathfrak{C}, s, t)$. It cannot be the first conjunct. Suppose, for contradiction, that the structure $(\mathfrak{C}, s, t)$ fails to make the first conjunct true. Then, by (3), L_s contains at least n strategies, contradicting $N_s = 1$.

Suppose $u(s, t) = 0$ on account of $(\mathfrak{C}, s, t)$ falsifying the second conjunct: $(\mathfrak{C}, s, t) \not\models x = y \to f(x) = g(y)$. Then, $x^t = y^t$ but not $f^s(x^t) = g^s(y^t)$. Consider the strategy $s^* \in S_2$ with $f^{s^*} = h_{ab,\vec{z}}$ and $g^{s^*} = h_{b,\vec{z}}$, where $b = x^t$ and $\vec{z}$ is any pair of distinct objects. From the second item of Fact 15, it follows that s^* loses only against t, just like s. Hence, s and s^* are payoff equivalent.

Suppose $u(s, t) = 0$ on account of $(\mathfrak{C}, s, t)$ falsifying the third conjunct: $(\mathfrak{C}, s, t) \not\models R(x, y) \to f(x) \neq g(y)$. Then, y^t is an R-successor of x^t but $f^s(x^t) = g^s(y^t)$. Consider the strategy $s^* \in S_1$ with $f^{s^*} = g^{s^*} = h_{ab,\vec{z}}$, where $a = x^t$, $b = y^t$ and $\vec{z}$ is any pair of distinct objects. Apply the first item of Fact 15 to show that s and s^* are payoff equivalent.

Case $N_s \geq 2$: It suffices to show that L_s contains at least one strategy t such that $x^t = y^t$ or $R^{\mathfrak{C}}(x^t) = y^t$. For suppose $t \in L_s$ with $R^{\mathfrak{C}}(x^t) = y^t$. Consider the strategy $s^* \in S_1$ with $f^{s^*} = g^{s^*} = h_{ab,\vec{z}}$, where $x^t = a$, $y^t = b$ and $\vec{z}$ is any pair of distinct objects. By the first item of Fact 15, s^* *only* loses against t. Since s loses against t and at least one other strategy ($N_s \geq 2$), s is weakly dominated by s^*. Similarly for $t \in L_s$ with $x^t = y^t$.

For the sake of contradiction, assume that L_s contains no strategy t for which $x^t = y^t$ or $R^{\mathfrak{C}}(x^t) = y^t$. Fix an arbitrary strategy $t \in L_s$. Since $t \in L_s$, $u(s, t) = 0$, and thus the structure $(\mathfrak{C}, s, t)$ automatically satisfies the second and third conjunct, given that $x^t \neq y^t$ and $R^{\mathfrak{C}}(x^t) = y^t$. This structure falsifies the first conjunct and by (3), $\{t \in T : x^t = a\} \subseteq L_s$. We conclude that the strategy (x^t, y^t) with $x^t = y^t = a$ is also contained in L_s. □

We shall remove some of Eloise's strategies once more. Let $\vec{c} = (c_0, c_1)$ be any pair of distinct objects. Let $S_i^{\vec{c}}$, $i = 1, 2$, be the subset of S_i that only has strategies s with functions f^s and g^s from $H_1^{\vec{z}} \cup H_2^{\vec{z}}$, where $z_0^s = c_0$ and $z_1^s = c_1$.

Fact 17. *The value of* $(S_1 \cup S_2, T, u)$ *is equal to the value of* $(S_1^{\vec{c}} \cup S_2^{\vec{c}}, T, u)$.

Proof. Let $s \in S_1^{\vec{d}}$, where $\vec{c} \neq \vec{d}$. So s is not in $S_1^{\vec{c}}$. Let a and b the objects for which $f^s = g^s = h_{ab,\vec{d}}$. Consider the strategy s^* in $S_1^{\vec{c}}$, for which $f^{s^*} = g^{s^*} = h_{ab,\vec{c}}$. The first item of Fact 15 implies that if Eloise plays either s or s^*, she only loses against the strategy t with $x^t = a$ and $y^t = b$. Hence, s and s^* are payoff equivalent.

Likewise for $s \in S_2^{\vec{d}}$. □

Each strategy in $s \in (S_1^{\vec{c}} \cup S_2^{\vec{c}})$ assigns to the function symbols f and g proper colorings, that is, functions with range $\{c_0, c_1\}$. Thus, each play of the game in which Eloise plays such an s induces an interpretation of the logical symbols that satisfies the first conjunct of $Sk(\psi)$. Eloise playing a strategy from $(S_1^{\vec{z}} \cup S_2^{\vec{z}})$, the only way for Abelard to win a play is by choosing a pair of objects that satisfies the antecedent of the second or third conjunct. Let $T' = \{t \in T : x^t = y^t \text{ or } R^{\mathfrak{C}}(x^t) = y^t\}$.

Fact 18. *The value of* $(S_1^{\vec{c}} \cup S_2^{\vec{c}}, T, u)$ *is equal to the value of* $(S_1^{\vec{c}} \cup S_2^{\vec{c}}, T', u)$.

Proof. Left as an exercise to the reader. □

Write $\Gamma^* = (S^*, T^*, u)$ for $(S_1^{\vec{c}} \cup S_2^{\vec{c}}, T', u)$. $S_1^{\vec{c}}$ and $S_2^{\vec{c}}$ are the sets that contain for every object a one strategy $h_{ab,\vec{c}}$ and one strategy $h_{b,\vec{c}}$, respectively, where b is the R-successor of a. Since $S_1^{\vec{c}}$ and $S_2^{\vec{c}}$ do not overlap, S^* contains $2n$ strategies. Similarly, T^* is the set that contains for each object a two strategies (x^t, y^t), where $x^t = a$ and y^t is either equal to a or to a's R-successor. Hence, T^* has size $2n$.

We can write u as a $2n \times 2n$ matrix, with a 0 in cell (s, t) if, for $a = x^t$ and $b = y^t$, either of the two cases holds:

- $a \neq b, f^s = h_{ab,\vec{z}}$ and $g^s = h_{ab,\vec{z}}$ or
- $a = b, f^s = h_{ab,\vec{z}}$ and $g^s = h_{b,\vec{z}}$;

and 1 otherwise. The matrix of u has a diagonal of 0s and is otherwise filled with 1s. It is easy to verify that the value of Γ^* equals $(1 - \frac{1}{2n})$.

Computing the semantic value of φ_{even} has unveiled several aspects of equilibrium semantics. In standard game-theoretic semantics, emphasis is on winning strategies that guide its owner through the extensive game tree. According to one favoured interpretation of strategic games, the players (randomly) select a strategy in parallel and hand them to a neutral arbiter who processes the strategies and returns the outcome to the players. In the present setting, the arbiter receives the best evidence that Eloise and Abelard can produce for their respective cases. If Eloise's strategy is winning, her evidence defies each of Abelard's challenges. In the case of φ_{even} a winning strategy would comprise a coloring that sends any pair of adjacent objects to distinct colors. Things get more complex, and interesting, when neither Eloise nor Abelard has a winning strategy. In that case, Eloise and Abelard draw strategies so as to maximize their respective expected utility.

It is probably worthwhile to note that, in the case of φ_{even}, equilibrium semantics does not give a quantitative account of truth. In a quantitative framework, the truth value of a sentence φ on a structure $\mathfrak{M}$ is regarded as the "degree" to which $\mathfrak{M}$ has property φ. From a quantitative point of view, a circular graph $\mathfrak{C}$ of odd size $n + 2$ would be "more even" than a circular graph $\mathfrak{C}'$ of size n. It follows from this section's analysis, namely, that the truth value of φ_{even} is strictly higher on $\mathfrak{C}$ than on $\mathfrak{C}'$: $(1 - \frac{1}{2n+4})$ vs. $(1 - \frac{1}{2n})$. We would say that this is an undesirable property of a quantitative interpretation for logic.

6. Strategic IF games realize all rationals

It is known that the class of win–loss, two-player games whose utility function return only rationals realize precisely the rationals [15, p. 739]. That is, (i) every win–loss game has a value in $\mathbb{Q}$ and (ii) for every $q \in \mathbb{Q}$, there is a win–loss game with value q. Even though the class of strategic IF games is only a subclass of the class of win–loss, two-player games, it can be shown that they too realize precisely the rationals in [0, 1]. Item (i) of this claim follows from (i) for general win–loss, two-player games. Item (ii) is implied by Theorem 19.

Theorem 19. *Let $0 \leq m < n$ be integers and $q = m/n$. There exists an IF sentence ζ_q that has truth value q on every structure $\mathfrak{M}$ with at least two objects.*

Proof. We will first give an auxiliary game with value q, and then give its definition in IF. The second game proves the theorem.

- *Game*—Let M be such that $|M| \geq n$. Let $C \subseteq M$ such that $|C| = n$. We describe a two-step game G^1:

  S1 Abelard picks up m objects $b_1, \ldots, b_m$ from M;
  S2 Eloise picks up one object c from M, not knowing $b_1, \ldots, b_m$.

  Eloise gets pay-off 1 if and only if at least one of the following conditions is met for some $1 \leq i < j \leq m$:

  1. $b_i = b_j$ (Abelard has chosen two identical objects);
  2. $b_i \notin C$ (Abelard has chosen outside C);
  3. $c = b_i$ (Eloise guesses one of Abelard's objects).

  Conditions (1) and (2) force Abelard to chose m distinct objects from C. Note that there is no need to hardwire in the winning conditions that Eloise choose c from C. Any strategy of Eloise that chooses c from outside C is weakly dominated by every strategy that chooses c from C.
- *Value*—Let σ be Eloise's uniform mixed strategy over $\{s_a : a \in C\}$. Let $\mathcal{B}$ be the set of sets B for which $B \subseteq C$ and $|B| = m$. Let τ be Abelard's uniform mixed strategy over $\{t_B : B \in \mathcal{B}\}$, where t_B is a strategy that lets Abelard pick precisely the objects in B in an arbitrary order. The pair (σ, τ) is an equilibrium in G^1 with value m/n.
- *Definition in IF*—Assume that $\mathfrak{M}$ interprets the constant symbols $c_1, \ldots, c_n$ in such a way that $C = \{c_1^{\mathfrak{M}}, \ldots, c_n^{\mathfrak{M}}\}$. Consider the IF sentence which defines G^1:

$$\forall x_1 (\forall x_2/\{x_1\}) \ldots (\forall x_m/\{x_1, \ldots, x_{m-1}\})(\exists y/\{x_1, \ldots, x_m\})\ \beta_1 \vee \beta_2 \vee \beta_3,$$

  where

$$\beta_1 = \bigvee_{i \in \{1,\ldots,m\}} \bigvee_{j \in \{1,\ldots,m\}-\{i\}} x_i = x_j$$

$$\beta_2 = \bigvee_{i \in \{1,\ldots,m\}} \bigwedge_{j \in \{1,\ldots,n\}} x_i \neq c_j$$

$$\beta_3 = \bigvee_{i \in \{1,\ldots,m\}} x_i = y.$$

  The slash-sets in the universal quantifiers reduce the set of Abelard's strategies. The IF sentence that is obtained by dropping the slash-sets in the universal quantifiers gives rise to a game in which Abelard picks an object for x_1 and then picks an object for x_2 *on the basis of* x_1, etc. This sentence has the same truth value as the above sentence, but it gives rise to larger games, see also our discussion in Section 5.

  The formulas β_1–β_3 encode the respective winning conditions (1)–(3) from G^1 in first-order logic. The ignorance of Eloise in S2 is codified by the slash.

The previous game assumes that $|M| \geq n$ and that we have n distinct objects at our disposal. We can drop both assumptions by letting Eloise pick n objects from which Abelard has to chose, *and* by letting them draw long enough "bitstrings" that encode their choice. In this way, we only need two dedicated letters, which can also be chosen by Eloise. It does not matter which two objects are chosen as letters, as long as they are distinct.

- *Game*—Let $\ell = \lceil {}^2 \log n \rceil$.

  S0′ Eloise picks up two (distinct) objects a_0 and a_1 from M;
  S1′ Eloise picks up $\ell \cdot n$ objects $\vec{b}_1, \ldots, \vec{b}_n$ from M;
  S2′ Abelard picks up $\ell \cdot m$ objects $\vec{c}_1, \ldots, \vec{c}_m$ from M;
  S3′ Eloise picks up ℓ objects $\vec{d}$ from M, not knowing $\vec{b}_1, \ldots, \vec{b}_m$,

  where $\vec{b}_i$ abbreviates $(b_i^1, \ldots, b_i^\ell)$, etc. Eloise gets pay-off 1 if and only if $a_0 \neq a_1, \vec{b}_1, \ldots, \vec{b}_n, \vec{d} \in A^\ell$ and at least one of the following conditions is met for some $1 \leq i < j \leq m$:

1. $c_i^k = c_j^k$, for all $1 \le k \le \ell$ (Abelard has chosen two equal bitstrings);
2. $\vec{c}_i \notin \{\vec{b}_1, \ldots, \vec{b}_n\}$ (one of Abelard's bitstrings is not in $\{\vec{b}_1, \ldots, \vec{b}_n\}$);
3. $c_i^k = d^k$, for all $1 \le k \le \ell$ (Eloise guesses one of Abelard's bitstrings).

Let G^2 be the game described by S0′–S3′ and the latter winning conditions.

- *Value*—It is readily observed that the value of G^2 is $\frac{m}{n}$.
- *Definition in IF*—Consider the sentence ζ_q that is given by

$$\exists z_1 (\exists z_2/\{z_1\})(\exists w_1^1/W_1^1)\ldots(\exists w_1^\ell/W_1^\ell)\ldots(\exists w_n^1/W_n^1)\ldots(\exists w_n^\ell/W_n^\ell)$$
$$(\forall x_1^1/X_1^1)\ldots(\forall x_1^\ell/X_1^\ell)\ldots(\forall x_m^1/X_m^1)\ldots(\forall x_m^\ell/X_m^\ell)$$
$$(\exists y^1/Y^1)\ldots(\exists y^\ell/Y^\ell)\,\gamma,$$

where for $1 \le i \le n$, $1 \le j \le m$ and $1 \le k \le \ell$,

$$W_i^k = \{z_1, z_2, \vec{w}_1, \ldots, \vec{w}_{i-1}, w_i^1, \ldots, w_i^{k-1}\}$$
$$X_j^k = \{\vec{x}_1, \ldots, \vec{x}_{j-1}, x_j^1, \ldots, x_j^{k-1}\}$$
$$Y^k = \{z_1, z_2, \vec{w}_1, \ldots, \vec{w}_n, \vec{x}_1, \ldots, \vec{x}_m, y^1, \ldots, y^{k-1}\}$$

and $\gamma = (z_1 \ne z_2 \wedge (\gamma_1 \vee \gamma_2 \vee \gamma_3))$ with

$$\gamma_1 = \bigvee_{i\in\{1,\ldots,m\}} \bigvee_{j\in\{1,\ldots,m\}-\{i\}} (x_i^1 = x_j^1 \wedge \cdots \wedge x_i^\ell = x_j^\ell)$$
$$\gamma_2 = \bigvee_{i\in\{1,\ldots,m\}} \bigwedge_{j\in\{1,\ldots,n\}} (w_j^1 \ne x_i^1 \vee \cdots \vee w_j^\ell \ne x_i^\ell)$$
$$\gamma_3 = \bigvee_{i\in\{1,\ldots,m\}} (x_i^1 = y^1 \wedge \cdots \wedge x_i^\ell = y^\ell). \quad \square$$

Corollary 20. *Let $0 < \varepsilon < 1$ be nonrational. Then, the "inclusive" definition of equilibrium semantics from Definition 9 coincides with its "exclusive" version, since $V(\Gamma) \ge \varepsilon$ iff $V(\Gamma) > \varepsilon$, for every strategic IF game Γ.*

With the help of Theorem 19, we can show that the expressive power of IF logic is independent of the threshold value ε under which we evaluate its sentences, given that ε is a rational and that $0 < \varepsilon < 1$. This follows from Theorem 22. In the proof of this result, we use the following helpful proposition.

Proposition 21. *Let φ be an IF sentence and let $0 \le q \le 1$ be a rational. Then,*

1. *there is an IF sentence φ' s.t. for every structure $\mathfrak{M}$, $V(\Gamma') = qV(\Gamma) + (1-q)$;*
2. *there is an IF sentence φ' s.t. for every structure $\mathfrak{M}$, $V(\Gamma') = qV(\Gamma)$,*

where $\Gamma = \Gamma(\mathfrak{M}, \varphi)$ and $\Gamma' = \Gamma(\mathfrak{M}, \varphi')$.

Proof. We give the rationale behind the first claim. Let v denote the value $V(\Gamma)$.

- *Game*—First, the game G^2 from the proof of Theorem 19 is played until all quantifier moves have been handled. Then, Eloise is shown all objects that have been chosen so far and she is offered the choice between (a) continuing the game of G^2 and (b) playing the game of φ on $\mathfrak{M}$ (in a way that is unaffected by the previous moves).
- *Value*—Let us consider the moment in this game where Eloise is shown all objects chosen so far. Either (i) the selected assignment satisfies γ (in which case Eloise would receive 1 if she would choose (a)) or (ii) the selected assignment makes γ false (in which case Eloise gets nothing if she would choose (a)). If (i), Eloise chooses (a) since this results in a guaranteed win for her. If (ii), she chooses (b) since the expected payoff of this choice is v. By the reasoning from the proof of Theorem 19, the odds that (i) occurs is q and the odds that (ii) occurs is $1-q$. Hence, the value of this game is $q + (1-q)v$.
- *Definition in IF*—Let us write $\vec{Q}\vec{x}\gamma$ for the sentence ζ_q from Theorem 19 and let us assume, without loss of generality, that none of the variables in $\vec{x}$ appears in φ. The sentence $\vec{Q}\vec{x}(\gamma \vee \varphi)$ defines the described game. The choice between (a) and (b) is triggered by the disjunction in $(\gamma \vee \varphi)$.

The sentence $\vec{Q}\vec{x}(\gamma \wedge \varphi)$ bears witness to the second item in the same way. $\square$

Theorem 22. *Let $0 < \varepsilon, \varepsilon' \le 1$ be rationals s.t. if $\varepsilon' = 1$ then $\varepsilon = 1$. Then, for every IF sentence φ in the vocabulary L there is an IF sentence φ' in the same vocabulary such that $\mathfrak{M} \models_\varepsilon \varphi$ iff $\mathfrak{M} \models_{\varepsilon'} \varphi'$, for every L-structure $\mathfrak{M}$.*

Proof. We distinguish two cases.

Case $\varepsilon < \varepsilon' < 1$: Let q be the rational $(\varepsilon'-1)/(\varepsilon-1)$. Let φ' be an IF sentence for which $V(\Gamma') = qV(\Gamma) + (1-q)$, on all $\mathfrak{M}$, where $\Gamma = \Gamma(\mathfrak{M}, \varphi)$ and $\Gamma' = \Gamma(\mathfrak{M}, \varphi')$. Such a φ' exists due to Proposition 21.1. An elementary algebraic argument shows that $\mathfrak{M} \models_\varepsilon \varphi$ iff $\mathfrak{M} \models_{\varepsilon'} \varphi'$.

Case $\varepsilon > \varepsilon'$: Let q be the rational ε'/ε. The sentence φ' for which $V(\Gamma') = qv(\Gamma)$ does the job (Proposition 21.2). $\square$

We will comment on the exception that $\varepsilon' = 1$ imply $\varepsilon = 1$ below.

7. Complexity of finding equilibria

In this section, we explore the expressive power of IF logic under equilibrium semantics for arbitrary $0 < \varepsilon < 1$ by drawing on descriptive complexity. In descriptive complexity (or finite model theory [11]), a logic is associated with the class $STR(\varphi)$ of finite structures in which φ is true, where φ ranges over the sentences of the logic at hand. In this framework, Fagin [6] showed that NP $= \Sigma_1^1$, that is, every NP-soluble problem is definable in Σ_1^1 and conversely, every Σ_1^1-definable property is soluble in NP. Here, as above, Σ_1^1 denotes the fragment of second-order logic with sentences $\exists X_1 \ldots \exists X_n \varphi$ with φ first-order.

Definition 23. Let $0 \leq \varepsilon \leq 1$. Let φ be an IF sentence. Define $STR_\varepsilon(\varphi)$ as the set

$$\{\mathfrak{M} : \mathfrak{M} \models_\varepsilon \varphi \text{ and } M \text{ is finite}\}.$$

Let IF_ε be the class of all $STR_\varepsilon(\varphi)$, where φ ranges over the IF sentences.

Results in descriptive complexity enable us to approach the expressive power of IF under equilibrium semantics in terms of the computational complexity of IF_ε.

Theorem 22 implies a lower bound on IF_ε.

Corollary 24. *Let $0 < \varepsilon \leq 1$ be rational. Then,* NP $\subseteq \text{IF}_\varepsilon$.

Proof. That NP $= \text{IF}_1$ is implied by Fagin's result, the fact that Σ_1^1 coincides with IF logic qua expressive power and Proposition 10. It follows from Theorem 22 that for every $STR_1(\varphi) \in \text{IF}_1$ there is a $STR_\varepsilon(\varphi') \in \text{IF}_\varepsilon$ for which $STR_1(\varphi) = STR_\varepsilon(\varphi')$. □

The converse direction (i.e. NP $\supseteq \text{IF}_\varepsilon$) does not follow from Theorem 22, since it does not cover the case in which $\varepsilon < \varepsilon' = 1$. Σ_2^{P} is the complexity class of problems that are soluble by a nondeterministic polynomial-time Turing machine M^L that has access to an oracle that can be queried about the NP computable language L. An alternative way of writing Σ_2^{P} is NP^{NP}. The class Π_2^{P} abbreviates coNP^{NP}. The class $\Sigma_2^{\text{P}} \cap \Pi_2^{\text{P}}$ consists of the languages that are solvable in Σ_2^{P} and in Π_2^{P}. In descriptive complexity, $\Sigma_2^1 = \Sigma_2^{\text{P}}$, where Σ_2^1 is the fragment of second-order logic of the sentences $\exists X_1 \ldots \exists X_m \forall Y_1 \ldots \forall Y_n \varphi$ with φ first-order. In the same sense, $\Pi_2^1 = \Pi_2^{\text{P}}$. The remainder of this section is dedicated to the proof of the following upper bound.

Theorem 25. *Let $0 \leq \varepsilon \leq 1$. Then,* $\text{IF}_\varepsilon \subseteq \Sigma_2^{\text{P}} \cap \Pi_2^{\text{P}}$.

Proof. Immediate from Lemmata 30 and 31. □

We prove Theorem 25 by analyzing an algorithm that computes the equilibrium for strategic IF games. Computing the Nash equilibrium in two-player strategic games (possibly nonwin–loss and nonconstant-sum) was recently shown [5] to be complete for the class PPAD, of which it is known that P $\subseteq$ PPAD $\subseteq$ NP. It is well known [16] that computing the value of constant-sum games is in P.

Proposition 26. *Let $N = \{0, 1\}$. Let $\Gamma = ((S_i)_{i \in N}, u)$ be a finite, two-player, constant-sum strategic game. Then, computing the value of Γ can be done in time polynomial in the size of Γ.*

Proposition 26 does not settle the complexity for strategic games, since the size of the strategic IF game $\Gamma(\mathfrak{M}, \varphi)$ is exponential in the size of the input: the cardinality of $\mathfrak{M}$.[4] A result in [12] guarantuees that there exists a polynomial-sized (or "polysized") subgame Γ' in $\Gamma(\mathfrak{M}, \varphi)$ that has the same value as $\Gamma(\mathfrak{M}, \varphi)$. (S', T', u) is a *subgame* in (S, T, u), if $S_1' \subseteq S_1$, $S_2' \subseteq S_2$ and u' is the restriction of u to $S' \times T'$. We shall simply write u for u'.

The *support* of a mixed strategy σ is the set of pure strategies to which σ assigns a nonzero probability. The following result helps us find polysized subgames.

The result addresses games in their extensive form. In this form, a state in the game is modelled as a *history*, that is, the list of moves that lead from the start of the game to the current situation. A *terminal history* is a history that cannot be extended, i.e. that corresponds to a state in which the game terminates.

Theorem 27 ([12]). *Let G be a finite, two-player, constant-sum game in extensive form. Let σ be a mixed strategy in G. Then, there exists a mixed strategy σ' in G that is equivalent modulo utility to σ s.t. the support of σ' contains at most ℓ pure strategies, where ℓ is the number of terminal histories in the game tree.*

A history in an extensive IF game $G = G(\mathfrak{M}, \varphi)$ can be seen as a list of operator-action pairs. The connectives $\vee$ and $\wedge$ are accompanied by an action *left* or *right*, the quantifiers $(\forall x/W)$ and $(\exists x/W)$ are accompanied by an object from M. Thus, a history is a list of at most n pairs, where n is the length of φ. Suppose φ contains n_0 connectives and n_1 quantifiers. Then, there are at most $2^{n_0} m^{n_1} < m^n$ many terminal histories in G, where m is the cardinality of M.

[4] In descriptive complexity, we consider the computational problem of deciding whether $\mathfrak{M} \in STR_\varepsilon(\varphi)$. The size of the input is $O(m^c)$ for some constant c, where m the size of $\mathfrak{M}$. Note that the size of φ, n, is a constant parameter in $STR_\varepsilon(\varphi)$.

Whence, if (σ, τ) is an equilibrium in $\Gamma = \Gamma(\mathfrak{M}, \varphi)$, then there is an equilibrium (σ', τ') s.t. S' (and T') is the support of σ' (and τ') for which the size of S' (and T') is polynomial in m. It follows from Theorem 27 that the value of Γ coincides with the value of its polysized subgame $\Gamma' = (S', T', u)$. Theorem 27 does not tell us how to obtain Γ' from Γ. To this end, we turn to von Neumann's Minimax Theorem, see [14].

Theorem 28. *Let $\Gamma = (S, T, u)$ be a finite, two-player, constant-sum strategic game. Then,*

$$V(\Gamma) = \max_{\sigma} \min_{\tau} U(\sigma, \tau) = \min_{\tau} \max_{\sigma} U(\sigma, \tau),$$

where σ (resp. τ) ranges over the mixed strategies in S (resp. T).

We give a reformulation of Theorem 28 in terms of subgames.

Lemma 29. *Let $\Gamma = (S, T, u)$ be a finite, two-player, constant-sum strategic game with value v. Then,*

1. *there exists $S' \subseteq S$, s.t. for every $T' \subseteq T$, $V(S', T', u) \geq v$;*
2. *there exists $T' \subseteq T$, s.t. for every $S' \subseteq S$, $V(S', T', u) \geq v$.*

Proof. We prove the first item. By Theorem 28, $\max_{\sigma} \min_{\tau} U(\sigma, \tau) = v$. Hence, there exists a mixed strategy σ^* maximizing $\min_{\tau} U(\sigma^*, \tau) = v$. Let τ^* be the mixed strategy minimizing $U(\sigma^*, \tau^*)$. Then, for all mixed strategies $\tau' \in T$, we have that $U(\sigma^*, \tau') \geq v$.

Let S^* be the support of σ^*. Then, $V(S^*, T', u) \geq v$, where T' the support of any mixed strategy τ'. □

Theorem 27 guarantees that we can restrict our attention in Lemma 29 to polysized sets S' and T'. It is left as an exercise to the reader to work out the equivalence between item (1) of Lemma 29 and its refinement (1′) by taking into account Theorem 27:

1′. there exists polysized $S' \subseteq S$, s.t. for every polysized $T' \subseteq T$, $V(S', T', u) \geq v$.

We are now ready to prove one part of Theorem 25.

Lemma 30. $\mathrm{IF}_{\varepsilon} \subseteq \Sigma_2^{\mathrm{P}}$.

Proof. Consider the game $\Gamma = \Gamma(\mathfrak{M}, \varphi) = (S, T, u)$ and suppose its value is v. We need to decide whether $v \geq \varepsilon$. It follows from (1′) that there exists a polysized $S' \subseteq S$, s.t. for every polynomial-sized $T' \subseteq T$, $V(S', T', u) \geq \varepsilon$ iff $\mathfrak{M} \in STR(\varphi)$.

Since a pure strategy in Γ can be described in polynomial time in m, we can describe all pure strategies in polysized subsets S' in polynomial time in m. Whence, an NP machine guesses a set S^* for which $V(S^*, T', u) \geq v$ for all T', if such an S^* exists.

Given S^*, we query an NP oracle whether there exists a polysized set T^*, for which the value v^* of $\Gamma^* = (S^*, T^*, u^*)$ is smaller than ε. That is, the oracle accepts iff $v^* < \varepsilon$. We decide that $\mathfrak{M} \in STR_{\varepsilon}(\varphi)$ iff the oracle rejects.

To prove that the above is an NP query, we need to show that we can compute in polynomial time the value of Γ^* from S^* and T^*. To this end, we first make the observation that u^*, i.e. the restriction of u to the strategy pairs in Γ^*, can be written out in polynomial time by iterating over all strategy pairs (s, t) in Γ, of which there are polynomially many, and computing the outcome $u(s, t)$. Now that we have Γ^* written out, we apply Proposition 26 and the claim follows. □

Likewise, we can show that item (2) of Lemma 29 is equivalent to the following condition:

2′. there exists polysized $T' \subseteq T$, s.t. for every polysized $S' \subseteq S$, $V(S', T', u) \geq v$.

The proof of the following lemma runs completely analogous to the proof of Lemma 30 using clause (2′).

Lemma 31. $\mathrm{IF}_{\varepsilon} \subseteq \Pi_2^{\mathrm{P}}$.

8. Discussion

In this paper, we outlined the various interpretations that have been given for IF logic. We developed a new interpretation that is obtained by considering semantic games for IF logic in strategic form $\Gamma(\mathfrak{M}, \varphi)$. We applied the notion of equilibrium to these games to assign a truth value of an IF sentence φ on a finite structure $\mathfrak{M}$. These efforts yield a truth relation $\mathfrak{M} \models_{\varepsilon} \varphi$, which holds iff the value of the equilibrium of the strategic IF game of φ on $\mathfrak{M}$ is at least ε. We saw in Proposition 10 that, for $\varepsilon = 1$, equilibrium semantics coincides with the standard game-theoretic semantics. Theorem 19 implies that strategic IF games realize precisely the rationals. We derived from this result that, from a model-theoretic point of view, $\models_{\varepsilon}$ is exchangeable for $\models_{\varepsilon'}$ as long as $0 < \varepsilon' < 1$ and ε and ε' are rationals (Theorem 22). The expressive power of IF logic under game-theoretic semantics coincides with Σ_1^1, the existential fragment of second-order logic. By drawing on results from descriptive complexity theory and combinatorial game theory we showed in Theorem 25 that the expressive power is bounded by $\Sigma_1^2 \cap \Pi_1^2$.

In this paper, we only scratched the surface of a number of interesting properties of equilibrium semantics. The first one is the restriction to finite structures. Every two-player, constant-sum game has an equilibrium if it is finite. If it is infinite this may not be the case. It would be very interesting to see what strategic IF games $\Gamma(\mathfrak{M}, \varphi)$ have equilibriums, for infinite $\mathfrak{M}$.

We saw that IF logic is at least as expressive as Σ^1_1 under equilibrium semantics and bounded, qua expressive power, by $\Sigma^2_1 \cap \Pi^2_1$. The precise expressive power of IF logic under equilibrium semantics is left open. One of the reasons why we find this problem interesting is that it may turn out that IF logic can express its own complement, that is, for every class $STR_\varepsilon(\varphi)$, where φ is an IF sentence, there is another IF sentence φ' that defines precisely the complement of $STR_\varepsilon(\varphi)$.

After submission, we learned that Galliani [7] devised a strategic approach to IF logic in terms of *behavioral strategies*. A behavioral strategy of a player in an extensive game $G = G(\mathfrak{M}, \varphi)$ can be understood as a probability distribution over the player's information sets in G. The truth value of a model-sentence pair is defined as the value of an equilibrium in behavioral strategies. Galliani gives a compositional interpretation for this semantic interpretation of IF logic. Galliani independently shows that precisely the rationals can be realized, using a game that is most similar to the one given in the proof of Theorem 19. By means of an example, Galliani showed us, in personal communication, that the semantic interpretations do not coincide.

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