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The algebraic structure of the set of solutions to the Thue equation

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ABSTRACT

Let F_n be a binary form with integral coefficients of degree $n \geqslant 2$, let d denote the greatest common divisor of all non-zero coefficients of F_n, and let $h \geqslant 2$ be an integer. We prove that if $d = 1$ then the Thue equation (T) $F_n(x, y) = h$ has relatively few solutions: if $\mathcal{A}$ is a subset of the set $\mathcal{T}(F_n, h)$ of all solutions to (T), with $r := \mathrm{card}(\mathcal{A}) \geqslant n + 1$, then

(#) *h divides the number* $\Delta(\mathcal{A}) := \prod_{1 \leqslant k < l \leqslant r} \delta(\xi_k, \xi_l)$,

where $\xi_k = \langle x_k, y_k \rangle \in \mathcal{A}$, $1 \leqslant k \leqslant r$, and $\delta(\xi_k, \xi_l) = x_k y_l - x_l y_k$. As a corollary we obtain that if h is a prime number then, under weak assumptions on F_n, there is a partition of $\mathcal{T}(F_n, h)$ into at most n subsets maximal with respect to condition (#).

1. Introduction

Let $F_n(x, y) = a_0 x^n + a_1 x^{n-1} y + \cdots + a_n y^n$ be a binary form of degree $n \geqslant 2$ with integral coefficients, and let $h \geqslant 2$ be an integer. This paper deals with the structure of the set $\mathcal{T}(F_n, h)$ of solutions $\langle x, y \rangle$ to the Diophantine Thue equation

$$F_n(x, y) = h \qquad (T)$$

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in integers x, y, where the greatest common divisor (GCD) of all non-zero coefficients a_k equals 1. We thus consider the case which cannot be reduced to equation $F_n(x, y) = 1$ addressed e.g. in [2,10]. We also assume that the set $\mathcal{T}(F_n, h)$ is not empty.

In 1909 Thue proved that if F_n is irreducible and $n \geqslant 3$ then Eq. (T) has a finite number of solutions. Since then this case of Eq. (T) has been called the *Thue equation*, and the Thue result has been strengthened by a number of authors. For the story of estimating the number $N_c(F_n, h)$ of all *coprime* solutions of (T) we refer the reader to two papers of Stewart [8,9]. (We recall that two integers a, b with $ab \neq 0$ are coprime if $(a, b) = 1$, where the symbol (a, b) denotes GCD of a, b.)

Today the best unconditional result is due to Stewart [8, Theorem 1]:

$$N_c(F_n, h) \leqslant Cn^{1+t},$$

where $C = 2800$ and t is the number of all distinct prime factors of the constant term h. This is an improvement of *conditional* results obtained earlier by Evertse [3] (that $N_c(F_n, h) \leqslant 7^{15(\binom{n}{3}+1)^2} + 6 \times 7^{2\binom{n}{3}(t+1)}$), Bombieri and Schmidt [1] (that $C = 215$ for h large enough and $F_n(x, 1)$ irreducible), and others [4–7].

Our theorems and their corollaries presented below deal with the algebraic structure of the set $\mathcal{T}(F_n, h)$ (*without any restriction on* F_n). However, the problem of estimating $N_c(F_n, h)$ by the use of Theorem 2, or its Corollary 2, of Section 4 seems to be interesting and is obviously open.

In order to present our main results we shall fix notation and introduce some terminology. An element $\langle x, y\rangle$ of $\mathcal{T}(F_n, h)$ will be also denoted by ξ or η, and $-\xi$ means $\langle -x, -y\rangle$. For a nonempty subset $\mathcal{A} = \{\xi_1, \xi_2, \ldots, \xi_r\}$ of $\mathcal{T}(F_n, h)$ with $r = \operatorname{card}(\mathcal{A}) \geqslant 2$ we let

$$\Delta(\mathcal{A}) := \prod_{1 \leqslant k < l \leqslant r} \delta(\xi_k, \xi_l),$$

where $\delta(\xi_k, \xi_l) = x_k y_l - y_k x_l$ and $\xi_k = \langle x_k, y_k\rangle$, $k = 1, 2, \ldots, r$. Obviously $\delta(\xi, \xi) = 0$ for all $\xi \in \mathcal{T}(F_n, h)$, and $\delta(\eta, -\eta) = 0$ for all $\eta \in \mathcal{T}(F_n, h)$ for n is even.

Our fundamental theorem reads as follows.

Theorem 1. *Let $\mathcal{A}$ be a subset of $\mathcal{T}(F_n, h)$. If* $\operatorname{card}(\mathcal{A}) \geqslant n + 1$ *then h is a divisor of $\Delta(\mathcal{A})$.*

This theorem yields an information about the number of solutions to the Thue equation (T) that fulfil the extra condition (1) below, and complements partially the above-mentioned conditional results of Silverman, Mueller and Schmidt, and Lorenzini and Tucker.

An immediate consequence of Theorem 1 is

Corollary 1. *Let $\mathcal{A}$ be a subset of $\mathcal{T}(F_n, h)$ with* $\operatorname{card}(\mathcal{A}) \geqslant 2$. *If*

$$\big(h, \Delta(\mathcal{A})\big) < h \tag{1}$$

(*in particular, if h and $\delta(\xi, \eta)$ are coprime for all distinct $\xi, \eta \in \mathcal{A}$*), *then* $\operatorname{card}(\mathcal{A}) \leqslant n$.

Corollary 1 allows us to define a number $\omega(F_n, h)$ which is useful for further purposes as follows. We put $\omega(F_n, h) = 1$ if

$$\delta(\xi, \eta) \equiv 0 \pmod h \tag{2}$$

for all $\xi, \eta \in \mathcal{T}(F_n, h)$, and $\omega(F_n, h) =$ the maximal cardinality of the subsets $\mathcal{A}$ of $\mathcal{T}(F_n, h)$ with $\operatorname{card}(\mathcal{A}) \geqslant 2$ that fulfil inequality (1) otherwise.

The results given in Theorem 1 and Corollary 1 can be now presented in a concise form.

Theorem 1′. *For every Thue equation* (T) *we have* $\omega(F_n, h) \leqslant n$.

As an application of this result we show in Theorem 2 below that if h is a prime number then, under weak assumptions on F_n, there is a partition of the set $\mathcal{T}(F_n, h)$ into at most n subsets maximal with respect to the condition

$$h \text{ divides } \Delta(\mathcal{A}).$$

2. The proof of Theorem 1

Let $\xi_k = \langle x_k, y_k \rangle$, $k = 1, \ldots, n+1$ be (pairwise distinct) elements of $\mathcal{A} \subset \mathcal{T}(F_n, h)$. We thus have $n+1$ equations

$$\begin{gathered} a_0 x_1^n + a_1 x_1^{n-1} y_1 + \cdots + a_n y_1^n = h, \\ a_0 x_2^n + a_1 x_2^{n-1} y_2 + \cdots + a_n y_2^n = h, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_0 x_{n+1}^n + a_1 x_{n+1}^{n-1} y_{n+1} + \cdots + a_n y_{n+1}^n = h \end{gathered} \tag{3}$$

and we treat $a_0, a_1, \ldots, a_n$ as unknown quantities. Hence the determinant W of the system (3) is of the form

$$W = \det \begin{pmatrix} x_1^n & x_1^{n-1} y_1 & \cdots & y_1^n \\ x_2^n & x_2^{n-1} y_2 & \cdots & y_2^n \\ \cdots & \cdots & \cdots & \cdots \\ x_{n+1}^n & x_{n+1}^{n-1} y_{n+1} & \cdots & y_{n+1}^n \end{pmatrix}.$$

Claim. W equals $\Delta(\mathcal{A})$. *For a proof, let us assume first that all x_k's are non-zero, and put $X = (x_1 \cdot x_2 \cdot \cdots \cdot x_{n+1})^n$, and $u_k = y_k / x_k$, $k = 1, 2, \ldots, n+1$. It is now obvious that $W = X \cdot V(u_1, u_2, \ldots, u_{n+1})$, where V is the Vandermonde's determinant for the u_k's. Hence, $W = X \cdot \prod_{1 \leqslant k < l \leqslant n+1} (u_k - u_l) = \Delta(\mathcal{A})$. This particular case suggests our claim is true in general (also when $X = 0$), what can be checked by the same method (i.e., by mathematical induction) as for the proof of the form of V.*

Since h divides 0, in our proof we may now assume that $\Delta(\mathcal{A}) \neq 0$, whence, by the Cramer theorem,

$$a_k = \frac{h \cdot W_k}{\Delta(\mathcal{A})}, \quad k = 1, 2, \ldots, n+1, \tag{4}$$

where W_k is a determinant with kth column consisting of 1's only, and the remaining columns the same as in W.

Assume, by way of contradiction, that h does not divide $\Delta(\mathcal{A})$, i.e., the number $h_0 := (h, \Delta(\mathcal{A}))$ is strictly less than h. Hence $h = h_0 \cdot h_1$, where $2 \leqslant h_1 \leqslant h$, and, by (4), there exist integers $r_1, r_2, \ldots, r_{n+1}$ such that $a_k = h_1 \cdot r_k$ for all k's. It follows that h_1 divides the greatest common divisor of the coefficients of F_n; but by assumption the latter is equal to 1. This contradicts that $h_1 \geqslant 2$.

3. The set $\mathcal{T}_+(F_n, h)$

In this section we define a subset $\mathcal{T}_+(F_n, h)$ of $\mathcal{T}(F_n, h)$ which permits us to eliminate in Theorem 1 the case $\Delta(\mathcal{A}) = 0$.

Since $F_n(-\xi) = (-1)^n F_n(\xi)$, it is more convenient to work (for n even) not on the whole set $\mathcal{T}(F_n, h)$ but on its subset $\mathcal{T}_+(F_n, h)$ defined as follows: for *n odd* we simply put $\mathcal{T}_+(F_n, h) := \mathcal{T}(F_n, h)$, and for *n even* we define $\mathcal{T}_+(F_n, h)$ by the rules:

$$\mathcal{T}_+(F_n, h) := \{\xi \in \mathcal{T}(F_n, h)\colon\ x_\xi \geqslant 1\} \quad \text{when } x_\xi \neq 0 \text{ for all } \xi \in \mathcal{T}(F_n, h),$$

and

$$\mathcal{T}_+(F_n, h) := \{\xi \in \mathcal{T}(F_n, h)\colon\ x_\xi \geqslant 1\} \cup \{\langle 0, |u|\rangle\} \quad \text{when } \langle 0, u\rangle \in \mathcal{T}(F_n, h),$$

where x_ξ denotes the element x of the first coordinate of ξ.

Put $\mathcal{T}_-(F_n, h) := \{\xi \in \mathcal{T}(F_n, h)\colon\ -\xi \in \mathcal{T}_+(F_n, h)\}$. Then, obviously, $\mathcal{T}_-(F_n, h) = \emptyset$ for n odd, and

$$\mathcal{T}_+(F_n, h) \cap \mathcal{T}_-(F_n, h) = \emptyset \quad \text{for all } n\text{'s}.$$

Hence we obtain a decomposition of $\mathcal{T}(F_n, h)$:

$$\mathcal{T}(F_n, h) = \mathcal{T}_+(F_n, h) \cup \mathcal{T}_-(F_n, h) \tag{5}$$

(with $\mathcal{T}(F_n, h) = \mathcal{T}_+(F_n, h)$ for n odd).

From the definition of $\Delta(\mathcal{A})$ it follows that if $\operatorname{card}(\mathcal{A}) \geqslant 2$ then $\Delta(\mathcal{A}) = 0$ if (and only if) $\delta(\xi, \eta) = 0$ for some distinct elements ξ, η of $\mathcal{A}$. In the lemma below we give a characterization of the latter case.

Lemma 1. *Let $\xi, \eta \in \mathcal{T}(F_n, h)$. Then $\delta(\xi, \eta) = 0$ if and only if $\eta = \xi$ or $\eta = -\xi$.*

Thus, the elements ξ, η of $\mathcal{T}_+(F_n, h)$ (and hence, of $\mathcal{T}(F_n, h)$ for n odd) are distinct if and only if $\delta(\xi, \eta) \neq 0$.

Proof. Since the "if" part is obvious, let us assume that $\delta(\xi, \eta) = 0$. Hence the elements ξ and η are linearly dependent:

$$\xi = a \cdot \eta \tag{6}$$

for some real number a. Now from (T) and (6) we obtain $h = F_n(\xi) = F_n(a\eta) = a^n F_n(\eta) = a^n h$, whence $a^n = 1$. $\square$

Let, for a subset $\mathcal{B}$ of $\mathcal{T}(F_n, h)$, the symbol $-\mathcal{B}$ denote the set $\{-\xi\colon\ \xi \in \mathcal{B}\}$. The above lemma motivates us to consider the auxiliary subset $\mathcal{A}_+$ of $\mathcal{T}(F_n, h)$ of the form

$$\mathcal{A}_+ := (\mathcal{A} \cap \mathcal{T}_+(F_n, h)) \cup (-(\mathcal{A} \cap \mathcal{T}_-(F_n, h)))$$

(which coincides with $\mathcal{A}$ for n odd, and consists of the elements ξ with $x_\xi \geqslant 0$ for n even). Then from Lemma 1 we obtain that

$$\text{if } \operatorname{card}(\mathcal{A}_+) \geqslant 2 \quad \text{then } \Delta(\mathcal{A}_+) \neq 0, \tag{7}$$

and it is also obvious that

$$\mathcal{A}_+ = \mathcal{A} \quad \text{for } \mathcal{A} \subset \mathcal{T}_+(F_n, h) \text{ or } n \text{ odd}. \tag{8}$$

For example, if n is even and ξ, η are two distinct elements of $\mathcal{T}(F_n, h)$ with $x_\xi \geqslant 1$ and $x_\eta \geqslant 1$, then for the set $\mathcal{A} := \{\xi, \eta, -\eta\}$ we have $\delta(\eta, -\eta) = 0$, and hence $\Delta(\mathcal{A}) = 0$. On the other hand, $\mathcal{A}_+ = \{\xi, \eta\}$, whence $\mathrm{card}(\mathcal{A}) = 3 > 2 = \mathrm{card}(\mathcal{A}_+)$ and $\Delta(\mathcal{A}_+) = \delta(\xi, \eta) \neq 0$.

From the definitions of $\Delta(\mathcal{A})$ and $\mathcal{A}_+$, and from Lemma 1 it also immediately follows that

$$\text{if } \Delta(\mathcal{A}) \neq 0 \quad \text{then } \mathrm{card}(\mathcal{A}) = \mathrm{card}(\mathcal{A}_+)$$

and

$$\Delta(\mathcal{A}) = \Delta(\mathcal{A}_+). \tag{9}$$

Remark 1. The properties (7), (8) and (9) allow us to calculate the number $\omega(F_n, h)$ with the help of subsets $\mathcal{A}$ of $\mathcal{T}_+(F_n, h)$ instead of $\mathcal{T}(F_n, h)$: we put $\omega(F_n, h) = 1$ as in the initial definition, and the maximal cardinality of the subsets $\mathcal{A}$ of $\mathcal{T}_+(F_n, h)$ with $\mathrm{card}(\mathcal{A}) \geqslant 2$ that fulfil inequality (1) otherwise.

Remark 2. The properties (7), (8), (9) imply also that both in Theorem 1 and Corollary 1 we need to consider only the case $\mathcal{A} \subset \mathcal{T}_+(F_n, h)$ (for which $\Delta(\mathcal{A}) \neq 0$); then, by Remark 1, the conclusion of Theorem 1′ remains unchanged.

4. h-Homogeneous subsets of $\mathcal{T}(F_n, h)$

We say that a subset $\mathcal{A}$ of $\mathcal{T}(F_n, h)$ is *h-homogeneous* provided that every pair $\xi, \eta \in \mathcal{A}$ fulfils condition (2). (Notice that all 1-element subsets of $\mathcal{T}(F_n, h)$ are h-homogeneous.) One of the tools describing the partition of $\mathcal{T}(F_n, h)$ into h-homogeneous subsets will be the number $\overline{\omega}(F_n, h)$ defined (in a similar way as $\omega(F_n, h)$) by means of condition (2) and the somewhat strengthened condition (1): *for a subset $\mathcal{A}$ of $\mathcal{T}(F_n, h)$ one has*

$$\delta(\xi, \eta) \not\equiv 0 \pmod h \quad \text{for all distinct } \xi, \eta \in \mathcal{A}. \tag{1′}$$

We put $\overline{\omega}(F_n, h) = 1$ if condition (2) holds on the set $\mathcal{T}_+(F_n, h)$, and $\overline{\omega}(F_n, h) =$ the maximal cardinality of the subsets $\mathcal{A}$ of $\mathcal{T}_+(F_n, h)$ with $\mathrm{card}(\mathcal{A}) \geqslant 2$ that fulfil the above condition (1′) otherwise. It is easily seen that

$$\overline{\omega}(F_n, h) \geqslant \omega(F_n, h),$$

and

$$\overline{\omega}(F_n, h) = \omega(F_n, h) \quad \text{for } h \text{ prime}, \tag{10}$$

and the first inequality in (10) can be strict (see Example 2 in Section 5). However, in contrast to Theorem 1′, we do not know if $\overline{\omega}(F_n, h) \leqslant n$ in general (i.e., if h is not a prime number, see (10)).

The remaining part of this section deals with the structure of the subset $\mathcal{T}_c(F_n, h)$ of all *coprime* elements $\langle x, y\rangle$ of $\mathcal{T}_+(F_n, h)$, with an application to $\mathcal{T}(F_n, h)$. The (non-zero) integers a_0 and a_n in the hypotheses of the next results are taken from the form of F_n.

One can easily check that *if h is nth power-free and $(a_0, h) = 1$ or $(a_n, h) = 1$ then*

$$\mathcal{T}_c(F_n, h) = \mathcal{T}_+(F_n, h). \tag{11}$$

The lemma below describes the basic properties of an equivalence relation on $\mathcal{T}_c(F_n, h)$; its second part follows from property (11).

Lemma 2. *Let the set $\mathcal{T}_c(F_n, h)$ be not empty. Then the relation $\sim$ on $\mathcal{T}_c(F_n, h)$ of the form*

$$\xi \sim \eta \quad \text{if and only if} \quad \delta(\xi, \eta) \equiv 0 \pmod{h}$$

is an equivalence relation, and the number $\omega_c(F_n, h)$ defined as the cardinality of $\mathcal{T}_c(F_n, h)/\sim$ fulfils inequality $\omega_c(F_n, h) \leqslant \overline{\omega}(F_n, h)$.

If, additionally, h is nth power-free then the relation $\sim$ holds on $\mathcal{T}_+(F_n, h)$, and

$$\operatorname{card}\bigl(\mathcal{T}_+(F_n, h)/\sim\bigr) = \overline{\omega}(F_n, h).$$

Proof. Since only the transitivity of $\sim$ is nontrivial, let $\xi_k = \langle x_k, y_k\rangle$, $k = 1, 2, 3$, and $\xi_1 \sim \xi_2$ & $\xi_2 \sim \xi_3$; equivalently,

$$h | x_1 y_2 - x_2 y_1 \quad \text{and} \quad h | x_2 y_3 - x_3 y_2. \tag{12}$$

Choose integers a, b with $by_2 - ax_2 = 1$, and define the matrix $A = \left(\begin{smallmatrix} a & y_2 \\ -b & -x_2 \end{smallmatrix}\right)$. Then A has determinant 1. Let $[u, v]$ denote a vector in $\mathbf{R}^2$, and set

$$\bigl[x'_k, y'_k\bigr] = [x_k, y_k]A$$

for $k = 1, 2, 3$. Then the number $x'_1 y'_2 - x'_2 y'_1 = (x_1 y_2 - x_2 y_1) \det A$ is divisible by h, and similarly $x'_2 y'_3 - x'_3 y'_2$ is divisible by h. Since $x'_2 = -1$ and $y'_2 = 0$, the two latter relations imply that both y'_1, y'_3 are divisible by h. This further implies $x'_1 y'_3 - x'_3 y'_1$ is divisible by h, and hence $x_1 y_3 - x_3 y_1 = (x'_1 y'_3 - x'_3 y'_1) \det A^{-1}$ is divisible by h, i.e., $\xi_1 \sim \xi_3$. We thus have proved that the relation $\sim$ is transitive. $\square$

In the next theorem we show that every nonempty set $\mathcal{T}_c(F_n, h)$ can be partitioned into maximal h-homogeneous subsets; its proof follows immediately from Lemma 2 applied to the cases $\omega_c(F_n, h) = 1$ and $\omega_c(F_n, h) \geqslant 2$.

Theorem 2. *Let the set $\mathcal{T}_c(F_n, h)$ be not empty. Then the following alternative holds*:

(i) *the set $\mathcal{T}_c(F_n, h)$ is h-homogeneous*;
(ii) *there is a partition of $\mathcal{T}_c(F_n, h)$ into $\omega_c(F_n, h) \geqslant 2$ nonempty and h-homogeneous subsets $\mathcal{A}_k$*

$$\mathcal{T}_c(F_n, h) = \bigcup_{k=1}^{\omega_c(F_n,h)} \mathcal{A}_k$$

such that, for all distinct k_1, k_2 and arbitrary $\xi_i \in \mathcal{A}_{k_i}$, $i = 1, 2$, we have $\delta(\xi_1, \xi_2) \not\equiv 0 \pmod{h}$.

If, additionally, h is nth power-free then the above alternative holds for $\mathcal{T}_+(F_n, h)$ and $\overline{\omega}(F_n, h)$, respectively, instead of $\mathcal{T}_c(F_n, h)$ and $\omega_c(F_n, h)$, respectively.

From Theorem 1′, and from the second part of Theorem 2, and properties (10) and (11) we obtain the following corollary.

Corollary 2. *Let h be a prime number. If the set $\mathcal{T}(F_n, h)$ is not h-homogeneous, then it has the form:*

(a) *for n odd,*

$$\mathcal{T}(F_n, h) = \bigcup_{k=1}^{\omega(F_n,h)} \mathcal{A}_k,$$

where $2 \leqslant \omega(F_n, h) \leqslant n$, all the sets $\mathcal{A}_k$ are h-homogeneous, nonempty and pairwise disjoint, and for all distinct k_1, k_2 and arbitrary $\xi_i \in \mathcal{A}_{k_i}$, $i = 1, 2$, we have $\delta(\xi_1, \xi_2) \not\equiv 0 \pmod h$;

(b) *for n even,*

$$\mathcal{T}(F_n, h) = \bigcup_{k=1}^{\omega(F_n,h)} (\mathcal{A}_k \cup \mathcal{B}_k)$$

where $\mathcal{A}_k \subset \mathcal{T}_+(F_n, h)$, $\mathcal{B}_k := -\mathcal{A}_k$, for all k's, with the same properties of $\omega(F_n, h)$ and $\mathcal{A}_k$'s as in item (a)*.*

Remark 3. It is not claimed in the above corollary that the sets $\mathcal{A}_k$ consist of at least two elements. However, if all $\mathcal{A}_k$'s were singletons then, by Theorem 1′, we would have the very strong bound for the cardinality $N(F_n, h)$ of $\mathcal{T}(F_n, h)$: $N(F_n, h) = \omega(F_n, h) \leqslant n$ for n odd, and $N(F_n, h) = 2 \cdot \omega(F_n, h) \leqslant 2n$ for n even (see (1)).

5. Examples

In this section we illustrate the notions introduced in Sections 1 and 3 and some relations between them. By Remarks 1 and 2, we can consider only subsets of $\mathcal{T}_+(F_n, h)$. The testing equation will have the form

$$F_n^{(s,t)}(x, y) = p^t (= h), \qquad (T^{(s,t)})$$

where $F_n^{(s,t)}(x, y) = \prod_{k=1}^{n}(x - kp^s y) + p^t y^n$, p is a prime number, and n, s, t are integers with $n \geqslant 3$, $s \in \{0, 1\}$, $t \in \{1, 2\}$.

Example 1. *The number $\omega(F_n, h)$ can attain the values both 1 and n, as well as every prime number between them* (we do not know if all the integers between 1 and n are admissible). For a proof of the case $\omega(F_n, h) = 1$ we shall use the equation $(T^{(1,1)})$ and its solutions ξ_k, $k = 1, 2, \ldots, n$, of the form $\xi_k = \langle pk, 1\rangle$. For all k, l we then have $\delta(\xi_k, \xi_l) = p \cdot (k - l)$ and hence, for $\eta = \langle x, y\rangle$ another (possible) solution, the number p divides $x - kpy = \delta(\eta, \xi_k)$ for some k. From Lemma 1 it now follows that $h(= p)$ divides all the numbers $\delta(\xi, \eta)$ with $\xi, \eta \in \mathcal{T}(F_n^{(1,1)}, h)$, and so $\omega(F_n^{(1,1)}, h) = 1$ (by definition).

In the remaining cases we consider the equation $(T^{(0,1)})$. Since $h = p$ is prime, we can use the definition of $\overline{\omega}$ instead of ω (see (10)). Let ξ_k denote the solution of the form $\langle k, 1\rangle$, $k = 1, 2, \ldots, n$. Then we have

$$p \mid \delta(\xi_k, \xi_l) \quad \text{if and only if} \quad p \mid (k - l), \qquad (13)$$

whence, for $p \geqslant n$ and $k \neq l$, the numbers p and $\delta(\xi_k, \xi_l)$ are coprime. By Theorem 1′ and (10), we have $\omega(F_n^{(0,1)}, h) = n$.

Now let $p < n$, and let $[\xi_k]$ denote the abstract class, $k = 1, \ldots, p$, of the relation $\sim$ in $\mathcal{T}_+(F_n^{(0,1)}, h)$ (see Lemma 2). By (13), every ξ_j with $j > p$ falls into $[\xi_k]$ for some $k \leqslant p$. Moreover, if $\eta = \langle x, y \rangle$ is another (possible) solution to $(T^{(1,1)})$ with $x > 0$, then p divides $x - ky = \delta(\eta, \xi_k)$, and hence $\eta \in [\xi_k]$ for some $k \leqslant p$. We thus have shown that the set $\mathcal{T}_+(F_n^{(0,1)}, h)$ is partitioned into p sets: $[\xi_1] \cup \cdots \cup [\xi_p]$, and each of them is h-homogeneous (with $h = p$). From the second part of Theorem 2 it now follows that $\omega(F_n^{(0,1)}, h) = p$.

Example 2. *Inequality in* (10) *is strict, in general.* We consider the equation $(T^{(1,2)})$ with $2 < p \leqslant n$. Then $h = p^2$ is nth power-free, but evidently not a prime number. It is obvious that our equation and equation $(T^{(1,1)})$ of Example 1 possess the same solutions ξ_k, $k = 1, \ldots, n$, with the property (in our case)

$$p | \delta(\xi_k, \xi_l), \quad \text{but} \quad p^2 \nmid \delta(\xi_k, \xi_l) \quad \text{for } 1 \leqslant k < l \leqslant p. \tag{14}$$

Hence $(h, \Delta(\mathcal{A})) = p < h$ for $\mathcal{A} = \{\xi_1, \xi_2\}$ (with $\operatorname{card}(\mathcal{A}) = 2$), and so $\omega(F_n^{(1,2)}, h) \geqslant 2$. Moreover, for every solution $\eta = \langle x, y \rangle$ not of the form ξ_k (if any) we have $p | x$, thus p divides $\delta(\eta_1, \eta_2)$, for all distinct η_1, η_2, and p divides $\delta(\eta, \xi_k)$, for all k's. It immediately follows that if $\operatorname{card}(\mathcal{A}) \geqslant 3$ then $p^3 | \Delta(\mathcal{A})$, and hence $\omega(F^{(1,2)}, h) = 2$ by definition.

On the other hand, for the set $\mathcal{B} = \{\xi_1, \ldots, \xi_p\}$ property (14) holds. Thus, from the definition of $\overline{\omega}$ it follows that $\overline{\omega}(F_n^{(1,2)}, h) \geqslant p$, and (by our hypothesis) this value is larger than $2 = \omega(F_n^{(1,2)}, h)$.

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