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Automorphisms of Finite Order on Rational Surfaces

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We classify minimal pairs (X, G) for smooth rational projective surface X and finite group G of automorphisms on X. We also determine the fixed locus X^G and the quotient surface $Y = X/G$ as well as the fundamental group of the smooth part of Y. The realization of each pair is included. Mori's extremal ray theory and recent results of Alexeev and also Ambro on the existence of good anti-canonical divisors are used.

Key Words: rational surface; automorphism; quotient singularity; fundamental group.

INTRODUCTION

More than one hundred years ago, Kantor wrote a book on finite birational automorphism groups of rational surfaces. From the sixties to eighties, Manin, Iskovskih, Gizatullin also thoroughly studied G-rational surfaces defined over non-closed fields. One aim of theirs was to reduce to G-minimal surfaces. In [Giz], G-pseudoprojective rational surfaces, which are not G-projective surfaces, are shown to be relative minimal elliptic surfaces; the same paper also shows that not every G-rational surface is G-pseudo-projective. Segre [Seg] did, among many other things, the classification of $Aut\, X$ for cubic surfaces X (see also [Ho2]).

Recently, $Aut\, X$ has also been classified again for the quartic del Pezzo surface in [Ho1]. In [Koit], automorphism groups of rational surfaces obtained by blowing up very general points in $\mathbf{P}^2$ are completely classified. It is very desirable to test the modern machineries on the old subject and obtain a simpler proof at the same time.

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In this note, we work over the complex numbers field **C** and consider pairs of (X, G) of an arbitrary smooth rational projective surface X with a fixed group G acting on it. To simplify the arguments, we assume also that G is cyclic of prime order. We believe that the general case could be handled similarly. Indeed, our last theorem deals with arbitrary G, where we reduce to either G-stable conic fibration or the del Pezzo case (see Remark 5).

Actually, this note is inspired by Bayle and Beauville's recent simple new classification of birational involutions of rational surfaces [BB]. Following it, we also adopt the latest Mori theory [Mor]. Though the theory has been developed along the course of classification of higher dimensional varieties (dimension at least 3), we will see how useful it is even for surfaces. First, it will help us to reduce to a G-minimal surface very quickly, which has either a G-stable conic-fibration (Mori fibration), or a Picard number one quotient surface. The first case is easy to treat.

For the second case, we have two approaches. The top down approach is based on known information on Weyl groups $W(E_n)$ of lattice E_n where we apply Manin's results in [Man2]; see also [Re1]. For the bottom up approach (more geometric), we will study the quotient surfaces; this approach is normally more difficult. To do so, we apply results of Alexeev and Ambro about the existence of a good member in the anti-canonical system [Alex, Am]; implicitly we are also using Fujita's theory of polarized varieties, Δ genus zero case [Fuj]. This way, we avoid referring to the classification list of automorphism groups of del Pezzo surfaces X; such a list is available if K_X^2 is bigger.

It turns out that all pairs (X, G) with G cyclic of prime order p, except the last 3 rows in Table I ($p = 5$), have minimal models $(X_{\min}, G)$, via a G-equivariant birational morphism (only smooth blow-downs of G-stable divisors but no blow-ups), such that at least one of X and $Y = X/G$ is a minimal rational surface (i.e., $\mathbf{P}^2$ or Hirzebruch surfaces $\mathbf{F}_e$, $e \neq 1$) or the projective cone $\overline{\mathbf{F}}_3$ ($p = 3$).

To be precise, denote by μ_p the multiplicative group of prime order p. By writing (X, μ_p), we mean that X is a smooth projective rational surface with a faithful μ_p-action. It is natural to assume that X is minimal in terms of G-equivariant birational morphisms (Definition 1.4).

We now state our results. For $X = \mathbf{F}_e$ in Theorem 1(I), X^{μ_p} should be well known and is also determined in Lemma 4.3.

THEOREM 1. *Let p be a prime number and let* (X, μ_p) *be a minimal pair of a smooth projective rational surface and the group* μ_p *acting faithfully on X.*

(I) *If p is odd prime (for* $p = 2$ *see Theorem* 4 *and Remark* 5 *below) and the* μ_p*-invariant sublattice* $(Pic X)^{\mu_p}$ *has rank* ≥ 2*, then X is a Hirze-*

TABLE I

p	X	X^{μ_p}; $\mu_p = \langle g \rangle$	$Y = X/\mu_p$; π_1	Sing Y	$r = r(Y)$	Details
$p \geq 3$	$\mathbf{P}^2$	Line $\{Z = 0\}$ & point $[0, 0, 1]$ $g = \mathrm{diag}[1, 1, \zeta_p]$	$\bar{\mathbf{F}}_p$	$\frac{1}{p}(1, 1)$	$(p + 2)/p$	Ex. 2.1a
$p \geq 3$	$\mathbf{P}^2$	$[1, 0, 0], [0, 1, 0]$ $[0, 0, 1]$; $2 \leq v < p$ $g = \mathrm{diag}[1, \zeta_p, \zeta_p^{\,v}]$	$\pi_1(Y^0) = \mu_p$	$\frac{1}{p}(1, v)$ $\frac{1}{p}(1, p + 1 - v)$ $\frac{1}{p}(1, (v - 1)/v)$	$r = \frac{3}{p}$ $(p \geq 3)$ or $r = \frac{1}{p}$ $(p \geq 5)$	Ex. 2.1b
3	Cubic del Pezzo	Smooth $\in \lvert -K_X \rvert$	$\mathbf{P}^2$	$\varnothing$	3	Ex. 2.2
3	deg 1 del Pezzo $\lvert -K_X \rvert$ has 6 cuspidals	Smooth of genus 2 & a point	$\bar{\mathbf{F}}_3$	$\frac{1}{3}(1, 1)$	$5/3$	Ex. 2.3
5	deg 1 del Pezzo $\lvert -K_X \rvert$ has 10 nodals & 1 cuspidal	Smooth $\in \lvert -K_X \rvert$ & a point	$K_Y^2 = 5$ $\pi_1(Y^0) = (1)$	$\frac{1}{5}(1, 4)$	1	Ex. 2.4 Lemma 2.12
5	deg 1 del Pezzo $\lvert -K_X \rvert$ has 6 cuspidals	Smooth $\in \lvert -K_X \rvert$ & a point	$K_Y^2 = 5$ $\pi_1(Y^0) = (1)$	$\frac{1}{5}(1, 4)$	1	Ex. 2.4 Lemma 2.12
5	deg 5 del Pezzo	2 points	$K_Y^2 = 1$ $\pi_1(Y^0) = \mu_5$	$\frac{1}{5}(1, 4)$ $\frac{1}{5}(1, 4)$	1	Ex. 2.5 Lemma 2.13

bruch surface $\mathbf{F}_e$ $(e \neq 1)$ *and* (X, μ_p) *is birationally equivariant to a pair* $(\mathbf{P}^2, \mu_p)$ *given in Example* 2.1.

(II) *Suppose that* $(\mathrm{Pic}X)^{\mu_p}$ *has rank* 1. *Then* (X, μ_p) *is equal to one of the pairs in Examples* 2.1–2.8. *The fixed locus* X^{μ_p}, $X, Y = X/\mu_p$, *Fano index* $r(Y)$, *the type of all singularities on* Y, *and the topological fundamental group of* $Y^0 = Y - \mathrm{Sing}\, Y$ *are summarized in Table* I (*for odd prime p only*).

COROLLARY 2. *Let p be an odd prime number and let* (X, μ_p) *be an arbitrary pair of a smooth projective rational surface and the group* μ_p *acting faithfully on* X. *Then* (X, μ_p) *is birationally equivariant to one of the pairs* $(X_{\min}, \mu_p)$ *in Table* I. *Set* $Y_{\min} = X_{\min}/\mu_p$.
In particular, either $X_{\min} = \mathbf{P}^2$, *or* $Y_{\min} = \mathbf{P}^2$, *or* $Y_{\min} = \overline{\mathbf{F}}_3$, *or* $(X_{\min}, \mu_p)$ $(p = 5)$ *is one of the pairs in rows* 5, 6, 7 *of Table* I *given in Examples* 2.4 *and* 2.5 (*see Lemmas* 2.12–2.13 *for the uniqueness of the pairs in rows* 6, 7).

In the result below, (2) is trivial, while (1) is not so obvious; there is a rational surface with at worst two (quotient) singularities such that its smooth part has infinite π_1 (see [GZ3, Sect. 4]). See Remark 4.7.

COROLLARY 3. *Let* (X, μ_p) *be as in Corollary* 2. *Set* $Y = X/\mu_p$ *and* $Y^0 = Y - \mathrm{Sing}\, Y$. *Then* $\pi_1(Y^0)$ *equals* (1) *or* μ_p. *Moreover, when* (X, μ_p) *is a minimal pair, we have*:

(1) *If* X^{μ_p} *contains a curve, then* Y^0 *is simply connected.*

(2) *If* X^{μ_p} *is a finite set, then* $\pi_1(Y^0) = \mu_p$ (*see also Lemma* 4.4).

In the following, we denote by $X^G = \{x \in X \mid gx = x \text{ for some } 1 \neq g \in G\}$ the fixed locus, $\sigma : X \to Y = X/G$ the quotient map, $Y^0 = Y - \mathrm{Sing}\, Y$, and $r(Y)$ the Fano index.

THEOREM 4. *Let* (X, G) *be a minimal pair of a smooth projective rational surface and an arbitrary finite group G acting faithfully on X. Then we have*:

(I) *Suppose that the G-invariant sublattice* $(\mathrm{Pic}X)^G$ *has rank* ≥ 2. *Then X has a G-stable conic fibration each singular fibre of which is a linear chain of two* (-1)-*curves*.

(II) *Suppose that* $(\mathrm{Pic}\, X)^G$ *has rank* 1. *Then X is a* (*smooth*) *del Pezzo surface and Y is a singular del Pezzo surface with at worst quotient singularities so that* $\pi_1(Y^0)$ *is finite*; *one has* $\pi_1(Y^0) = G$ *if the fixed locus* X^G *is a finite set*. *Moreover, the following are true*.

(1) *If* $r(Y) = 1$ *and* X^G *is a finite set, then* (X, G) *is equal* (*modulo G-equivariant isomorphism*) *to one of the pairs in Examples* 2.1b $(p = 3)$, 2.5, 2.9–11. $X^G, X, Y = X/G$, K_Y^2 *and the types of all singularities of* Y *are*

TABLE II

G	X	X^G; $G = \langle g_1, \dots \rangle$	K_Y^2	Sing Y	Details
$\mathbf{Z}/(3)$	$\mathbf{P}^2$	$[1,0,0], [0,1,0]$ $[0,0,1]$; $g_1 = \mathrm{diag}[1, \zeta_3, \zeta_3^2]$	3	$(1,2)/3$ $(1,2)/3$ $(1,2)/3$	Ex. 2.1b
$\mathbf{Z}/(4)$	$\mathbf{P}^1 \times \mathbf{P}^1$	4 points	2	$(1,1)/2$ $(1,3)/4$ $(1,3)/4$	Ex. 2.11 Lemma 2.15
$\mathbf{Z}/(5)$	deg 5 del Pezzo	2 points	1	$(1,4)/5$ $(1,4)/5$	Ex. 2.5 Lemma 2.13
$\mathbf{Z}/(6)$	deg 6 del Pezzo	6 points	1	$(1,1)/2$ $(1,2)/3$ $(1,5)/6$	Ex. 2.9 Lemma 2.14
$\mathbf{Z}/(3) \oplus \mathbf{Z}/(3)$	$\mathbf{P}^2$	12 points; $g_1 = \mathrm{diag}[1, \zeta_3, \zeta_3^2]$ $g_2 = (a_{ij})$ $a_{13}a_{21}a_{32} = 1$ other $a_{ij} = 0$	1	$(1,2)/3$ $(1,2)/3$ $(1,2)/3$ $(1,2)/3$	Ex. 2.10
$\mathbf{Z}/(4) \oplus \mathbf{Z}/(2)$	$\mathbf{P}^1 \times \mathbf{P}^1$	12 points	1	$(1,1)/2$ $(1,1)/2$ $(1,3)/4$ $(1,3)/4$	Ex. 2.11 Lemma 2.15

summarized in Table II (*see Lemmas* 2.13–2.15 *for the uniqueness of the pairs in the rows* 3, 4, 2, 6).

(2) *If* $r(Y) > 1$, *then* Y *is either* $\mathbf{P}^2$ *or the projective cone* $\overline{\mathbf{F}}_e$ $(e \geq 2)$ *with* $e \mid |G|$.

Remark 5. (1) If $G = \mu_2$ in Theorem 4(I), then it is birationally equivariant to some De Jonquieres involution of degree $d \geq 2$ [BB]; when $d = 2$, it is given in Example 2.6.

(2) For a normal surface S (like Y in Theorem 4(II)) with at worst quotient singularities, $\mathbf{Q}$-ample anti-canonical divisor $-K_S$ and rank *Pic* S $= 1$, the Fano index $r(S) = 1$ holds if and only if S is a Gorenstein log del Pezzo surface other than the projective cone $\overline{\mathbf{F}}_2$ (this cone has Fano index 2) (Lemma 1.9 and [MZ1, Lemma 6]); for such S, it is also shown in [MZ, Lemma 6] that $\pi_1(S^0)$ is abelian of order ≤ 9.

(3) Kantor had classified automorphism groups of del Pezzo surfaces, though it was not told which automorphism is lifted from a del Pezzo surface of bigger degree. In this sense, the result in Theorem 4 and Kantor's book together complete the picture of automorphism groups of

rational surfaces. In particular, we have to refer to [Kan] for the case where $r(Y) < 1$. See also [MZ, Zh2].

(4) The difference of our approach from others lies in two aspects: (i) we determine also the fixed locus X^{μ_p} and the quotient surface X/μ_p and (ii) we include both the geometric approach (bottom up), and the algebraic approach as an Appendix, though the uniqueness and realizability of pairs are only treated in the geometric approach.

1. PRELIMINARY RESULTS

1.1. Let (X, G) be a pair of smooth rational projective surface and a non-trivial finite group G acting faithfully on X. Denote by $Y = X/G$ the quotient surface and $\sigma : X \to Y$ the quotient map. Let $f : Z \to Y$ be the minimal resolution. Note that Y is a rational surface by Luroth's theorem and Y has at worst quotient singularities and hence is simply connected [Ko, Theorem 7.8].

We assert that X^G is non-empty. Indeed, if X^G is empty then the quotient map $X \to Y$ is an unramified finite morphism of degree $|G|$ over the simply connected surface Y, whence $|G| = 1$, a contradiction.

The following is well known [Bri, Satz 2.11]; for the smoothness of X^{μ_p} we diagonalize the action locally and see that the μ_p fixed part is defined by a local coordinate (the eigenvector w.r.t. to the eigenvalue $\neq 1$).

LEMMA 1.2. (1) *The fixed locus X^G is non-empty. If $G = \mu_p$ then X^{μ_p} is a disjoint union of smooth curves R_i and finitely many points p_j* $(1 \le j \le s; s \ge 0)$.

(2) *The surface $Y = X/G$ is a* **Q**-*Gorenstein normal rational surface with singularities. If $G = \mu_p$, then $q_i := \sigma(p_i)$ is a cyclic quotient singularity of type $\frac{1}{p}(1, k_i)$ for some $1 \le k_i \le p - 1$; one has* $\operatorname{Sing} Y = \{q_1, \dots, q_s\}$ *and* $\sigma^{-1}(q_i) = p_i$.

(3) *Suppose that $G = \mu_p$. Then $Y = X/\mu_p$ is Du Val at q_i (i.e., Gorenstein in the present quotient singularity case) if and only if $k_i = p - 1$ (this is always true when $p = 2$). In general, pK_Y is Cartier.*

(4) *The quotient map σ is unramified outside the fixed locus X^G. If $G = \mu_p$, the ramification formula has the form* (**Q**-*linear equivalence*): $K_X \sim_{\mathbf{Q}} \sigma^*(K_Y) + (p-1)\Sigma_i R_i$.

(5) *The σ-invariant sublattice $(\operatorname{Pic} X)^G \otimes \mathbf{Q}$ has rank equal to the Picard number $\rho(Y) = \operatorname{rank} \operatorname{Pic} Y$ of the quotient surface Y.*

1.3. On surfaces, quotient singularity and log terminal singularity are equivalent [Kaw, Corollary 1.9]. So there is a **Q**-effective divisor Δ supported on $f^{-1}(\mathrm{Sing}\,Y)$ and with the integral part $[\Delta] = 0$, such that

$$K_Z = f^*(K_Y) - \Delta.$$

Write $\Delta = \Sigma_i \Delta_i$ where Δ_i is supported on $f^{-1}(q_i)$. Then $\Delta_i = 0$ if and only if q_i is Du Val.

1.4. Definition. Fix a group G. Let (X_i, G) be two pairs where G acts faithfully on X_i. A *G-equivariant birational morphism* $\tau : (X_1, G) \to (X_2, G)$ is a birational morphism $\tau : X_1 \to X_2$, satisfying $\tau(gx) = g\tau(x)$ for every $g \in G$. The existence of such τ is equivalent to that of a G-stable divisor on X_1 which can be smoothly blown down. If the G-invariant sublattice $(\mathit{Pic}\,X)^G \otimes \mathbf{Q}$ has rank 1, then there is no such τ and (X, μ_p) is minimal in the sense below.

Two pairs (X_i, G) are birationally G-equivariant if there is a birational map $X_1 \cdots \to X_2$ which can be decomposed as $f_1 \circ \cdots \circ f_n$ such that for each i either f_i or f_i^{-1} is a G-equivariant birational morphism.

(X, G) is called a *minimal pair*, if for any G-equivariant birational morphism $\tau : (X, G) \to (X_2, G)$, one has $\tau = \mathrm{id}$.

Let (X, G) be a pair (with X rational and G finite) and let $Y = X/G$. Suppose that $-K_Y$ is $\mathbf{Q}$-ample. Write $-K_Y \sim_{\mathbf{Q}} rP$, where r is a positive rational number and P a Cartier ample divisor. Let $r(Y)$ be the largest (hence P is the "smallest") among such expression, noting that $\mathit{Pic}\,Y$ is a torsion free $\mathbf{Z}$-module of finite rank (Y is simply connected). By the same reasoning, the divisor class of P is uniquely determined by $-K_X$ or X. This $r(Y)$ is called the *Fano* index of Y. When $G = \mu_p$, one can write $r(Y) = m/p$ with a positive integer m because pK_Y is Cartier.

Remark 1.5. If X is a smooth Fano n-fold (i.e., $-K_X$ is ample) then $r(X) \le n + 1$, and $r(X) = n$ (resp. $r(X) = n + 1$) if and only if X is a smooth quadric hypersurface in $\mathbf{P}^{n+1}$ (resp. $X = \mathbf{P}^n$) [KO].

Let $\overline{\mathbf{F}}_e$ ($e \ge 2$) be the projective cone, with vertex q_1, over a (smooth) rational curve of degree e in $\mathbf{P}^e$. Then the resolution of the vertex is the Hirzebruch surface $\mathbf{F}_e$ (see [Hart]), where the $(-e)$-curve is the inverse of the vertex. $\overline{\mathbf{F}}_e$ can also be embedded into $\mathbf{P}^{e+1}$ as a non-degenerate surface of degree e (see [Nag]). The hyperplane section $\overline{H}$ of $\overline{\mathbf{F}}_e \subseteq \mathbf{P}^{e+1}$ is the generator of the Picard lattice and is the image of a section on $\mathbf{F}_e$ disjoint from the $(-e)$-section; so $\overline{H}^2 = e$. One sees that $r = (e+2)/e > 1$.

1.6. Suppose that $G = \mu_p$. The induced μ_p action on $\mathit{Pic}\,X \otimes \mathbf{C}$ can be diagonalized. Since μ_p acts on the integral lattice $\mathit{Pic}\,X$, there is a generator h of μ_p satisfying, where $\zeta_p = \mathit{exp}(2\pi\sqrt{-1}/p)$,

$$h^* \mid \mathit{Pic}\,X \otimes \mathbf{C} = \mathrm{diag}\left[I_c, M_p^{\oplus t}\right], \qquad M_p = \left[\zeta_p, \zeta_p^2, \ldots, \zeta_p^{p-1}\right].$$

LEMMA. (1) $\operatorname{rank}(\operatorname{Pic} X)^{\mu_p} \otimes \mathbf{Q} = c$; $K_X^2 = 10 - c - (p-1)(2 + c - s - \Sigma_i(2 - 2g(R_i))$.

(2) *Writing* $X^{\mu_p} = \coprod R_i \coprod \{p_1, \dots, p_s\}$, *we have* $s + \Sigma_i(2 - 2g(R_i)) = c + 2 - t$.

(3) *Let* k_i $(1 \le i \le p-1)$ *be the number of isolated* μ_p*-fixed points at which a generator of* μ_p *can be diagonalized as* (ζ, ζ^i) *with* ζ *a primitive pth root of* 1 (*so* $s = \Sigma_i k_i$). *Then*

$$1 = \sum_j \left(\frac{1 - g(R_j)}{2} + \frac{(p+1)R_j^2}{12} \right) + \frac{1}{p-1} \sum_{i=1}^{p-1} \sum_{\zeta} k_i/(1-\zeta)(1-\zeta^i)$$

$$= \sum_j \left(\frac{1 - g(R_j)}{2} + \frac{(p+1)R_j^2}{12} \right) + k_1 \frac{5-p}{12} + k_2 \frac{11-p}{24}$$

$$+ k_3 a_3 + k_4 a_4 + \cdots,$$

where ζ *runs over the set of primitive pth root of* 1, *where* $a_3 = 1/4$, $a_4 = 1/2$, *when* $p = 5$.

Proof. Applying the topological fixed-point formula, we obtain

$$s + \sum_i (2 - 2g(R_i)) = \chi_{top}(X^h) = \sum_i (-1)^i Tr(h^* \mid H^i(X, \mathbf{C}))$$

$$= 2 + c + tTr(M_p) = 2 + c - t.$$

The Picard number $\rho(X)$ is $10 - K_X^2$ and also equals $c + t(p-1)$. So (1) and (2) are proved.

By the holomorphic Lefschetz fixed point formula [ASIII, p. 567], one has

$$1 = \sum_i (-1)^i Tr(h^* \mid H^i(X, \mathcal{O}_X))$$

$$= \sum_i 1/\det(1 - h \mid T_{p_i}) + \sum_j (1 - g(R_j))/(1 - \zeta_p^{n_j})$$

$$- \sum_j R_j^2 \zeta_p^{n_j}/(1 - \zeta_p^{n_j})^2,$$

where T_{p_i} is the tangent space of X at p_i, h is the generator of μ_p, and h^* acts on the normal bundle of R_j by a multiple $\zeta_p^{n_j}$ (a primitive pth root of 1). Letting h run in the set of generators of μ_p and taking sums for both sides of the above equality, to prove (3) we only need to show

$$(p-1)/2 = \sum_{i=1}^{p-1} 1/(1 - \zeta_p^i), \qquad (1 - p^2)/12 = \sum_{i=1}^{p-1} \zeta_p^i/(1 - \zeta_p^i)^2.$$

These can be checked by using $p = (1-x)(x^{p-2} + 2x^{p-3} + \cdots + (p-2)x + p - 1)$ with $x = \zeta_p$ to get rid of the denominators. The equalities above were originally calculated by Cay Horstman and were kindly brought to our attention by Jonghae Keum.

In Lemmas 1.7–1.10 below, except Lemma 1.7(4) and Lemma 1.8, we assume only that X is a smooth rational surface and G a non-trivial finite group acting on it.

LEMMA 1.7. *Suppose that* $\operatorname{rank}(\mathit{Pic}\,X)^G \otimes \mathbf{Q} = 1$. *Then we have*:

(1) *X is a del Pezzo surface. Hence $d = K_X^2$ satisfies $1 \le d \le 9$. $d = 9$ if and only if $X = \mathbf{P}^2$; $d = 8$ if and only if X is the Hirzebruch surface $\mathbf{F}_e$ with $e = 0$ or $e = 1$; $d \ge 2$ if and only if X is the blow-up of $\mathbf{P}^2$ at $9 - d$ points in general position.*

(2) *One has $\mathit{Pic}\,Y = \mathbf{Z}P$ (see Definition 1.4). If $K_X^2 \le 7$, then $(\mathit{Pic}\,X)^G = \mathbf{Z}K_X$.*

(3) *$-K_Y$ is $\mathbf{Q}$-ample. A general member of $|P|$ is smooth and irreducible, which does not pass through the singular locus of Y; one has also $2g(P) - 2 = P.(K_Y + P)$.*

(4) *Suppose further that $G = \mu_p$. Then X^{μ_p} is either a finite set, or a union of a smooth irreducible curve R and finitely many points.*

Proof. Clearly, both the pull-back H on X of an ample divisor on Y and $-K_X$ are generators of the rank one $\mathbf{Q}$-module $(\mathit{Pic}\,X)^G \otimes \mathbf{Q}$. Noting that the Kodaira dimension $-\infty = \kappa(X) < 2$, $-K_X$ is a positive multiple of H and (1) follows [Man2].

The first part of (2) is true because $\mathit{Pic}\,Y$ is a rank one lattice and $-K_Y$ is ample (see (3)). Let C be any G-stable Cartier divisor. Then $C = (m/n)(-K_X)$ for some coprime positive integers. Intersecting this with a (-1)-curve E on X, one obtains $n(C.E) = m$ and $n \mid m$, whence $n = 1$ and (2) is proved.

For (4), if X^{μ_p} contains two (disjoint) curves R_1, R_2 then both R_i are positive multiples of H and this leads to that $0 = R_1.R_2 = H^2 \times$ (a positive number), a contradiction.

For (3), since $\rho(Y) = 1$, either K_Y or $-K_Y$ is $\mathbf{Q}$-ample. By the ramification formula (similar to the one in Lemma 1.2) and the fact that the divisor $-K_X$ is ample, we see that $-K_Y$ is $\mathbf{Q}$-ample.

By the main theorem of [Am] or [Alex], $\dim |P| \ge 1$ and (a general member) P is smooth irreducible. Since P is Cartier and Y has at worst rational singularities (for the second equality) and by the Riemann–Roch theorem (for the third) one has (cf. [Art, Theorem 2.3])

$$\begin{aligned}\chi(\mathcal{O}_P) &= \chi(\mathcal{O}_Y) - \chi(\mathcal{O}_Y(-P)) = \chi(\mathcal{O}_Z) - \chi(\mathcal{O}_Z(-f^*P))\\ &= \frac{-1}{2} f^*P.(K_Z + f^*P)\\ &= \frac{-1}{2}(P' + D)(K_Z + P') = \chi(\mathcal{O}_{P'}) + \frac{-1}{2}D.(K_Z + P'),\end{aligned}$$

where P' is the proper transform on Z of P and D an effective divisor with support in $f^{-1}(\operatorname{Sing} Y)$ such that $f^*P = P' + D$. Since $P \cong P'$, we get $0 = D.(K_Z + P') \geq D.P'$, where the inequality is true because each component of D is a $(-n)$-curve with $n \geq 2$ [Bri, Satz 2.11]. Thus $D^2 = D.f^*P = 0$ and $D = 0$ for D is negative definite [Mum, p. 230]. So $f^*P = P'$ and P is away from the singular locus of Y. Now $2g(P) - 2 = 2g(P') - 2 = f^*P.(K_Z + f^*P) = P.(K_Y + P)$ because $f_* K_Z = K_Y$.

LEMMA 1.8. *Suppose that the quotient surface $Y = X/\mu_p$ satisfies $\rho(Y) = 1$ and $r(Y) < 1$. Then the fixed locus X^{μ_p} is a finite set.*

Proof. Write $-K_Y = rP$ with $r = r(Y) < 1$. Suppose the contrary that X^{μ} contains an irreducible curve R. Then X^{μ} is a union of R and points p_i (Lemmas 1.2 and 1.7). Note that $B = \sigma(R)$ is away from $\operatorname{Sing} Y$ (Lemma 1.2), is Cartier, and satisfies $\sigma^*B = pR$. Write $B = bP$, where $b \geq 1$ by the maximality of r. Then the ramification formula implies that

$$-K_X = -(\sigma^*K_Y + (p-1)R) = [rp - b(p-1)]/p\sigma^*P.$$

Since $-K_X$ is ample (Lemma 1.7), $r > b(p-1)/p \geq (p-1)/p$. This and the fact that $r = m/p$ with an integer m would imply that $m \geq p$ and $r \geq 1$. This contradiction proves the lemma.

The following two results are essentially proved in [Fuj, Chap. 1, Sect. 5]. For the convenience of the reader we give a kind of new proof here.

LEMMA 1.9. *Suppose that the quotient surface $Y = X/G$ satisfies $r(Y) = 1$. Then Y is Du Val (i.e., Gorenstein in the present case) and a general member of $|-K_Y|$ is a smooth elliptic curve which does not pass through the singular locus of Y.*

Proof. In view of Lemma 1.7, we only need to show that Y is Gorenstein and $P \sim -K_Y$. By the assumption $K_Y + P$ is **Q**-linearly equivalent to zero. Then $0 \sim_{\mathbf{Q}} f^*(K_Y + P) = (K_Z + P') + \Delta$; here $P' = f^*P$ is a smooth curve with $p_a(P') = 1 > 0$ (cf. Subsection 1.3 and Lemma 1.7), and hence the Riemann–Roch theorem implies that $|K_Z + P'| \neq \emptyset$. Thus $\Delta = 0$, whence Y has only Du Val singularities (cf. Subsection 1.3). Finally, since the two Cartier divisors $-K_Y$ and P are **Q**-linear equivalent, they are linear equivalent because the rational surface Y is simply connected and hence $Pic\, Y$ is torsion free. The lemma is proved.

LEMMA 1.10. *Suppose that the quotient surface $Y = X/G$ satisfies $r = r(Y) > 1$. Then (a general member) P is a smooth rational curve away from the singular locus of Y. Moreover, $(r-1)P^2 = 2$.*

Proof. Substituting $-K_Y = rP$ into the equality in Lemma 1.7, we get $2g(P) - 2 = (1-r)P^2 < 0$. Thus $g(P) = 0$ and the current lemma follows from Lemma 1.7.

2. EXAMPLES

In this section, we shall construct examples of pairs (X, μ_p) (see Theorems 1 and 4 in the Introduction).

2.1. Suppose that μ_p acts faithfully on $X = \mathbf{P}^2$ with homogeneous coordinates X, Y, Z. Then one can diagonalize a suitable generator g of μ_p as one of the following, where $\zeta_p = exp(2\pi\sqrt{-1}/p)$,

2.1a. $g = \mathrm{diag}\left[1, 1, \zeta_p\right]$; 2.1b. $\mathrm{diag}\left[1, \zeta_p, \zeta_p^{\,v}\right]$ $(2 \le v \le p-1)$.

In 2.1a, X^μ is a union of the line $Z = 0$ and the point $p_1 = [0,0,1]$. This p_1 dominates a singularity q_1 of $Y := X/\mu_p$ of type $\frac{1}{p}(1,1)$. It is easy to see that Y is the projective cone $\overline{\mathbf{F}}_p$ with the vertex at q_1. The $\mathbf{Z}/(p)$-covering map $\sigma: X \to Y$ is branched along the vertex and a smooth hyperplane B ($\sim \overline{H}$ in notation of Remark 1.5). One has $r(Y) = (p+2)/p$ (Remark 1.5).

In 2.1b, one must have $p \ge 3$ and X^μ is a union of three points $p_1 = [1,0,0]$, $p_2 = [0,1,0]$, $p_3 = [0,0,1]$. These p_i dominate singular points q_i of $Y := \mathbf{P}^2/\mu_p$. The q_i are respectively of type $\frac{1}{p}(1,v)$, $\frac{1}{p}(1, p+1-v)$, and $\frac{1}{p}(1,u)$ with $uv \equiv v - 1 \pmod p$. One has $\pi_1(Y^0) = \mu_p$ (Lemma 4.4) One sees also that Y is Du Val if and only if $p = 3$; if this is the case then $g = \mathrm{diag}[1, \zeta_3, \zeta_3^2]$ and Y is a Gorenstein log del Pezzo surface of rank 1 with 3 type $\frac{1}{3}(1,2)$ singularities and also $r(Y) = 1$ [MZ1, Lemma 6]. If $p \ge 5$, then $r(Y) = 1/p, 3/p$ (Lemma 1.9 and Propositions 3.1 and 3.3).

2.2. Let $Y = \mathbf{P}^2$, $p = 3$, and $\sigma: X \to Y$ the triple cover totally branched along a smooth plane cubic B; X is a del Pezzo surface with $K_X^2 = 3$ and $R \in |-K_X|$, where $\sigma^* B = 3R$.

2.3. Let $Y = \overline{\mathbf{F}}_p$, $p = 3$, and $\sigma: X \to Y$ be the triple cover branched along a smooth genus-2 curve B and the vertex q_1 ($\notin B$) of the cone Y.

Then X is a del Pezzo surface with $K_X^2 = 1$ and $|-K_X|$ contains 6 members of cuspidal rational curves lying over the 6 generating lines of the cone Y tangent to the branch curve B. Blowing up the unique base point of $|-K_X|$ [Dem, Proposition 2, p. 40] with E the exceptional curve, one gets a relatively minimal rational elliptic surface $\varphi : \tilde{X} \to \mathbf{P}^1$ with a section E and six type II singular fibres; so the Mordell Weil group of the fibration φ is torsion free and of full rank 8. There is an induced μ_3-action on $\tilde{X}$ fixing (point wise) the section E.

2.4 (the rows 5 and 6 of Table I). Here we construct a 1-dimensional family (X_s, μ_5) $(s \in \mathbf{P}^1 \backslash \{0, \pm 1\})$ and a unique pair (X_{II}, μ_5), where each surface is a degree 1 del Pezzo surface on which μ_5 acts faithfully and fixes (point wise) a smooth member in the anti-canonical linear system. When $X = X_s$ (resp. $X = X_{II}$), $|-K_X|$ has 10 nodal members forming two μ_5-orbits, and one cuspidal member (resp. 5 + 1 cuspidal members forming two μ_5-orbits).

Let $\tilde{Z} \to \mathbf{P}^1$ with $\tilde{Z} = \tilde{Z}_I$ (resp. $\tilde{Z} = \tilde{Z}_{II}$) be the unique elliptic surface with (only one) section E, a type II^* fibre $\tilde{Z}_{t=0}$, and two type I_1 fibres at $t = \pm 1$ (resp. a single type II fibre at $t = \infty$) [MP, Theorem 5.4]. Express the type II^* fibre as $D_1 + 5C + D_2$, where C is a (-2)-curve, and $\operatorname{Supp} D_i$ are *the* two disjoint chains of (-2)-curves of length 4 so that the section E meets a tip component of D_2. Let $\tilde{Z} \to Y$ (later referred to as $Y = Y_I$, $Y = Y_{II}$, respectively) be the contraction of $E + D_2$, D_1 to a smooth point q and a type $\frac{1}{5}(1,4)$ singular point q_1; both points lie on the image of C (also denoted by C).

Let B be the image on Y of a smooth fibre $\tilde{Z}_{t=s}$, $s \neq 0, \pm 1$ (resp. $\tilde{Z}_{t=1}$) when $Y = Y_I$ (resp. $Y = Y_{II}$). Since fibres on $\tilde{Z}$ are linearly equivalent, pushing down, we get an induced relation $\mathscr{O}_Y(C)^{\otimes 5} \cong \mathscr{O}_Y(B)$. This gives rise to a $\mu_4 \cong \mathbf{Z}/(5)$-Galois cover,

$$\sigma : X = Spec \bigoplus_{i=0}^{4} \mathscr{O}_Y(-iC) \to Y,$$

(referred to as $X = X_s$, $X = X_{II}$, respectively) which is etale outside the smooth elliptic B in $|-K_Y|$ and the only singularity q_1 of Y; along B the map σ is totally branched.

2.4.1. Conversely, suppose that $5\overline{M} \sim B$ is a relation on Y with $\overline{M}$ a Weil divisor. We now show that $\mathscr{O}_Y(\overline{M}) \cong \mathscr{O}_Y(C)$. Pulling it back by the minimal resolution $f : Z \to Y$ ($\tilde{Z} \to Z$ is the contraction of $E + D_2$), one has $5M + D_1' \sim B$ on Z, where M is mapped to $\overline{M}$, the same B denotes its preimage on Z and D_1' is supported on the support of (the image of Z of) D_1. On Z, (the image of) C satisfies $C.B = 1$, while each component of

D_1 has zero intersection with B. Using these to intersect the above relation with C and components of D_1, one sees that $D'_1 - D_1 = 5D$ with D a $\mathbf{Z}$-combination of irreducible components of D_1. Thus $5(M + D) + D_1 \sim B$ on Z. On the other hand, one has $5C + D_1 \sim B$ on Z. These relations imply that $5(M + D - C) \sim 0$ and hence $M + D \sim C$ for the rational surface Z has torsion free $\mathit{Pic}\, Z$. Passing to Y, one gets $\mathcal{O}_Y(\overline{M}) \cong \mathcal{O}_Y(C)$.

It is easy to see that X is smooth and $-K_X = \sigma^*(-K_Y - 4C) = \sigma^*(5C - 4C) = \sigma^* C$ so that $-K_X$ is nef and big with $K_X^2 = 1$ because $C^2 = 1/5$ on Y; every member F $(\neq B, 5C)$ in $|-K_Y|$ has total transform on X splitting into 5 elliptics meeting at the unique point lying over q $(= B \cap C)$, while $\sigma^* B = 5R$ with a smooth elliptic $R \in |-K_X|$ and $\sigma^* C$ is a cuspidal curve in $|-K_X|$ with a cusp at the point $p_1 = \sigma^{-1}(q_1)$. Thus σ is totally ramified exactly along R and the point p_1, and X is a del Pezzo surface with $X^{\mu_5} = R \coprod \{p_1\}$; indeed, X has no (-2)-curves and the only singular members $(\neq \sigma^* C)$ in $|-K_X|$ are 10 nodal curves lying over the two type I_1 fibres on $\tilde{Z}_I$ (resp. five cuspidal curves lying over the type II fibre on $\tilde{Z}_{II}$).

Finally, we have rank $\mathit{Pic}\, Y = 1$ and $K_Y^2 = 5$ (noting that Y is Du Val). One has also $r(Y) = 1$ and $\pi_1(Y - \mathrm{Sing}\, Y) = (1)$ [MZ, Lemma 6].

2.5 (the row 7 of Table I). We shall calculate X^{μ_5} and determine the type of singularities of $Y = X/\mu_5$ for the unique pair (X, μ_5), where X is the unique del Pezzo surface with $K_X^2 = 5$ (Lemma 2.13).

Let $\tilde{X} \to \mathbf{P}^1$ be a relatively minimal elliptic surface with a section and with two type I_5 fibres F_1, F_2; such an $\tilde{X}$ is unique and the fibration has the Mordell Weil group $\mathbf{Z}/(5)$ [MP, Theorem 5.4]; it has also two type I_1 singular fibres. Using Shioda's height pairing [Sh, Theorem 8.6], one can verify that the five sections P_i $(0 \le i \le 4)$ are disjoint and meet distinct fibre components of F_1, F_2. Let $\tilde{X} \to X$ be the blow-down of the sections P_i. Then X is the del Pezzo surface with $K_X^2 = 5$. The translation automorphism given by the section P_1 induces an automorphism g on X so that X^{μ_5} consists of two points (the images of the nodes of the two type I_1 fibres on $\tilde{X}$), where $\mu_5 = \langle g \rangle$. Set $Y = X/\mu_5$ and $Y^0 = Y - \mathrm{Sing}\, Y$. Then Y is a Gorenstein log del Pezzo surface with two type $\frac{1}{5}(1,4)$ singularities, $K_Y^2 = 1$, rank $\mathit{Pic}\, Y = 1$, and $r(Y) = 1$ (see [MZ, Lemma 6]).

Conversely, one can show that such a kind of singular del Pezzo surface Y is unique modulo isomorphism and that $\pi_1(Y^0) = \mu_5$; hence the quotient map $\sigma : X \to X/\mu_5$ here is the completion for the universal covering map of the smooth part of the *unique* Gorenstein log del Pezzo surface with two type $\frac{1}{5}(1,4)$ singularities (see Lemma 4.4).

2.6. μ_p ($p = 2$) acts on $X = \mathbf{P}^1 \times \mathbf{P}^1$ by $(x, y) \mapsto (y, x)$ (switching the fibrations). One sees that $Y = X/\mu_p$ is $\mathbf{P}^2$ and the quotient map $\sigma : X \to Y$ is branched along a smooth conic, whose inverse on X is the diagonal. This μ_p is birationally equivalent to *De Jonquieres involution* of degree 2 [BB, Example 1.6].

2.7. Let $Y = \mathbf{P}^2$, $p = 2$, and $\sigma : X \to Y$ the double cover branched along a smooth quartic curve. μ_p is called *Geiser's involution* on the del Pezzo surface X with $K_X^2 = 2$. Conversely, if X is a del Pezzo surface with $K_X^2 = 2$ then $\Phi_{|-K_X|}$ is the σ above [Dem, Chap. V, Sect. 4].

2.8. Let $Y = \overline{\mathbf{F}}_p$ ($p = 2$) and $\sigma : X \to Y$ the double cover branched along a smooth genus-4 curve B and the vertex q_1 ($\notin B$) of the cone Y. Then X is a del Pezzo surface with $K_X^2 = 1$. μ_p is called the *Bertini involution*. Conversely, if X is a del Pezzo surface with $K_X^2 = 1$ then $\Phi_{|-2K_X|}$ is the σ above [Dem, Chap. V, Sect. 5].

2.9 (the row 4 of Table II). We construct a pair (X, μ_6) with X the (unique) del Pezzo surface with $K_X^2 = 6$ and the group μ_6 acting faithfully on X such that X^{μ_6} is a finite set and $Y = X/\mu_6$ has exactly 3 singularities of type $\frac{1}{2}(1,1)$, $\frac{1}{3}(1,2)$, $\frac{1}{6}(1,5)$ (types A_1, A_2, A_5 in other notation) as all of its singularities.

Let $\tilde{X} \to \mathbf{P}^1$ be a relatively minimal rational elliptic fibration with type I_1, I_2, I_3, I_6 singular fibres F_0, F_1, F_2, F_3 and a section P_0. Such an elliptic surface is unique [MP, Theorem 5.4.]. Write $F_1 = C_0 + C_1$, $F_2 = \Sigma_{i=0}^2 D_i$, $F_3 = \Sigma_{j=0}^5 E_j$, with $D_i.D_{i+1} = E_j.E_{j+1} = 1$ so that P_0 meets C_0, D_0, E_0. Using the height-pairing in [Sh, Theorem 8.6], one sees that the Mordell Weil group of the fibration is $\mathbf{Z}/(6)$ and its generator P_1 meets C_1, D_1, E_1 after suitable relabelling.

Denote by g the translation automorphism given by the section P_1. One sees that $\tilde{X}^{\mu_6}$, with $\mu_6 = \langle g \rangle$, consists of 6 points: g (resp. g^3; g^2) fixes the node p_1 (resp. the two nodes p_2, p_3; the three nodes p_4, p_5, p_6) of the type I_1 (resp. I_2; I_3) fibre. Let $\tilde{X} \to X$ be the smooth blow-down of the six (disjoint) sections so that X is the degree 6 del Pezzo surface. Then X has an induced μ_6 action so that X^{μ_6} is still a 6-point set (the image of $\tilde{X}^{\mu_6}$). One sees that $Y = X/\mu_6$ has exactly three singular points of type A_5, A_2, A_1 (the images of $p_1, \{p_2, p_3\}, \{p_4, p_5, p_6\}$), rank $Pic\, Y = 1$, and $K_Y^2 = 1$.

2.10 (the row 5 of Table II). Suppose that $G \cong (\mathbf{Z}/(3))^{\oplus 2}$ acts faithfully on $\mathbf{P}^2$ with coordinates X, Y, Z, so that X^G is a finite set. Take generators g_1, g_2 of G. By the assumption on X^G, one has $g_1 = \mathrm{diag}[1, \zeta_3, \zeta_3^2]$ after a change of coordinates. Now the commutativity of g_1, g_2 in $PGL_2(\mathbf{C})$ implies that $g_2 = (a_{ij})$ with $a_{13}a_{21}a_{32} = 1$ and $a_{ij} = 0$ for other entries. One sees easily that each order 3 subgroup of G fixes exactly 3 points, and $|X^G| = 12$. For instance, $X^{g_1} = \{[1,0,0],[0,1,0],[0,0,1]\}$ and $X^{g_2} = \{[1, a_{21}\zeta_3^i, \zeta_3^{2i}/a_{13}] \mid 0 \le i \le 2\}$.

2.11 (the rows 2 and 6 of Table II). We shall construct:

(1) a pair (X, μ_4) with $X = \mathbf{P}^1 \times \mathbf{P}^1$ and the group μ_4 acting faithfully on X such that X^{μ_4} is a finite set and $Y = X/\mu_4$ has exactly 3 singularities of type $\frac{1}{2}(1,1), \frac{1}{4}(1,3), \frac{1}{4}(1,3)$ (i.e., types A_1, A_2, A_3) as all of its singularities; and

(2) a pair (X, G) with $X = \mathbf{P}^1 \times \mathbf{P}^1$ and the group $G \cong \mu_2 \times \mu_4$ acting faithfully on X such that X^G is a finite set and $Y = X/G$ has exactly 4 singularities of type $\frac{1}{2}(1,1), \frac{1}{2}(1,1), \frac{1}{4}(1,3), \frac{1}{4}(1,3)$ as all of its singularities.

Let $\tilde{X} \to \mathbf{P}^1$ be a relatively minimal rational elliptic fibration with type I_2, I_2, I_4, I_4 fibres F_1, F_2, F_3, F_4 and a section P_0. Such an elliptic surface is unique [MP, Theorem 5.4]. Write $F_1 = C_0 + C_1$, $F_2 = D_0 + D_1$, $F_3 = \Sigma_{i=0}^3 E_i$, and $F_4 = \Sigma_{j=0}^3 G_i$ with $E_i.E_{i+1} = G_j.G_{j+1} = 1$ so that P_0 meets C_0, D_0, E_0, G_0. Using the height-pairing in [Sh, Theorem 8.6], one sees that the Mordell Weil group of the fibration is $\mu_2 \times \mu_4$; after suitable relabelling, two (disjoint) sections P_1, P_2 meet fibres in this way: P_1 meets C_0, D_1, E_1, G_1 and P_2 meets C_1, D_0, E_1, G_3. Clearly P_1, P_2 have order 4 and generate the Mordell Weil group with P_0 as the origin.

Denote by g_i the translation automorphism given by the section P_i. One sees that $\tilde{X}^G$, with $G = \langle g_1, g_2 \rangle \cong \mu_2 \times \mu_4$, consists of 12 points: g_1 (resp. g_2) fixes the two nodes p_1, p_2 (resp. p_3, p_4) of F_1 (resp. F_2), while g_2 (resp. g_1) switches p_1, p_2 (resp. p_3, p_4); $g_2 g_1^{-1}$ (resp. $g_2 g_1$) fixes the four nodes $p_5, \ldots, p_8$ (resp. $p_9, \ldots, p_{12}$) of F_3 (resp. F_4). Let $\tilde{X} \to X$ be the smooth blow-down of the eight (disjoint) sections so that $X = \mathbf{P}^1 \times \mathbf{P}^1$. Then there is an induced G actions so that X^G is still a 12-point set (the image of $\tilde{X}^G$).

One sees that $Y_1 = X/H$ with H ($\cong \mu_4$) generated by (the image) of g_1 has exactly three singular points of type A_3, A_3, A_1 (the images of $p_1, p_2, \{p_3, p_4\}$), $K_{Y_1}^2 = K_X^2/4 = 2$, and rank $Pic\, Y_1 = 1$.

One can also verify that $Y = X/G$ has exactly four singular points of type A_3, A_3, A_1, A_1 (the images of $\{p_1, p_2\}, \{p_3, p_4\}, \{p_5, \ldots, p_8\}, \{p_9, \ldots, p_{12}\}$), rank $Pic\, Y = 1$, and $K_Y^2 = 1$.

LEMMA 2.12. *Let X be a del Pezzo surface with $K_X^2 = 1$ and a faithful μ_5-action such that μ_5 fixes (point wise) a (smooth) elliptic curve $R \in |-K_X|$. Then modulo μ_5-equivariant isomorphism the pair (X, μ_5) is equal to either (X_s, μ_5) $(s \in \mathbf{P}^1 \backslash \{0, \pm 1\})$ or (X_{II}, μ_5) in Example* 2.4.

Proof. It suffices to show that the covering map $X \to Y = X/\mu_5$ here coincides with one in Example 2.4. We shall show that the relative minimal model of the induced elliptic fibration on Y has a type II^* singular fibre and (only one) section. To begin with, let $\tilde{X} \to X$ be the blow-up of the unique base point of $|-K_X|$ with $\tilde{E}$ the exceptional curve. Then $\tilde{X}$ is a relatively minimal elliptic surface with a section $\tilde{E}$. Since X contains no (-2)-curves for $-K_X$ is ample, each fibre of the elliptic fibration on $\tilde{X}$ is irreducible. The induced μ_5-action on $\tilde{X}$ fixes the proper transform of R (also denoted by R, which is a smooth fibre now) and stabilizes $\tilde{E}$. Clearly the rational curve $\tilde{E}$ has exactly two μ_5-fixed points: the intersection $\tilde{E} \cap R$ and one more point p on another fibre F_1.

If F_2 is a singular fibre ($\neq F_1$) then $\{gF_2 \,|\, g \in \mu_5\}$ is a set of 5 singular fibres of the same type. Hence the Euler number $12 = \chi(\tilde{X}) = \chi(F_1) + 5t$ with $t \geq 0$. Thus $\chi(F_1) = 2, 7$ because there should be at least two singular fibres if one calculates the Picard number in terms of contributions from fibres and the rank of the Mordell Weil group [Sh, Corollary 5.3].

Let $\tilde{Z} \to \mathbf{P}^1$ be the smooth relative minimal model of the elliptic fibration on the quotient $\tilde{X}/\mu_5$ induced from the one on $\tilde{X}$. Since μ_5 acts on the base curve of the fibration of $\tilde{X}$ as an automorphism of order 5, we see that if $T(\overline{F}_1)$ is the monodromy of the fibre $\overline{F}_1$ on $\tilde{Z}$ dominated by the fibre F_1 on $\tilde{X}$, then the monodromy $T(F_1)$ equals $T(\overline{F}_1)^5$. This and $\chi(F_1) = 2, 7$ imply that F_1 is of type II and its image $\overline{F}_1$ is of type II^* at $t = 0$ with t the inhomogeneous coordinate of the base curve (we arrange t this way) [BPV, Table 6, p. 159]. So $\tilde{Z}$ is a rational elliptic surface with only one section E (the image of $\tilde{E}$) and we can identify it with either $\tilde{Z}_I$ or $\tilde{Z}_{II}$ in Example 2.4. Then the fibre B on $\tilde{Z}$ dominated by the fibre R is at $t = s$ with $s \neq 0, \pm 1$ (resp. at $s = 1$).

One sees that $\tilde{Z} \to \tilde{X}/\mu_5$ is the contradiction of D_1, D_2 in notation of Example 2.5. Contracting further $\tilde{E}$ on $\tilde{X}$ and (the image of) E on $\tilde{X}/\mu_5$ to get X and $Y = X/\mu_5$, we see that our $\sigma : X \to Y$ is also a $\mathbf{Z}/(5)$-Galois cover totally branched at the only singular point q_1 of Y and the curve B (the image of R). It is known that such a cover is given by a relation $\mathcal{O}_Y(\overline{M})^{\otimes 5} \cong \mathcal{O}_Y(B)$ for some Weil divisor $\overline{M}$. Since $\mathcal{O}_Y(\overline{M}) \cong \mathcal{O}_Y(C)$ by Subsection 2.4.1, our σ here coincides with the one in Example 2.4. This proves the lemma.

LEMMA 2.13. *There is only one pair (X, μ_5) of the del Pezzo surface X with $K_X^2 = 5$ and the group μ_5 acting faithfully on X modulo equivariant μ_5-isomorphism.*

Proof. A degree 5 del Pezzo surface X is the blow-up of 4 points p_i on $\mathbf{P}^2$ (no three of them are collinear), and hence there is only one such X

modulo isomorphism (these 4 points p_i form a frame of $\mathbf{P}^2$, and any other frame is mapped to this by a projective transformation). It is known that $Aut(X)$ is the symmetric group S_5 in 5 letters. Since all sylow-5 groups of S_5 are conjugate to each other, the lemma follows.

LEMMA 2.14. *Modulo equivariant μ_6-isomorphism, there is only one pair (X, μ_6) of the (unique) del Pezzo surface X with $K_X^2 = 6$ and the group μ_6 acting faithfully on X such that X^{μ_6} is a finite set and $Y = X/\mu_6$ has exactly 3 singularities of type $\frac{1}{2}(1,1)$, $\frac{1}{3}(1,2)$, $\frac{1}{6}(1,5)$ as all of its singularities.*

Proof. Note that Y is a Gorenstein log del Pezzo surface with 3 singularities of type A_1, A_2, A_5 in other notation. In view of Lemma 4.4, it is enough to show that there is only one such Y modulo isomorphism. Let $f: Z \to Y$ be the minimal resolution. Then Z is an almost del Pezzo surface with $K_Z^2 = 1$ so that $|-K_Z|$ has exactly one base point, $\dim|-K_Z| = K_Z^2 = 1$ (Riemann–Roch and Kawamata–Viehweg vanishing) and a general member of $|-K_Z|$ is irreducible [Dem, Theorem 1, p. 39]. Let $\tilde{Z} \to Z$ be the blow-up of the unique base point of $|-K_Z|$ with P_0 the exceptional curve. Then $\tilde{Z}$ is a relatively minimal elliptic surface with P_0 as a section. One sees that the inverse of $\operatorname{Sing} Y$ is contained in three different fibres F_1, F_2, F_3 of types I_2, I_3, I_6 [MP, Theorem 4.1]. Now the uniqueness of Y follows from the uniqueness of such an elliptic surface [MP, Theorem 5.4] and also the uniqueness of the pair $(\tilde{Z}, P_0)$ modulo translation automorphism. This proves the lemma.

LEMMA 2.15. (1) *Modulo equivariant H-isomorphism ($H \cong \mu_4$) there is only one pair (X, H) with $X = \mathbf{P}^1 \times \mathbf{P}^1$ and the group H acting faithfully on X such that X^H is a finite set and $Y_1 = X/H$ has exactly 3 singularities of type $\frac{1}{2}(1,1)$, $\frac{1}{4}(1,3)$, $\frac{1}{4}(1,3)$ as all of its singularities and rank $\operatorname{Pic} Y_1 = 1$.*

(2) *Modulo equivariant G-isomorphism ($G \cong \mu_4 \times \mu_2$), there is only one pair (X, G) with $X = \mathbf{P}^1 \times \mathbf{P}^1$ and the group G acting faithfully on X such that X^G is a finite set and $Y = X/G$ has exactly 4 singularities of type $\frac{1}{2}(1,1)$, $\frac{1}{2}(1,1)$, $\frac{1}{4}(1,3)$, $\frac{1}{4}(1,3)$ as all of its singularities.*

(3) *There is a subgroup H_1 of G such that $(X, H_1) = (X, H)$ modulo μ_4-equivariant isomorphism (identity $H = H_1 = \mu_4$) and hence $Y = X/G = (X/H_1)/\overline{G} = Y_1/\overline{G}$, where $\overline{G} = G/H_1 \cong \mu_2$.*

Proof. (1) Note that Y_1 is a Gorenstein log del Pezzo surface of Picard number 1 and with singularities of type A_1, A_2, A_3. It suffices to show such Y_1 is unique (Lemma 4.4). Let $f: Z_1 \to Y_1$ be the minimal resolution. Then $f^{-1}(\operatorname{Sing} Y_1)$ is a disjoint union of linear chains of (-2)-curves of length $1, 3, 3$. As in Lemma 2.14, Z_1 is an almost del Pezzo surface with $K_{Z_1}^2 = 2$, $\dim|-K_{Z_1}| = 2$, and a general member A of $|-K_{Z_1}|$ smooth irreducible [Dem, Theorem 1, p. 39]. Pick up any (-1)-curve D_4' on Z_1 such that $D_4' + f^{-1}(\mathrm{Sing}\, Y_1)$ is a disjoint union of two linear chains of length $1, 7$ (D_4' connects the two length-3 chains). One can find such D_4' by playing with $\mathbf{P}^1$-fibrations, or from [Zh1, Lemmas 3.5, 4.2, 4.3] we see that $(Z_1, f^{-1}(\mathrm{Sing}\, Y_1))$ fits Case (9) in Lemma 4.2 there and the picture at [Zh1, p. 454], and we just let $D_4' = E_2$ in the notation there.

Let $Z_2 \to Z_1$ be the blow-up of the point $A \cap D_4'$ (with A fixed for the time being) with P_2' the exceptional curve. Then Z_2 is again an almost del Pezzo surface with $K_{Z_2}^2 = 1$. As in Lemma 2.14, let $Z_3 \to Z_2$ be the blow-up of the only base point of $|-K_{Z_2}|$ with P_0 the exceptional curve. Then Z_3 is a relatively minimal elliptic surface so that the strict inverse of $D_4' + f^{-1}(\mathrm{Sing}\, Y_1)$ is contained in two different fibres F_1, F_2 of types I_2, I_8 [MP, Theorem 4.1]. Now the uniqueness of Y follows from the uniqueness of such an elliptic surface [MP, Theorem 5.4] and also the uniqueness of the triplet $(Z_3; P_0, P_2)$ modulo translation automorphism, noting that the Mordell Weil group of the fibration is $\mathbf{Z}/(4)$ and when we choose P_0 as the origin then P_2 is the unique element of order 2. This proves (1).

(2) Note that Y is a Gorenstein log del Pezzo surface with 4 singularities of type A_1, A_1, A_3, A_3. As in Lemma 2.14, let $f : Z \to Y$ be the minimal resolution and let $\tilde{Z} \to Z$ be the blow-up of the unique base point of $|-K_Z|$ with P_0 the exceptional curve. Now the uniqueness of Y follows from the uniqueness of such an elliptic surface [MP, Theorem 5.4] and also the uniqueness of the pair $(\tilde{Z}, P_0)$ modulo translation automorphism. This proves (2).

(3) This is shown in Example 2.11.

3. CASE: THE INVARIANT SUBLATTICE IS OF RANK 1

In this section, we consider the case $(\rho(Y) =)$ rank $(Pic\, X)^{\mu_p} = 1$. We first treat the case $r(Y) < 1$.

PROPOSITION 3.1. *Suppose that the quotient surface satisfies $\rho(Y) = 1$ and $r = r(Y) < 1$. Then $p \geq 5$, (X, μ_p) equals a pair in Example* 2.1b, *and $r = 1/p$ or $r = 3/p$.*

Proof. By Lemmas 1.2 and 1.8, X^{μ_p} is a finite set $\{p_1, \dots, p_s\}$ with $s \geq 1$. Set $q_i = \sigma(p_i)$ so that $\mathrm{Sing}\, Y = \{q_1, \dots, q_s\}$. Note that if $X = \mathbf{P}^2$ and $s = 3$ then $p \geq 5$ because $r < 1$ (see Example 2.1). Write $r = m/p$ with an integer $1 \leq m \leq p - 1$.

We shall frequently apply Lemma 1.6 to the extent that $9 - (p-1)(3 - s) = K_X^2$, which is between 1 and 9 because X is del Pezzo (Lemma 1.7).

Since $2g(P) - 2 = P.(P + K_Y) = (p - m)P^2/p$ is an integer (Lemma 1.7), $p \,|\, P^2$. One has also $K_X^2 = (\sigma^* K_Y)^2 = pK_Y^2 = m^2 P^2/p \geq m^2$.

If $m \geq 3$, then $K_X^2 = 9$, $r = 3/p$, $P^2 = p$, and $s = 3$; so the proposition is true. If $m = 2$ and $P^2/p \geq 2$, then $K_X^2 = 8$, $r = 2/p$ and $(p, s) = (2, 2)$, which leads to that $r = 1$, a contradiction. If $m = 2$ and $P^2 = p$, then $K_X^2 = 4$, which leads to $4 = 9 - (p - 1)(3 - s)$, a contradiction.

We now assume that $r = 1/p$. If $s = 3$, then $K_X^2 = 9$, $P^2 = 9p$; so the proposition is true.

Suppose that $s = 1$. Then $K_X^2 = 9 - 2(p - 1)$, whence $p = 2, 3, 5$. In the notation of Lemma 1.6, one has $k_u = s = 1$ for some $1 \leq u \leq p - 1$ and $1 = k_u(1/(p - 1))\Sigma_\zeta 1/(1 - \zeta)(1 - \zeta^u)$, where ζ runs over the set of all primitive pth root of 1. This is impossible by the calculation of the right hand side in Lemma 1.6.

Suppose that $s = 2$. Then $K_X^2 = 10 - p$ and $p = 2, 3, 5, 7$. Since μ_7 cannot act on a cubic del Pezzo (see [Man2, Table 1, p. 176]), $p \neq 7$. Applying Lemma 1.6, we can show that $p = 5$, $k_4 = s = 2$. But then the quotient surface Y is Gorenstein and hence $r(Y) \geq 1$, a contradiction. This completes the proof of the proposition.

Next we consider the case $r(Y) = 1$.

PROPOSITION 3.2. *Suppose that the quotient surface satisfies $\rho(Y) = 1$ and $r(Y) = 1$. Then (X, μ_p) is equal to one of the pairs in Examples* 2.1b (*with $p = 3$*), 2.4, *and* 2.5.

Proof. We note that the minimal resolution Z of Y is neither $\mathbf{P}^2$ nor a Hirzebruch surface $\mathbf{F}_e$, for otherwise, either $Y = Z = \mathbf{P}^2$, or $Y = \overline{\mathbf{F}}_e$ $(e \geq 2)$ because $\rho(Y) = 1$, which would lead to $r(Y) > 1$, a contradiction (Remark 1.5); in particular, $K_Z^2 \leq 7$. If $K_X^2 \geq 8$, then $X = \mathbf{P}^2$ or $X = \mathbf{F}_e$ $(e = 0, 1)$ because X is del Pezzo (Lemma 1.7). If $X = \mathbf{F}_1$ then μ_p stabilizes the (-1)-section and the divisor class of a fibre, which contradicts that $\rho(Y) = 1$ (Lemma 1.2). By the same reasoning, if $X = \mathbf{F}_0 = \mathbf{P}^1 \times \mathbf{P}^1$, then $p = 2$ and μ_p switches the two fibrations, but then $Y = \mathbf{P}^2$ and $r = 3 > 1$, a contradiction (see Example 2.6). If $X = \mathbf{P}^2$, then the proposition is true by Lemma 1.9 (see Example 2.1). So we may assume the following, noting that $f^* K_Y = K_Z$ (Lemmas 1.3 and 1.9):

Condition 3.2.1. Given $1 \leq K_X^2 \leq 7$ and $1 \leq K_Y^2 = K_Z^2 \leq 7$.

Consider first the case X^{μ_p} contains an irreducible curve R. We shall show that this fits Example 2.4. By Lemma 1.7, X^μ is a union of the irreducible (smooth) curve R and s points p_i. As in Lemma 1.8, writing $\sigma(R) = B$ and $B = bP = b(-K_Y)$ with b a positive integer (Lemma 1.7), we get $-K_X = (m/n)\sigma^* P$, where $m/n = [p - b(p - 1)]/p$ with coprime positive integers m, n. So $1 \le b = [1 - (m/n)][p/(p-1)] < p/(p-1) \le 2$. Thus $b = 1$ and $m/n = 1/p$. So B ($\sim -K_Y$) and its (reduced) preimage R are isomorphic elliptic curves. Now $-K_X = (1/p)\sigma^* P \sim R$ and $K_X^2 = P^2/p = K_Y^2/p$. By Lemma 1.6, one has also $K_X^2 = 9 - (p-1)(3-s)$.

Since each $q_i \in \operatorname{Sing} Y$ ($1 \le i \le s$) is Du Val, $D_i := f^{-1}(q_i)$ is a chain of $p-1$ of (-2)-curves. So $\rho(Z) = \rho(Y) + s(p-1) = 1 + s(p-1)$. Thus $K_Y^2 = K_Z^2 = 10 - \rho(Z) = 9 - s(p-1)$, which is an integer between 1 and 7 (cf. Condition 3.2.1). So $2 \le s(p-1) \le 8$. Solving $p[9-(p-1)(3-s)] = pK_X^2 = K_Y^2 = 9 - s(p-1)$, one obtains $s = 3 - 12/(p+1)$. So only $(p, s) = (5, 1)$ is possible. Our pair here is equal to the pair in Example 2.4 modulo μ_5-equivariant isomorphism (Lemma 2.12).

Next we consider the case where X^{μ_p} is a finite set $\{p_1, \dots, p_s\}$ with $s \ge 1$ (Lemma 1.2). Then $-K_X = \sigma^* K_Y$ and $K_X^2 = pK_Y^2$. As above, one obtains $9 - (p-1)(3-s) = K_X^2 = pK_Y^2 = p[9 - s(p-1)]$ with $2 \le (p-1)(3-s) \le 8$ (cf. 3.2.1), $s = 12/(p+1)$, and $(p, s) = (5, 2)$. Our pair here is now equal to the pair in Example 2.5 modulo μ_5-equivariant isomorphism (Lemma 2.13). This completes the proof of the proposition.

Now we treat the case were $r(Y) > 1$.

PROPOSITION 3.3. *Suppose that the quotient surface satisfies $\rho(Y) = 1$ and $r(Y) > 1$. Then (X, μ_p) is equal to one of the pairs in Examples* 2.1a, 2.2, 2.3, 2.6, 2.7, 2.8.

Proof. By Lemma 1.10 (a general member) P is a smooth rational curve away from the singular locus of Y. Let $P' = f^* P$ and $m = P^2$. Applying the cohomology exact sequence arising from the exact sequence,

$$0 \to \mathcal{O}_Z \to \mathcal{O}(P') \to \mathcal{O}_{P'}(m) \to 0,$$

we obtain $h^0(Z, P') = (P')^2 + 2$. As long as L is a smooth rational curve with $L^2 \ge 0$ on a smooth rational surface, one always has $h^0(L) = L^2 + 2$; thus by induction on L^2 (to reduce to $L^2 = 0$ case) one can deduce that $Bs|L| = \emptyset$. See [DZ, Lemma 1.7].

So the linear system $|f^* P|$ gives rise to a well-defined morphism $\Phi : Z \to \mathbf{P}^{m+1}$, with the image W a non-degenerate surface (noting that $P^2 > 0$) and hence $\deg W \ge m$. On the other hand, $m = (P')^2 = (\deg \Phi)(\deg W)$. Thus Φ is a birational morphism onto a degree m surface in $\mathbf{P}^{m+1}$. Clearly, Φ factors as $\varphi \circ f$, with a birational morphism $\varphi : Y \to W$ which is given by the linear system $|P|$. Since P is ample, φ is an isomorphism (W is normal; see below). So we can identify $Y = W$.

Non-degenerate surfaces W of degree m in $\mathbf{P}^{m+1}$ are well classified (cf. [Nag]). W is either $\mathbf{P}^2$ ($m = 1$), or the Veronese embedding of $\mathbf{P}^2$ in $\mathbf{P}^5$

($m = 4$), or the embedding (with the negative section C contracted if $a = n = m$) of the Hirzebruch surface $\mathbf{F}_n$ in $\mathbf{P}^{m+1}$ by the linear system $|C + aF|$, where $m = 2a - n$, $a \geq n$, C is the section with $C^2 = -n$, and F a fibre. If W is smooth then $Z = Y = W \cong \mathbf{P}^2$ because $\rho(Y) = 1$. If W is singular, then $a = n = m \geq 2$, $Z = \mathbf{F}_m$, and $Y = \overline{\mathbf{F}}_m$, the projective cone (see Remark 1.5). Clearly, $m = p$ and the only singularity in $\overline{\mathbf{F}}_p$ is of type $\frac{1}{p}(1,1)$.

Suppose that $Y = \mathbf{P}^2$. Then X^{μ_p} is a single smooth curve R (Lemma 1.7). Let d be the degree in Y of $B = \sigma(R)$. Then the $\mathbf{Z}/(p)$-Galois cover $\sigma : X \to Y$ is given by a relation $B \sim pF$. Set $d = \deg(B) = p\deg(F)$. Now $K_X = \sigma^*(K_Y + (p-1)F) = [(p-1)(deg F) - 3]\sigma^* P$, where P is a line. Since $-K_X$ is ample, $(p, \deg F) = (2,1), (2,2), (3,1)$. Thus (X, μ_p) is as in Examples 2.6, 2.7, and 2.2.

Suppose that $Y = \overline{\mathbf{F}}_p$. If X^{μ_p} contains no curve, then it is a single point p_1 and σ is unramified over $Y - \{q_1\}$, where $q_1 = \sigma(p_1)$ is the vertex of the cone Y; this is impossible because $Y - \{q_1\}$ is simply connected. Write $B = bP$. This P is the generator of $Pic\, Y$, is the hyperplane of $Y \subseteq \mathbf{P}^{p+1}$, and satisfies $P^2 = p$ ($P = \overline{H}$ in notation of Remark 1.5). In the present case, $b \geq 1$ is an integer for $B \in Pic\, Y$. As in Lemma 1.8, $-K_X = [(rp - b)(p-1)]/p\sigma^* P = [(p+2) - b(p-1)]/p\sigma^* P$, noting that $r = (p+2)/p$ (Remark 1.5). Since $-K_X$ is ample, either $b = 1$, or $(b, p) = (2,2), (2,3), (3,2)$.

Note that the $\mathbf{Z}/(p)$-Galois cover $\sigma : X \to Y$ is totally branched along the smooth curve B and the vertex q_1 ($\notin B$). If $b = 1$, then one sees easily that (X, μ_p) is equal to a pair in Example 2.1a.

If $(b, p) = (2,2)$, one can verify that the $\sigma^{-1}(q_1)$ has to split into two singularities of the same type, i.e., σ is not branched at q_1, a contradiction. If $(b, p) = (2,3)$ or $(3,2)$ then (X, μ_p) is as in Example 2.3 or 2.8. This proves Proposition 3.3.

4. THE PROOFS OF THEOREMS AND COROLLARIES

Let (X, G) be a pair with X a smooth projective rational surface and G a finite group acting effectively on X. We follow the approach in [BB] using the Mori theory. The cone theory [Mor, Theorems 1.5 and 2.1] implies the decomposition of the closed cone of effective cycles with coefficients in $\mathbf{R}$ and modulo numerical equivalence,

$$\overline{NE}(X) = \overline{NE}(X)_{K_X \geq 0} + \sum_{C \in \mathscr{E}} \mathbf{R}_+ [C],$$

where $\mathscr{E}$ is a countable set of smooth rational curves C satisfying $C^2 = -1, 0, 1$. Passing to the G-invariant part, we get [Mor, Proposition 2.6]

$$\overline{NE}(X)^G = \overline{NE}(X)^G_{K_X \geq 0} + \sum_{C \in \mathscr{F}} \mathbf{R}_+ \left[\sum_{g \in G} gC \right],$$

where $\mathscr{F}$ is the subset of curves C in $\mathscr{E}$ such that $\mathbf{R}_+[\Sigma_{g \in G} gC]$ is an extremal ray in the G-invariant cone $\overline{NE}(X)^G$.

For a curve C on X, denote by G_c the maximum subgroup of G stabilizing C and let k_c be the index $|G : G_c|$ and $\{g_i G_c \mid 1 \leq i \leq k_c\}$ (with $g_1 = \mathrm{id}$) the k_c cosets. For the lemma below, we are essentially proceeding along the idea in [Mor, Theorem (2.7)].

LEMMA 4.1. *Assume that (X, G) is a minimal pair such that*

$$\operatorname{rank}(\mathit{Pic}\, X)^G \geq 2.$$

Then there is a G-stable cone fibration $\varphi : X \to \mathbf{P}^1$ with a smooth rational curve as its general fibre, such that every singular fibre is a linear chain of two (-1)-curves. If φ is not smooth, i.e., X is not a Hirzebruch surface, then $|G|$ is even.

Proof. Since X is rational, K_X has negative intersection with a curve E and hence with the G-stable effective 1-cycle $\Sigma_{g \in G} gE$. So the cone $\overline{NE}(X)^G$ has an extremal ray $\mathbf{R}_+[L]$ where $L = \Sigma_{g \in G} gC$ with a smooth rational curve C.

Note that $L_{\mathrm{red}} = \Sigma_{i=1}^k g_i C$ and $L = |G_c| L_{\mathrm{red}}$, where $k = k_c$. First $L^2 \leq 0$, for otherwise L would belong to the interior of $\overline{NE}(X)^G$ [Mor, Lemma (2.5)] and L could not be extremal; here we use the fact that $\operatorname{rank}(\mathit{Pic}\, X)^G \geq 2$. Since $L^2 \leq 0$, we have $C^2 \leq 0$; and if $C^2 = 0$ then L_{red} is a disjoint union of k smooth rational curves of self intersection 0 and $\varphi := \Phi_{|C|}$ is a G-stable $\mathbf{P}^1$-fibration.

Consider the case $C^2 \leq -1$. Then C is a (-1)-curve for $C.K_X < 0$. If $k = 1$, then C is G-stable, a contradiction to the minimality of the pair. So $k \geq 2$. Now $0 \geq L^2 = |G|(C.L) = |G|\,|G_c|(C.L_{\mathrm{red}}) = |G|\,|G_c|(-1 + \Sigma_{i \geq 2} C.g_i C)$. If $\Sigma_{i \geq 2} C.g_i C = 0$ then L_{red} is a disjoint union of k of (-1)-curves which contradicts the minimality of the pair. So we may arrange $C.g_2 C = 1$ and $C.g_i C = 0$ $(i \geq 3)$. Since $g_2^{-1} C.C = 1$, one has $g_2^{-1} C = g_2 C$ and $g_2^2 \in G_c$ because C meets only one component $g_2 C$ among $g_i C$ $(2 \leq i \leq k)$.

If $k \geq 3$, we see that $g_3(C + g_2 C)$ is a linear chain of two intersecting (-1)-curves disjoint from $C + g_2 C$. We can easily arrange L_{red} as a disjoint unions of pairs of intersecting (-1)-curves $g_i(C + g_2 C)$ $(1 \leq i \leq k/2)$ so that an arbitrary element of G either stabilizes each of two components in $C + g_2C$, switches them, or maps them to some $g_i(C + g_2C)$. Thus $\varphi = \Phi_{|C+g_2C|}$ is a G-stable $\mathbf{P}^1$-fibration. Note that 2 divides $k = k_c = |G : G_c|$ and hence $|G|$.

To finish the proof, we still need to determine the singular fibres of the G-stable conic fibration φ. Take any (-1)-curve E in a fibre of φ and set $L = \Sigma_{g \in G} gE$. Then $L^2 \le 0$ because L is supported by fibres and hence negative semi-definite [Re2, A.7]. Now the same argument above will imply the lemma.

COROLLARY 4.2. *Assume that p is an odd prime number. Let (X, μ_p) be a minimal pair with X a smooth projective rational surface and μ_p acting faithfully on X. Suppose that $(\mathrm{Pic}\, X)^{\mu_p}$ has rank ≥ 2. Then X is a Hirzebruch surface $\mathbf{F}_e$ $(e \ne 1)$ and every ruling (there are two only when $e = 0$) is μ_p-stable.*

Proof. By Lemma 4.1, $X = \mathbf{F}_e$. If $e = 1$, then the unique (-1)-curve would be μ_p-stable and we reach a contradiction to the minimality assumption. The rest is clear.

Before we proceed to prove theorems, we need two results.

LEMMA 4.3. *Suppose that the group μ_p of prime order p acts faithfully on the Hirzebruch surface $X = \mathbf{F}_e$ and stabilizes its fixed ruling φ. Then X^{μ_p} is one of the following, where μ_p stabilizes exactly two out of all fibres in the first three cases,*

(1) *a union of two fibres,*

(2) *a union of a fibre and two points in another fibre,*

(3) *four points in two distinct fibres, and*

(4) *a disjoint union of two sections (one of which is the $(-e)$-section if $e > 0$).*

Proof. This result must be well known but we do not have a reference. Suppose that X^{μ_p} is contained in fibres. Note that there is an induced μ_p-action on the base rational curve of the ruling. So either μ_p stabilizes exactly two fibres (then Case (1), (2), or (3) of the lemma occurs), or μ_p stabilizes all fibres. If the second situation happens, then $X = \mathbf{F}_0 = \mathbf{P}^1 \times \mathbf{P}^1$, for otherwise the $(-e)$-section is μ_p-stable only and has exactly two μ_p-fixed points so that μ_p would stabilize only the two fibres containing these two points; on the other hand, μ_p stabilizes also the second ruling as well as its fibre F_2 through a point in X^{μ_p} (which is non-empty; see Lemma 1.2), so that F_2 is a section of the first ruling and has exactly two μ_p-fixed points, a contradiction again.

Next we consider the case where X^{μ} contains a (multi-)section. Then each fibre is μ_p-stable. Thus either X^{μ} is the union of two disjoint sections (one of which is the $(-e)$-curve if $e > 0$) so that Case (4) of the lemma occurs, or X^{μ} is the union of a double section D and a few points (a general fibre of the ruling has exactly two μ_p-fixed points). In the second case, $X = \mathbf{F}_0$ and μ_p stabilizes also the second ruling. Since D intersects all fibres of both rulings, an arbitrary fibre of any ruling is μ_p-stable, whence the diagonal of X (a section of both rulings) is also contained in X^{μ_p}, a contradiction to the assumption that X^{μ_p} is the union of a double section D of the first ruling φ and a few points. This proves the lemma.

LEMMA 4.4. *For $i = 1, 2$, let (X_i, G) be a pair of a simply connected smooth algebraic surface (e.g., a rational surface) and a finite group G acting faithfully on X_i such that X_i^G is a finite set. Let $Y_i = X_i/G$ and $Y_i^0 = Y_i -$* Sing Y_i. *Then we have*:

(1) *One has $\pi_1(Y_i^0) = G_i$ and the quotient map $\sigma_i : X_i \to Y_i$ is the completion of the universal covering map $U_i^0 \to Y_i^0$; in other words, X_i is the normalization of Y_i in the function field $\mathbf{C}(U_i^0)$.*

(2) *Two pairs (X_i, G) are equal modulo G-equivariant isomorphism if and only if the Y_i are isomorphic to each other.*

Proof. Since X_i with a few points removed is still simply connected, (1) follows. Part (2) is a consequence of (1).

Now we prove Theorem 1. Theorem 1(II) is a consequence of Propositions 3.1, 3.2, and 3.3. For Theorem 1(I), in view of Corollary 4.2, we only need to show that every pair $(\mathbf{F}_e, \mu_p)$ is μ_p-birationally equivariant to a pair $(\mathbf{P}^2, \mu_p)$ in Example 2.1. This can be proved as in [BB, (2.5)]. For reader's convenience, we give a sketch here. Let φ be as in Corollary 4.2.

Case 4.5. μ_p stabilizes each fibre. When $e > 0$ the unique $(-e)$-section is μ_p-fixed and X^{μ} contains one more disjoint section. We blow up a point p_1 on the second (positive) section and blow down the proper transform of the fibre containing p_1. Then we get a μ_p-birational equivariance between our original pair $(\mathbf{F}_e, \mu_p)$ and a new pair $(\mathbf{F}_{e-1}, \mu_p)$. Inductively we reduce to the case $e = 1$ and further blow down the μ_p-stable (-1)-curve on $\mathbf{F}_1$ to proceed μ_p-birationally equivariantly to a pair $(\mathbf{P}^2, \mu_p)$ in Example 2.1.

Case 4.6. μ_p acts non-trivially on the set of fibres (and hence on the base rational curve of ruling). Then there are exactly two μ_p-stable fibres (lying over two μ_p-fixed points of the base rational curve; see Lemma 4.3). Each stable fibre contains at least two μ_p-fixed points. We blow up the one not lying on the $(-e)$-curve and then blow down the proper transform of the fibre; we reduce to a pair $(\mathbf{F}_{e-1}, \mu_p)$. The rest is the same as in Case 4.5.

For both Cases 4.5 and 4.6, when $e = 0$, the argument is similar (see [BB, (2.5)]). This completes the proof of Theorem 1.

For Corollary 2, we first proceed μ_p-birationally equivariantly to a minimal pair and then apply Theorem 1.

For Corollary 3, it suffices to consider minimal pairs. Indeed, start with a pair (X, μ_p) and let $(X_{\min}, \mu_p)$ be a minimal pair with a μ_p-equivariant birational *morphism* $\tau : X \to X_{\min}$; then τ induces a birational morphism $\tau_y : Y = X/\mu_p \to Y_{\min} = X_{\min}/\mu_p$; the images of Sing Y and the τ_y-exceptional divisor form a finite subset Σ of $Y_{\min}$, and $Y^0_{\min} \setminus \Sigma$ can be regarded as a Zariski-open subset of Y^0, whence we have a surjective homomorphism $\pi_1(Y^0_{\min}) = \pi_1(Y^0_{\min} \setminus \Sigma) \to \pi_1(Y^0)$.

Remark 4.7. In particular, if X^{μ_p} contains a curve R with $R^2 \geq 0$ or $g(R) \geq 1$ then the image on $X_{\min}$ or R is still a *curve* in $X^{\mu_p}_{\min}$, whence $\pi_1(Y^0) = \pi_1(Y^0_{\min}) = (1)$ by the statement for minimal pairs.

We now prove Corollary 3 for a minimal pair (X, μ_p). If the lattice $(\mathit{Pic}\, X)^{\mu_p}$ has rank 1, then Corollary 3 is true by Table I in Theorem 1. Suppose that this lattice has rank ≥ 2. Then X is a Hirzebruch surface $\mathbf{F}_e$ and the fixed ruling $\varphi : X \to \mathbf{P}^1$ is μ_p-stable (Corollary 4.2). If X^{μ_p} is a finite set then $X \setminus X^{\mu_p}$ is simply connected and equals the universal cover of Y^0, whence Corollary 3 is true. If X^{μ_p} is a (disjoint) union of smooth curves then $Y = X/\mu_p$ is smooth rational and hence $Y^0 = Y$ is simply connected.

It remains to consider Lemma 4.3, Case (2). Then Y is rational with two singular points q_i (images of two isolated μ_p-fixed points) and a ruling $Y \to \mathbf{P}^1$ (induced from the one on X) such that both q_i are on the same fibre F_1. Thus $Y \setminus F_1$ is a $\mathbf{P}^1$-bundle over the affine line $\mathbf{A}^1$ and hence simply connected. Now the inclusion $Y \setminus F_1 \subseteq Y^0$ induces a surjective map $(1) = \pi_1(Y \setminus F_1) \to \pi_1(Y^0)$, whence $\pi_1(Y^0) = (1)$. This proves Corollary 3.

We prove Theorem 4. Part (I) follows from Lemma 4.1. Next we do (II) and so assume $(\mathit{Pic}\, X)^G$ has rank 1. Then by Lemma 1.7, X is del Pezzo and Y is singular del Pezzo. Now the main theorem in [GZ1, 2] shows that $\pi_1(Y^0)$ is finite (see [FKL] for a differential geometric proof and also [MS] for a new proof). Since the rational surface X with a few points removed, is still simply connected, it is clear that $\pi_1(Y^0) = G$ and $\sigma : X \to Y$ is the completion of the universal covering map of Y^0 provided that X^G is a finite set (Lemma 4.4). Theorem 4(II)(2) follows from the first half of the arguments in Proposition 3.3.

We now prove Theorem 4(II)(1). So assume $r(Y) = 1$ and X^G is finite. Then Y is a Gorenstein log del Pezzo surface of Picard number 1 so that $G = \pi_1(Y^0)$ and $\sigma : X \to Y$ is the completion of the universal covering map of Y^0 (see Lemma 4.4). Such Y is classified in [Fur, Theorem 2; MZ1, Lemma 6]; see also [BBD, p. 593; Ura]. Since our X is smooth, by [MZ1, Table 1, p. 71], (X, G) fits one of the rows in Table II, but the column on X^G; $G = \langle g_1, \dots \rangle$ is still to be verified. For rows 2, 3, 4, 6, this is done in the examples in Section 2 since we have the uniqueness by Lemmas 2.13–2.15. For rows 1 and 5, the generator(s) of G can be easily diagonalized as in Table II (see Examples 2.1 and 2.10). This completes the proof of Theorem 4.

Final Remark 4.8. In [Zh2, Appendix], there are examples of a non-abelian finite group acting faithfully on $X = \mathbf{P}^2$, such that the fixed locus $X^G = \{x \in X \mid gx = x \text{ for some } 1 \neq g \in G\}$ is a finite set. For instance, the non-abelian group of order 21 can act on $\mathbf{P}^2$ this way. Also shown are examples with $\mathbf{P}^2$ replaced by smooth del Pezzo surfaces or projective cones $\overline{\mathbf{F}}_e$; e.g., $X = \mathbf{P}^1 \times \mathbf{P}^1$ with a faithful action by a non-abelian group G of order 16 or 20. See [MM], [MZ3] for new developments.

APPENDIX: WEYL GROUP BASED PROOF OF TABLE I

I. Dolgachev[1]

Let (X, μ_p) be a minimal pair of a smooth rational projective surface and the group μ_p of odd prime order p acting faithfully on X. Write $\mu_p = \langle g \rangle$. Assume that $X \neq \mathbf{P}^2$ and $(Pic\, X)^{\mu_p}$ has rank 1. In this section we shall deduce Table I (the columns on X, X^{μ_p}) in an approach different from Section 3. We shall use the following information:

(1) X is a del Pezzo surface (Lemma 1.7). The minimality and rank assumption of the pair imply that μ_p acts faithfully on the sublattice $M = K_X^{\perp}$ of $Pic\, X$, and also $K_X^2 \leq 5$, noting that there are exactly 3 (resp. 6) (-1)-curves on X when $K_X^2 = 7$ (resp. $K_X^2 = 6$).

(2) The lattice M is isomorphic to the root lattice E_n, where $E_5 = D_5$, $E_4 = A_4$. Here $n = 9 - K_X^2 \geq 4$.

(3) The image g^* of g in $O(E_n)$ belongs to the Weyl group $W(E_n)$.

(4) All conjugacy classes in Weyl groups are known; see the tables in the Atlas of finite groups or [Car].

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(5) E_n embeds naturally into E_{n+1}, corresponding to the natural embeddings of the Dynkin diagrams. Any "old" conjugacy class in $W(E_n)$ coming from $W(E_{n-1})$ leaves a disjoint union of (-1)-curves invariant and then the pair cannot be minimal [Man2, Theorem 6.3].

(6) Denote by C the unique, if it exists, (smooth) irreducible curve in X^{μ_p} and write $C = -mK_X$ (Lemma 1.7), where $2g(C) - 2 = m(m-1)K_X^2$; we put $m = 0$ if X^{μ_p} is a finite set. The topological and holomorphic Lefschetz fixed point formulae in Lemma 1.6 give ($a_3 = 1/4$, $a_4 = 1/2$ when $p = 5$)

$$9 - K_X^2 = (p-1)\left[3 - \sum_{i=1}^{p-1} k_i + m(m-1)K_X^2\right],$$

$$1 = \frac{mK_X^2}{12}[(p-2)m+3] + k_1\frac{5-p}{12} + k_2\frac{11-p}{24} + k_3a_3 + k_4a_4 + \cdots.$$

(7) We have $(Pic\,X)^{\mu_p} = \mathbf{Z}K_X$ (Lemma 1.7).

Now we are in business.

Step 1. We know that only $p = 2, 3, 5, 7$ can divide $\#W(E_n)$.

Step 2. $p = 7$ can divide only $\#W(E_7)$, $\#W(E_8)$. The conjugacy class of g in $W(E_8)$ is coming from the subgroup $W(E_7)$ since there is only one each for $n = 7$ and $n = 8$. So $n = 7$ and $K_X^2 = 2$ by the minimality of the pair. The number of unordered sets of 7 disjoint (-1)-curves (an Aronhold set) on a degree 2 del Pezzo surface X is equal to $\#Sp(6, F_2)/7!$ (see, for example, [DO, p. 167]). Since the number 36×8 is congruent to $1 \bmod 7$, there is a g-invariant Aronhold set, a contradiction to the minimality of the pair.

Step 3. Assume $p = 5$. It is a new conjugacy class for $n = 4$ ($W(E_4) = S_5$) and for $n = 8$ (for $n \leq 7$ there is only one conjugacy class, so it is always old). If $n = 4$, then X is a del Pezzo surface of degree 5. Hence (X, μ_5) fits the last row of Table I (Lemma 2.13).

Step 4. Assume $p = 5$ and $n = 8$. Then $K_X^2 = 1$. The formulae in (6) above imply $(m; k_1, \ldots, k_4) = (1; 0, 0, 0, 1)$. So (X, μ_5) fits the rows 5 and 6 of Table I.

Step 5. Assume $p = 3$. There is a new conjugacy class of order 3 for every $n = 3, 6, 8$. If $n = 6$, there is only one new conjugacy class of order 3; it is c_{11} in [Man2, Table 1, p. 176]. In Carter's classification it corresponds to the graph $3A_2 \subset E_6$. Its trace on E_6 is equal to -3 (see also Lemma 1.6). Now the formulae in (6) imply $(m; k_1, k_2) = (1; 0, 0), (0; 0, 0)$.

The second case says that $X^{\mu_3} = \emptyset$, which is impossible (Lemma 1.2); the first case is the row 3 of Table I.

Finally, consider the case $n = 8$. There are 2 new conjugacy classes of order 3. In Carter's notation they are $4A_2$ and $E_8(a_8)$ (they should generate the same group). As above the formulae in (6) imply $(m; k_1, k_2) = (2; 1, 0)$. This is the row 4 of Table I.

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