Nilpotence in finitary skew linear groups

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Communicated by K.W. Gruenberg
Received 29 January 1992
Revised 13 March 1992

Abstract


We study nilpotence properties (upper central series, Engel elements, central heights, etc.) of groups of finitary linear transformations of a vector space over a division ring that is locally finite-dimensional over its centre. Not surprisingly properties of skew linear groups over these divisions rings do not usually extend to this much more general setting. However, not all is lost. Although the group itself may not have the property, frequently it has a local system of normal subgroups with the property. In many ways this is a surprisingly strong conclusion.

Most of the difficulties involve unipotent elements. This paper really concerns the development of bounds for unipotent skew linear groups of various kinds that are independent of the degree and hence are potentially meaningful in the finitary, infinite-dimensional situation. Some of these give new insights even in the ordinary linear case. Once this is done the nilpotence results hinted at above are then easy corollaries.

Introduction

A continuing theme in the theory of linear groups has been the computation of bounds for the central height of a unipotent subgroup of the hypercentre of a linear group, starting at least with work of M.S. Garascuk in 1960 and continuing to the present day. There are many applications and the computation of precise bounds for these central heights seem to be intrinsically important. So much so that over the years I have come to think of these bounds as the theorems and the applications as mere corollaries, whereas originally the applications were the theorems and the computations were lemmas on the way. The best bounds to date, as far as I am aware, are in [9] in both the linear and the skew linear case.

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The general bounds involve the degree of the matrix group. For a given group these can frequently be improved but then the bound involves quantities such as the composition length of the natural module and the dimensions of the composition factors, see [9, 1.5, 4.5(b) and 5.1]. All these are clearly useless for finitary linear or skew linear groups, where some or all of these numbers are typically infinite. We need new kinds of bounds, bounds that are still meaningful in the infinite-dimensional case. The bulk of this paper is devoted to developing such bounds. In tune with the philosophy suggested above we describe this as our theorem. The applications are relegated to corollaries. We state them in full, but their proofs, which are largely routine, we only sketch. Although designed to give information about the finitary case, the theorem below does give further insight even in the finite-dimensional linear case.

1. The results

Throughout this paper $F$ denotes a field, $D$ a division $F$-algebra and $V$ a left vector space over $D$. Sometimes we assume $F$ is the centre of $D$. The corresponding finitary general linear group $FGL(V)$ or $FAut_DV$ is the subgroup of $Aut_DV$ of $D$-automorphisms $g$ of $V$ such that $[V, g] = V(g - 1)$ has finite (left) dimension over $D$. A finitary skew linear group is then a subgroup of $FGL(V)$ for some $D$ and $V$. For the basic properties of such groups see Sections 1 and 2 of [11].

Let $D$ be a finite-dimensional central division $F$-algebra of characteristic $p \geq 0$. If $p = 0$ set $\delta = 1$. If $p > 0$ let $\delta$ denote the largest power of $p$ to divide $(D : F)^{1/2}$. (If $F$ is perfect then $\delta = 1$, see [1, Theorems 5.18 and 7.22], but we do not need to quote this result. If $F$ is perfect simply define $\delta$ to be 1 and the proofs below work with no modification.)

Let $G$ be a subgroup of $GL(n, D)$ and suppose $E$ is a unipotent normal subgroup of right Engel elements of $G$. Then $E, G = \langle 1 \rangle$ for some $r$ by the linear case; for example $r = n(D : F)^{1/2}$ would do. Let $X$ be a subset of $G$ and set $N = \langle X \rangle$, the normal subgroup of $G$ generated by $X$, $V = D^{m}$, row $n$-space over $D$, and $m = \max\{1, \dim_D[V, \langle X \rangle]\}$. Note that $m = \dim_D[V, \langle X \rangle]$ unless $X = \emptyset$ or $\langle 1 \rangle$.

**Theorem 1.1.** With the above notation suppose $U$ is a unipotent normal subgroup of right Engel elements of $N$ with either $F$ perfect or $N$ soluble. Then $[U, \delta f(m) N] = \langle 1 \rangle$, where $f(m) = 2m$ if $p = 0$ and $f(m) = m + (m - 1)p^\sigma + 1$ for $p^\sigma \leq m^3 < p^\sigma + 1$ otherwise. If $N$ is not unipotent then $[U, \delta f(m - 1) N] = \langle 1 \rangle$ and if $N$ is unipotent then $[U, \delta m^6 N] = \langle 1 \rangle$.

Clearly $m \leq n$ in Theorem 1.1, so the theorem does also yield bounds in terms of $n$ and $\delta$. These are something like twice the best known bounds in terms of $n$ in most cases, see [9]. This suggests that the bounds of Theorem 1.1 are unnecessarily large, but not grossly so.
As our first corollary we have the following. (Note that if $X$ below is finite then $\dim_D[V, \langle X \rangle]$ is always finite.)

**Corollary 1.2.** Let $F$ be a perfect field (possibly of characteristic zero), $D$ a locally finite-dimensional division $F$-algebra, $V$ a left vector space over $D$ and $G$ a subgroup of $\text{FGL}(V)$. Let $X$ be a subset of $G$ such that $m = \max\{1, \dim_D[V, \langle X \rangle]\}$ is finite. Set $N = \langle X^G \rangle$ and suppose $U$ is a unipotent normal subgroup of right Engel elements of $N$. Then with $f(m)$ as in Theorem 1.1 we have $[U, f(m)N] = \langle 1 \rangle$.

To state the second corollary we need to introduce some considerable notation. In the main it is standard. Commutators are all left normed. Let $G$ be any group.

$L(G)$ denotes the set of left Engel elements of $G$.
$
\tilde{L}(G)$ the set of bounded left Engel elements of $G$.
$R(G)$ the set of right Engel elements of $G$.
$\tilde{R}(G)$ the set of bounded right Engel elements of $G$.
$\eta(G)$ the Hirsch–Plotkin radical of $G$.
$\eta_1(G)$ the Fitting subgroup of $G$.
$\eta_2(G)$ the subgroup of $G$ generated by all normal subgroups of $G$ that are hypercentral in their own right as groups.
$\sigma(G)$ the Gruenberg radical of $G = \{x \in G : \langle x \rangle \text{ asc } G\}$.
$\tilde{\sigma}(G)$ the Baer radical of $G = \{x \in G : \langle x \rangle \text{ sn } G\}$.
$\rho(G) = \{x \in G : \forall g \in G, g \in o(\langle g, x^G \rangle)\}$.
$\tilde{\rho}(G) = \{x \in G : (\exists k \in \mathbb{N})(\forall g \in G), \langle g \rangle \text{ is subnormal in } \langle g, x^G \rangle \text{ in } k \text{ steps}\}$.
$\xi(G)$ the hypercentre of $G$.
$\zeta_\alpha(G)$ the $\alpha$th term of the upper central series of $G$, so $\zeta_0(G)$ denotes the centre of $G$.

For the general theory of the above objects, with the exception of $\eta_2(G)$, see [6]. Further, for any (finitary) (skew) linear group $G$, the unipotent radical of $G$, if it exists, we denote by $u(G)$ and the stability radical by $s(G)$, see [8] for a discussion of the linear case, [7] for the skew linear case and [11] for the finitary skew linear case. A $0'$-group is a periodic group.

**Corollary 1.3.** Let $F$ be a perfect field of characteristic $p \geq 0$, $D$ a locally finite-dimensional division $F$-algebra, $V$ a left vector space over $D$ and $G$ a subgroup of $\text{FGL}(V)$. Then:

(a) $L(G) = \eta(G) = \sigma(G) = \eta_2(G) = \langle M \triangleleft G : M = \zeta_{w2}(M) \rangle = \langle M \triangleleft G : \exists \alpha < \omega_2, M = \zeta_\alpha(M) \rangle$.

(b) $\tilde{L}(G) = \tilde{\sigma}(G) = \eta_1(G) = \eta_1(G)$.

(c) $R(G) = \rho(G) = \xi(G)$. For each finite subset $X$ of $G$ there is a normal subgroup $K$ of $G$ with $K \supseteq X$ with $R(G) \cap K \leq \zeta_{w1}(K)$.

(d) $\tilde{R}(G) = \tilde{\rho}(G) = \xi_1(G)$. For each finite subset $X$ of $G$ there is a normal subgroup $K$ of $G$ with $K \supseteq X$ with $\tilde{R}(G) \cap K \leq \zeta_{w}(K)$.
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(e) Modulo its unipotent radical $G$ has central height at most $\omega_2$.

(f) If $p > 0$ then $[R(G), G]$ and $R(G)/(R(G) \cap \xi_1(G))$ are locally finite groups.

If the unipotent radical of $R(G)$ is trivial then $[R(G), G]$ and $R(G)/(R(G) \cap \xi_1(G))$ are locally finite $p'$-groups.

Unlike the skew linear case [7, 3.5.3] there is no need for $\eta_1(G)$ in Corollary 1.3 to lie in $(QD^n)\cap$ or even in $QD^n$, since clearly $QD^n$-groups are hypercentral and $\eta_1(G)$ need not be hypercentral, even in the finitary linear case, see Section 1 of [10]. The heart of this present paper is Section 2, much of the remainder being modifications of earlier arguments that we only sketch.

2. Unipotent Engel elements in positive characteristic

Lemma 2.1. Let $D$ be a finite-dimensional central division $F$-algebra, where $F$ is a perfect field of characteristic $p > 0$, $G$ an irreducible subgroup of $GL(n, D)$ and $H$ a homogeneous normal subgroup of $G$ such that $G/H$ is a locally finite $p'$-group.

Suppose $G$ centralizes the centre $Z$ of the $F$-subalgebra $R = F[H]$ of $D^{n \times n}$ generated by $H$. Then $H$ too is irreducible.

Proof. We break the proof into five pieces.

(a) We may assume that $G/H$ is finite.

Since $\dim_{F} D$ is finite there are finitely generated subgroups $G_i$ and $H_i$ of $G$ and $H$ respectively such that $F[G] = F[G_i] \leq D^{n \times n}$ and $F[H] = F[H_i]$. Set $G_0 = \langle G_i, H_i \rangle$ and $H_0 = G_0 \cap H$, so $F[G] = F[G_0]$ and $F[H] = F[H_0]$. Then $H_0$ is a normal subgroup of $G_0$ and $G_0/H_0$ is a finite $p'$-group. Set $V = D^{n \times n}$, regarded as $D^{-}D^{n \times n}$ bimodule in the obvious way and suppose $W$ is a $D^{-}G_0$ submodule of $V$. Then $WG \leq W, F[G_0] \leq W$ and $W$ is also a $D^{-}G$ submodule of $V$. Thus $G_0$ is irreducible. Similarly the irreducible $D^{-}H$ submodules of $V$ are $D^{-}H_0$ irreducible and trivially $D^{-}H_0$ isomorphic. Consequently $H_0$ is homogeneous, and is irreducible if and only if $H$ is irreducible. Therefore we may replace $G$ and $H$ by $G_0$ and $H_0$ and assume that $G/H$ is finite.

(b) If $G = UH$ for some unipotent subgroup $U$ of $G$, then $H$ is irreducible.

$U$ is a unitriangularizable [7, 1.3.5]. Hence with $V$ as above there is a nonzero element $v$ of $V$ with $vU = \{v\}$. Since $V$ is $D^{-}G$ irreducible, we have $V = DvG = DvUH = DvH$, and so $V$ is $D^{-}H$ cyclic. Thus $V$ is a $D^{-}H$ image of $R^D = D^{op} \otimes_F R$; here $D^{op}$ denotes the opposite ring of $D$. Also $H$ is homogeneous, so $R$ is simple (cf. [7, 1.1.12]), $D$ is central simple and hence $R^D$ is also simple (e.g. [3, p. 259]). Note that since $vU = \{v\}$ the action of $U$ on $V = vR^D = DvR$ is given by conjugation with $U$ on $R$.

Now $G$ centralizes the centre $Z$ of $R$. By the Skolem–Noether Theorem $U$ acts on $R$ as inner automorphisms, and $U$ is a $p'$-group. Also $Z$ is perfect, so $Z \setminus \{0\}$ is a central, $p$-divisible subgroup of the group of units of $R$ with no elements of order
Thus the group of units of $R$ contains a $p$-subgroup $U$ inducing by conjugation on $R$ the same group of automorphisms as $U$. Now $R^p = E^{k\times k}$ for some finite-dimensional division algebra $E$ and integer $k$. Also $p$-subgroups of $\text{GL}(k, E)$ are unitriangularizable [7, Section 1.3] and $E^{k\times k}$ has a right $E^{k\times k}$ composition series invariant under conjugation by the upper unitriangular group $\text{Tr}^t(k, E)$ (with terms generated by $e_{ii}$ up to $e_{ii}$ for $i = 1, 2, \ldots, k$ where the $e_{ii}$ denote the standard $k \times k$ matrix units). Thus $R^p$ has a right composition series invariant under conjugation by $U$ and hence also invariant under the action of $U$. Thus the image $V$ of $R^p$ contains an irreducible $D-H$ submodule $W$ with $WU \leq W$. But then $V = WU = WUH = W$ and $H$ is irreducible as claimed.

(c) We may assume that $H$ is Zariski closed in $\text{GL}(n, D)$.

We compute the Zariski topology over the centre $F$ of $D$; that is, we consider the topology induced on $\text{GL}(n, D)$ by the Zariski topology of $\text{Aut}_F V$ and the obvious embedding of $\text{GL}(n, D) \cong \text{Aut}_F V$ into $\text{Aut}_F V$. Let $G^*$ and $H^*$ denote the closures in $\text{GL}(n, D)$ of $G$ and $H$ respectively. Since $(G: H)$ is finite, $G^* = GH^*$ and so $G^*/H^*$ is an image of $G/H$; in particular $G^*/H^*$ is a finite $p$-group.

By hypothesis $V = \bigoplus V_i$ for certain isomorphic irreducible $D-H$ submodules $V_i$. Clearly the normalizers of the $V_i$ are closed. Consequently $H^*$ normalizes each $V_i$. Also there exists $x$ in $\text{GL}(n, D)$ permuting the $V_i$ cyclicly and inducing an $H$-isomorphism of $V$. Then $x \in C_{\text{GL}(n, D)}(H)$ and centralizers are also closed. Hence $[x, H^*] = \langle 1 \rangle$ and so $x$ induces a $D-H^*$ isomorphism of $V$. Therefore the $V_i$ are all $D-H^*$ isomorphic and hence $H^*$ is homogeneous. If $G$ is irreducible clearly $G^*$ is too. Suppose $H^*$ is irreducible. Then $V = V_i$ and $H$ is irreducible. Finally $F[H]$ is closed in $D^{n\times n}$, being a subspace, so $F[H^*] = F[H]$. This proves (c).

(d) Suppose $H$ is irreducible. Then $G$ centralizes $E = C_{\text{GL}(n, D)}(H)$.

$R = F[H]$ is simple and now $E = \text{End}_p V_H$ is a division ring, by Schur’s Lemma, normalized by $G$, and by a theorem of Brauer [3, p. 263] the centre of $E$ is $Z$. Let $g \in G$. By the Skolem–Noether Theorem [3, p. 262] there exists a unit $e$ of $E$ such that $g^{-1}e = z$ centralizes $E$. For some power $q$ of $p$ we have that $g^q \in H$ also centralizes $F$. Consequently $g^q = g^q z^q \in C_z(F) = Z$. But $Z$ is perfect, being a finite field extension of $F$ and $Z[e] \leq E$ is a field. Therefore $e \in Z$ and $g$ centralizes $E$, as required.

(e) is irreducible.

By hypothesis $V = \bigoplus V_i$, where the $V_i$ are $D-H$ irreducible and isomorphic. Assume $H \neq G$. Then since by (a) the group $G/H$ is a finite $p$-group, there is a normal subgroup $K$ of $G$ with $K/H$ cyclic of order $p$. By (c) we may assume that $H$ is closed; whence $K$ is too. Now $\text{GL}(n, D)$ is closed in $\text{Aut}_F V$ and $F$ is perfect, so the Jordan components of any element of $K$ lie in $K$, see [7, 3.1.6] and [8, 7.3].

Therefore $K = \langle k \rangle H$ for some unipotent element $k$ of $K$. By Clifford’s Theorem [7, 1.1.7] $K$ is completely reducible. Apply (b) to each irreducible $D-K$ submodule of $V$. Each of these are then irreducible as $D-H$ module; that is, the $V_i$.
above may be assumed to be $K$-submodules. Also, if $K$ is homogeneous and $G$ centralizes the centre of $F[K]$, then induction on the index $(G : H)$ yields that $K$ is irreducible. Consequently $V$ will be equal to $V_i$ and $H$ will be irreducible as required.

Recall $R = F[H] \leq D^{n \times n}$ and $Z$ is the centre of $R$. Set $S = F[K]$. Certainly $K$ centralizes $Z$. Then by the Skolem–Noether Theorem again there is a unit $l$ of $R$ such that $k$ and $l$ induce the same automorphism of $R$ by conjugation. Let $\sigma_i$ denote the natural projection of $S$ into $\text{End}_D V_i$. Note that $\sigma_i$ is one-to-one on $R$. Also $\text{End}_D V_i$ is a central simple $F$-algebra, so Brauer’s Theorem [3, p. 263] again applies and $R\sigma_i$ is equal to its double centralizer in $\text{End}_D V_i$. But (d) yields that $K\sigma_i$ centralizes the centralizer of $R\sigma_i$. Therefore $K\sigma_i \leq R\sigma_i$ for each $i$. Hence $k \sigma_i = l \sigma_i z_i \sigma_i$ for some $z_i \in Z$. Now $k^q = 1$ for some power $q$ of $p$. Then $l'^q \sigma_i = z_i l'^q \sigma_i$ and so $l'^q = z_i$ for each $i$, since $\sigma_i$ is one-to-one on $R$. It follows that $z_i^{-1} z_j$ is the $q$th root of unity in the field $Z$ of characteristic $p$ and so $z_i = z_j = z$ for all $i$ and $j$. Then $k = l z \in R$, any $D-H$ isomorphism of $V_i$ to $V_j$ is also a $D-K$ isomorphism and $F[K] = R$. In particular, $K$ is homogeneous and the proof of the lemma is complete. \[ \square \]

**Lemma 2.2.** Let $n$ be a positive integer, $D$ any division ring and $G$ an irreducible subgroup of $GL(n, D)$ such that for some positive integer $m$ the group $G$ is generated by a set $X$ of elements $x$ satisfying $\dim_D[D^{(n)}, x] \leq m$. If $H$ is a normal subgroup of $G$ such that every finite image of $G/H$ is soluble, then either $n \leq 8m^3$ or $H$ is homogeneous.

In this paper we only use the case where $G/H$ is locally nilpotent; clearly then the finite images of $G/H$ are soluble. But the lemma has much wider application. For example, $G/H$ could lie in either of the most useful of the wide generalizations of the class of soluble groups, namely the class of locally soluble groups and the class of Kurosh radical groups (the class $PL\forall$ in P. Hall’s notation). Other possibilities are the Kurosh classes of $\bar{SN}$-groups and of $\bar{SI}$-groups (namely $(p^\infty)^{OS}$ and $(q^\infty)^{OS}$ resp.); see [6, Vol. 2] for details. Further $G/H$ could be infinite simple for example.

**Proof.** Suppose $H$ is inhomogeneous, but otherwise as in the lemma. By Clifford’s Theorem [7, 1.1.7] the group $H$ is completely reducible. Let $V = \bigoplus_{i=1}^s H_i$, where the $H_i$ are the nonzero homogeneous components of the $D-H$ bimodule $V = D^{(n)}$. Then $G$ permutes the $H_i$ transitively, see [7, 1.1.6]. Let $\pi: G \to \text{Sym}(s)$ be the resulting transitive permutation representation of $G$. We claim that $|\text{supp} x \pi| \leq 2m$ for every $x$ in $X$. If so then $s \leq 8m^2$ by 4.1.1 of [5]; note that $H \leq \ker \pi$, so $G\pi$ is finite and soluble by hypothesis. Since $H$ is not homogeneous, $s > 1$ and $G$ does not normalize any $H_i$. Thus for each $i$ there exists $x \in X$ with $H_i x \neq H_i$, and so $\dim_D H_i \leq \dim_D V, x \leq m$. Consequently $n \leq 8m^3$ and the proof of the lemma will be complete.
Let $x \in X$. We need to prove that $|\text{supp } x\pi| \leq 2m$. Suppose $(1,2,\ldots,t)$ is an orbit of $x\pi$ and set $W = H_1 \oplus \cdots \oplus H_t$. Then $W = H_1 + [W,x]$ and hence $\dim_D[W,x] \geq (t-1) \dim_D H_i$. But $\dim_D H_i \geq 1$ for each $i$ and $x\pi$ has at most $\lfloor |\text{supp } x\pi|/2 \rfloor$ nontrivial orbits. Therefore

$$|\text{supp } x\pi| \leq 2 \dim_D[V,x] \leq 2m,$$

as required. □

**Lemma 2.3.** Assume $D$, $G$, $X$ and $m$ are as in Lemma 2.2. Let $H$ be a normal subgroup of $G$ such that $G/H$ is a locally finite $p$-group for $p = \text{char } D > 0$. Suppose also that the centre $F$ of $D$ is perfect with $\dim_F D$ finite and set $A = C_{D^{\times\times n}}(H)$. Then:

(a) either $n \leq 8m^3$ or $G = C_G(A) \leq F[H] \leq D^{\times\times n}$,

(b) $[A, \rho^*G] = \{0\}$, where $\rho^* \leq 8m^3 < \rho^{*+1}$.

**Proof.** (a) Suppose $n > 8m^3$. The group $H$ is homogeneous by Lemma 2.2 and hence the $F$-subalgebra $F[H]$ of $D^{\times\times n}$ is simple [7, 1.1.12]. Also $A \cong \text{End}_D V_H$. By Brauer's Theorem [3, p. 263] $Z = A \cap F[H]$ is the centre of both $A$ and $F[H]$ and $F[H] = C_{D^{\times\times n}}(A)$.

Put $V = D^{(m)}$. If $G$ does not centralize $Z$ there exist $x$ in $X$ and $y$ in $Z\backslash\{0\}$ with $1 \neq Z = [x,y] \in Z$. Then $[V,z] \leq [V,x]z + [V,x']$ and so $\dim_D V(z-1) \leq 2m$. Since $n > 2m$ we have $V > V(z-1)$. But then $V/V(z-1)$ contains an irreducible $D-H$ submodule, upon which $z$ will act trivially. As $H$ is homogeneous this yields that $z$ acts trivially on $V$ and so $z = 1$. This contradiction shows that $G$ centralizes $Z$. Consequently $H$ is irreducible by Lemma 2.1 and in particular $A$ is a division ring.

Let $g \in G$. By the Skolem–Noether Theorem [3, p. 262] there is a nonzero $a$ in $A$ such that $g^{-1}a$ centralizes $A$. Now $g^q \in H$ centralizes $A$, for some power $q$ of $p$. Hence $a^q \in C_A(A) = Z$. But $Z$ is a perfect field, being a finite extension of $F$ and $Z[a]$ is a subfield of the division ring $A$. Therefore $a \in Z$ and so $g$ centralizes $A$. Consequently $G = C_G(A)$. Finally $C_G(A) \leq C_{D^{\times\times n}}(A) = F[H]$.

(b) Note first that the split extension of $A$ by $G/H$ is a locally finite $p$-group, so $G$ acts locally nilpotently on $A$. Now apply (a). If $n \leq 8m^3$ then $[A, \rho^*G] = \{0\}$ by 4.5(b) of [9]. If $G = C_G(A)$ then $[A, G] = \{0\}$. □

Let $D$ be a finite-dimensional central division $F$-algebra of positive characteristic $p$ and let $\delta$ denote the largest power of $p$ to divide $(D:F)^{1/2}$. Consider a subgroup $G$ of $\text{GL}(n, D)$. For the subset $X$ of $G$ set $N = \langle X^{\sigma} \rangle$, $V = D^{(n)}$ and

$$m = \max\{1, \dim_D[V, \langle X \rangle] \}.$$

**Theorem 2.4.** With the notation above let $U$ be a unipotent normal subgroup of
right Engel elements of $N$ and suppose that either $F$ is perfect or $N$ is soluble. Then 
$[U, N] = \langle 1 \rangle$, where $l = 2m\delta$ if $N$ is unipotent and $l = m + (m - 1)p^h$ for $p^h \leq 8m^3 < p^{r+1}$ otherwise. In particular, $[U, hN] = \langle 1 \rangle$ for $h = m + (m - 1)p^h + 1$ in all cases.

The positive characteristic case of Theorem 1.1 follows immediately from Theorem 2.4. For Theorem 2.4 we prove first that $l = 2m\delta p^h$ works, which is sufficient if one only requires the existence of a bound. Then we indicate how to modify the proof to obtain the better bound given in the statement of Theorem 2.4.

**Proof.** We can replace $U$ by the unipotent radical of $R(N)$; hence we may assume that $U$ is normal in $G$. Also $[U, N] = \langle 1 \rangle$ for some $l$ by the linear case. Consider the intersection of $[V, \langle X \rangle]$ with a $D$-$G$ composition series of $V$. Then there is a series

$$
\{0\} = V_0 \leq V_1 \leq \cdots \leq V_m \leq V_{2m+1} = V
$$

of $D$-$G$ submodules of $V$ such that each $V_{2i}/V_{2i-1}$ is $D$-$G$ irreducible or $\{0\}$ and

$$
\left( \bigcup_i (V_{2i}/V_{2i-1}) \cup \{0\} \right) \supseteq [V, \langle X \rangle].
$$

Then $[V_{2i+1}, \langle X \rangle] \leq V_{2i}$ for each $i$. But the $V_i$ are $G$-submodules, so in fact $[V_{2i+1}, N] \leq V_{2i}$. Since $V_{2i}/V_{2i-1}$ is $D$-$G$ irreducible or $\{0\}$, it is $U$-trivial and completely $D$-$N$ reducible into $D$-$N$ irreducible components of equal dimension [7, 1.1.6]. If $N$ is soluble this dimension is at most $8m^3$ by 3.3 of [11].

Consider first the following special case. Let $W$ be a proper $D$-$N$ submodule of $V$ such that $U$ acts trivially on $W$ and $V/W$. If $W$ and $V/W$ are both $N$-trivial then $N$ is abelian [4, 1.C.1] and $[U, N] = \langle 1 \rangle$. Suppose $V/W = \bigoplus X_i$ and $W = \bigoplus Y_i$, where the $X_i$ and $Y_i$ are $D$-$N$ irreducible of dimension over $D$ at most $s$. By stability theory $U$ embeds as $N$-module into

$$
\text{Hom}_D(V/W, W) \cong \bigoplus_{i,j} \text{Hom}_D(X_i, Y_j),
$$

where we have used that $V$ is finite-dimensional, cf. [4, Section 1.C]. Let $U_{ij}$ denote the image of $U$ in $\text{Hom}_D(X_i, Y_j)$. Then $[U_{ij}, N] = \langle 0 \rangle$, where

- $l = 1$ if $X_i$ and $Y_j$ are both $N$-trivial,
- $l = 0$ if $X_i$ and $Y_j$ are not isomorphic as $D$-$N$ modules, see [9, 2.6], and
- $l = \delta p^s$ for $p^s \leq s < p^{r+1}$ if $X_i \cong Y_j$ as $D$-$N$ modules, see [9, 4.5(b)].

Suppose $F$ is perfect and $X_i \cong Y_j$ as $D$-$N$ modules. Let $A$ denote the obvious image of $U_{ij}$ in $\text{End}_D I \cong \text{Hom}_D(X_i, Y_j)$ and $B$ the obvious image of $N$ in $\text{End}_D I$. The action of $N$ on $U_{ij}$ is given by conjugation of $A$ by $B$. For some $r$ we have $[A, B] = \langle 0 \rangle$ by the linear case, so $B/H$ for $H = C_n(A)$ is a nilpotent $p$-group.
Clearly \( A \subseteq C_{\text{End}}(H) \). If \( x \in X \) and \( g \in G \) then \( I \) is a \( D-N \) section of \( V \) and

\[
\dim_D[I, x^g] \leq \dim_D[V, x^g] = \dim_D[V, X] = m.
\]

By Lemma 2.3(b) we have \( \{A, B\} = \{0\} = [U_{ij}, iN] \) for \( l = p'' \) and \( p'' \leq 8m^3 < p''^{+1} \).

We now return to the general case and to our earlier notation. We induct on the length of the series \( \{V_i\} \) in order to prove that \( [U, iN] = \langle 1 \rangle \) for \( h = 2mp'' \) and \( p'' \leq 8m^3 < p''^{+1} \). Thus by induction assume that \( [U, iN] \) centralizes \( V_{2m} \) and \( V/V_i \) for \( k = (2m - 1)\delta p'' \). Then \( [U, iN] \) embeds as \( N \)-module into \( \text{Hom}_{D}(V/V_{2m}, V_i) \) and the above case of a series of length 2 applied to the obvious action of \( 1 + [U, kN] \) and \( N \) on \( V_i \oplus (V/V_{2m}) \) yields that \( [[U, kN], \delta pN] = \langle 1 \rangle \). Therefore \( [U, kN] = \langle 1 \rangle \) for \( h = 2mp'' \) as claimed.

This proves the existence of a bound involving only \( m, p, \delta \) and \( \sigma \). However the bound is unnecessarily large, since at least \( m + 1 \) of the factors \( V_i/V_{i-1} \) are \( N \)-trivial. At the 2-step stage (effectively the induction step) we only need to add \( \delta p'' \) if both factors are not \( N \)-trivial. If both are \( N \)-trivial we add 1 and if exactly one factor is \( N \)-trivial we add 0. Since \( 1 \leq \delta p'' \) we obtain \( [U, iN] \langle 1 \rangle \) for \( l = 2m \) if all the factors \( V_i/V_{i-1} \) are \( N \)-trivial, which happens exactly when \( N \) is unipotent, and for \( l = m + (m - 1)\delta p'' \) if at least one factor is not \( N \)-trivial. The proof is complete. \( \square \)

**Corollary 2.5.** Let \( F \) be a perfect field of characteristic \( p > 0 \), \( D \) a locally finite-dimensional division \( F \)-algebra, \( V \) a left vector space over \( D \) and \( G \) a subgroup of \( \text{FGL}(V) \). Assume \( X \) is a subset of \( G \) such that \( m = \max\{1, \dim_D[V, \langle X \rangle]\} \) is finite. Set \( N = \langle X^G \rangle \) and suppose \( U \) is a unipotent normal subgroup of right Engel elements of \( N \). Then \( [U, iN] = \langle 1 \rangle \) for \( l = 2m \) if \( N \) is unipotent for \( l = m + (m - 1)p'' \) and \( p'' \leq 8m^3 < p''^{+1} \) otherwise.

If \( X \) in Corollary 2.5 is finite then \( m \) is always finite, see Section 1 of [11]. Also, Corollary 2.5 yields the positive characteristic case of Corollary 1.2.

**Proof.** If \( X \supseteq Y \) then \( \dim_D[V, \langle Y \rangle] \leq m \) and if \( N \) is not unipotent it contains a nonunipotent element. Further the functions \( 2m \) and \( m + (m - 1)p'' \) are increasing functions of \( m \). A simple localization argument reduces us to the case where \( G \) is finitely generated. Then \( G \) embeds into \( \text{GL}(n, D) \) in a natural way for some finite \( n \) and some finitely generated \( F \)-subalgebra \( D_1 \) of \( D \). Necessarily \( D_1 \) is a division ring and as remarked in the Introduction, since \( F \) is perfect, \( p \) and \( (D_1 : \mathbb{Q}(D_1)) \) are coprime. Then Corollary 2.5 follows from Theorem 2.4. \( \square \)

If \( F \) is perfect and \( G \) is locally soluble there is a different approach to results like Corollary 2.5, which we now outline. The first three lemmas below will be used elsewhere.
Lemma 2.6. Let $F$ be a perfect field of characteristic $p > 0$ and $G$ a (Zariski) closed soluble-by-periodic subgroup of $\text{GL}(n, F)$. Suppose $U$ is a maximal unipotent subgroup of $G$ and $N$ is a closed subgroup of $G$ containing every diagonalizable element of $G$. Then $G = UN$.

Proof. Replacing $N$ by the closure of the subgroup generated by the diagonalizable elements of $G$, we may assume that $N$ is a normal subgroup of $G$. In view of the proof of [10, 3.1] it suffices to prove that the maximal unipotent subgroups of $G$ are all conjugate. Let $U$ and $U_1$ be two such subgroups. There is a closed soluble normal subgroup $S$ of $G$ with $G/S$ periodic. $G/S$ is isomorphic to a periodic linear group, so its maximal $p$-subgroups are conjugate and soluble, see [8, Chapter 9]. Thus for some $g \in G$ we have that $\langle U^g, U_1^g \rangle S/S$ is a soluble $p$-group. Consequently $\langle U^g, U_1^g \rangle S$ is also soluble. The result now follows from [7, 3.4.3].

Lemma 2.7. Let $F$ be a field, $D$ a locally finite-dimensional division $F$-algebra, $V$ a left vector space over $D$ and $G$ a subgroup of $\text{FGL}(V)$. Then the $F$-subalgebra $F[G]$ of $\text{End}_D V$ generated by $G$ is a locally finite-dimensional $F$-algebra.

Proof. Suppose $G$ is finitely generated and nontrivial. There is a finite-dimensional $D$-subspace $W$ of $V$ with $[V, G] \leq W$ and $V = W + C \langle G \rangle$, see Section 1 of [11]. If $\dim_D V$ is finite, set $W^* = V$. If not pick $w$ in $C \langle G \rangle \backslash \{0\}$ and set $W^* = W + Dw$. Let $x \in F[G]$. Since $W^*$ is $G$-invariant $W^*x \leq W^*$. Suppose $W^*x = \{0\}$. If $x = \sum \alpha_i g_i$, where the $\alpha_i$ are in $F$ and the $g_i$ are in $G$, then $wx = 0$ implies that $\sum \alpha_i = 0$. Then $C \langle G \rangle x = \{0\}$ and so $Vx = \{0\}$ and $x = 0$. Therefore $F[G]$ acts faithfully on $W^*$, in both cases, and consequently $F[G]$ embeds into $D^{\ast \ast n}$ for $n = \dim_D W^*$. But $D$ is locally finite-dimensional over $F$, so $D^{\ast \ast n}$ is too and $\dim_F F[G]$ is finite. The lemma follows.

For the next two results assume that $D$ is a locally finite-dimensional division algebra over the perfect field $F$ of characteristic $p \geq 0$ and that $V$ is a left vector space over $D$. The finite-dimensional case [7, pp. 84–85] applied locally shows that for each $g$ in $\text{FGL}(V)$ there exists a unique unipotent element $g_u$ of $\text{FGL}(V)$ and a unique $d$-element $g_d$ of $\text{FGL}(V)$ satisfying $g = g_u g_d$. (An element $d$ of $\text{FGL}(V)$ is a $d$-element if $\bar{F} \otimes_F F[d]$ is semisimple Artinian for $\bar{F}$ the algebraic closure of $F$. Since $F$ is perfect this is equivalent to $F[d]$ being semisimple Artinian and hence to $V$ being completely reducible as $F[d]$-module.) Now [7, 3.1.7] applied locally yields the following lemma:

Lemma 2.8. Let $G$ be a locally nilpotent subgroup of $\text{FGL}(V)$. Then $g \mapsto g_u$ and $g \mapsto g_d$ are homomorphisms of $G$ onto subgroups $G_u$ and $G_d$ of $\text{FGL}(V)$. Also $[G_u, G_d] = \{1\}$ and $GG_u = GG_d = G_u \times G_d$. 

Proposition 2.9. Assume that $p > 0$. Let $U$ be a unipotent normal subgroup of right Engel elements of the locally soluble-by-periodic subgroup $G$ of $\text{FGL}(V)$. Then:

(a) $G/C_G(U)$ is a Fitting group;

(b) for every finite subset $X$ of $G$ there is a normal subgroup $K$ of $G$ such that $K \supset X$ and $U \cap K$ lies in some finitely indexed term of the upper central series of $K$.

Part (b) is a weak form of Corollary 2.5; apart from the restriction on $G$ we have here no bound on the central height of $U \cap K$, while we do in Corollary 2.5 if we set $K = \langle X^G \rangle$. However we do have part (a) of Proposition 2.9, which does not seem to be a consequence of Corollary 2.5. Thus the two approaches yield slightly different information. Later we will see the same phenomenon in the characteristic zero case.

Proof. Repeat the proof of [10, 3.3], without assuming $F$ is algebraically closed, using [7, 3.4.4], the finite-dimensional case, and using Lemma 2.8 in place of [10, 2.3] and Lemma 2.6 in place of [10, 3.1]. Note that a unipotent subgroup of $G$ is a stability group by [11, 2.1(d)(iii)] and hence is a Fitting group [11, 2.4].

3. Unipotent Engel elements in characteristic zero

In this section $F$ denotes a field of characteristic zero, $D$ a locally finite-dimensional division $F$-algebra and $V$ a left vector space over $D$. As with the positive characteristic case we have two possible approaches yielding slightly different information.

Lemma 3.1. Let $G$ be a completely reducible subgroup of $\text{FGL}(V)$ and $u \in \text{End}_D V$ an endomorphism of $V$ such that $u^2 - u$ and $y$ commute for all $x$ and $y$ in $G$. Then $u$ and $G$ commute.

Proof. Let $x \in G$. Note that $x$ is algebraic over $F$, for example by Lemma 2.7. Now copy the proof of [7, 3.4.1] with $V$ in place of $D^n$. $\square$

Lemma 3.2. Let $U$ be a unipotent normal subgroup of the subgroup $G$ of $\text{FGL}(V)$. Suppose

$$
\{0\} = V_0 < V_1 < \cdots < V_r = V
$$

is a series of $D$-$G$ submodules of $V$ such that each $V_i/V_{i-1}$ is completely $D$-$G$ reducible and assume that $[U, G] = \langle 1 \rangle$ for some positive integer $s$. Then $[U, r, G] = \langle 1 \rangle$. 

Proof. By 2.1(d)(iii) and 2.2(d) of [11] each $V_i/V_{i-1}$ is trivial as $U$-module. In particular $U = \langle 1 \rangle$ if $r = 1$, which starts an induction on $r$.

Suppose $r = 2$. Then $U$ stabilizes the series $\{0\} < V_1 < V$ and there is a $G$-embedding of $U$ into $\text{Hom}_D(V/V_1, V_1)$. The latter embeds into $\prod_{i,j} \text{Hom}_D(X_i, Y_j)$, where $V/V_1 = \bigoplus X_i$ and $V_1 = \bigoplus Y_j$ and the $X_i$ and the $Y_j$ are $D$-$G$ irreducible. We wish to prove that $[U, G] = \langle 1 \rangle$. If this is not the case there exist $i$ and $j$ such that for $U_{ij}$, the projection of $U$ into $\text{Hom}_D(X_i, Y_j)$, we have $[U_{ij}, G] \neq \langle 0 \rangle$. But $[U_{ij}, G] = \langle 0 \rangle$ by hypothesis. Hence there is some $c$ in $U_{ij} \setminus \{0\}$ fixed by $G$. Then $c$ is a nontrivial $D$-$G$ homomorphism of $X_i$ to $Y_j$ and both of these are irreducible. Therefore $c$ is an isomorphism and we can interpret $U_{ij}$ as a subgroup of $\text{Hom}_D(X_i, X_i)$. Since $[U_{ij}, G] \neq \langle 0 \rangle = [U_{ij}, sG]$, there exists $u$ in $U_{ij}$ such that $u$ is not fixed by $G$ but $[u, x] = u^x - u$ is fixed by $G$ for all $x \in G$. This contradicts Lemma 3.1 and completes the proof that $[U, G] = \langle 1 \rangle$. (Note that in the above proof if no $X_i$ and $Y_j$ are $D$-$G$ isomorphic then $c$ cannot exist, all the $U_{ij}$ are zero and $U = \langle 1 \rangle$.)

Suppose now that $r > 2$. By induction $[U, r-2G]$ acts trivially on $V/V_1$ and on $V_{r-1}$. Therefore by stability theory [4, Section 1.C] there is a $G$-embedding of $[U, r-2G]$ into $\text{Hom}_D(V/V_{r-1}, V_1)$. Apply the case $r = 2$ to the obvious action of

$$\langle 1 + [U, r-2G], G/(C_G(V/V_{r-1}) \cap C_G(V_1)) \rangle$$

on $(V/V_{r-1}) \oplus V_1$. This yields that $[U, r-1G] = [[U, r-2G], G] = \langle 1 \rangle$. \[ \square \]

Lemma 3.3. Let $X$ be a subset of the subgroup $G$ of $\text{FGL}(V)$ such that $m = \dim_D[V, \langle X \rangle]$ is finite and set $N = \langle X^G \rangle$. Then $[U, 2mN] = \langle 1 \rangle$ for any unipotent normal subgroup $U$ of right Engel elements of $N$.

Again note that $\dim_D[V, \langle X \rangle]$ is always finite if $X$ is finite.

Proof. We may assume that $G$ is finitely generated, that $G \cong \text{GL}(n, D)$ for some finite $n$, so now $V = D^{(n)}$, and that $D$ is finite-dimensional over $F$. By the linear case $[U, G] = \langle 1 \rangle$ for some integer $s$, for example $s = n(D : F)$ would do. Intersecting $[V, \langle X \rangle]$ with a $D$-$G$ composition series of $V$ yields a series

$$\{0\} = V_0 \leq V_1 \leq \cdots \leq V_{2m+1} = V$$

of $D$-$G$ submodules of $V$ such that each $V_{2i}/V_{2i-1}$ is $D$-$G$ irreducible or $\{0\}$ and

$$\left( \bigcup_j (V_{2j}/V_{2j-1}) \cup \{0\} \right) \supseteq [V, \langle X \rangle].$$

Then $[V_{2i+1}, \langle X \rangle] \leq V_{2j}$ for each $i$. But the $V_j$ are $G$-submodules, so in fact $[V_{2i+1}, N] \leq V_{2j}$ for each $i$. By Clifford’s Theorem [7, 1.1.7] each $V_{2i}/V_{2i-1}$ is
completely $D-N$ reducible (possibly trivially so). Then Lemma 3.2 yields that $[U, z_m N] = \langle 1 \rangle$. □

**Lemma 3.4.** In Lemma 3.3, if $N$ is not unipotent then $[U, z_m N] = \langle 1 \rangle$.

**Proof.** Consider the proof of Lemma 3.3. Since $N$ is not unipotent there is some $i$ for which $[V_{2i}, N] \not= V_{2i-1}$. Then (the proof of) Clifford's Theorem yields that $V_{2i}/V_{2i-1}$ is a direct sum of irreducible $D-N$ modules, none of which are $N$-trivial (they are all $G$-conjugate). The proof of the $r = 2$ case of Lemma 3.3 yields, for example that $[V_{2i+1}, U] \subseteq V_{2i-1}$ and $[V_{2i}, U] \subseteq V_{2i-2}$. A suitable modification of the proof of Lemma 3.3 gives the desired conclusion. (There are three cases to consider according to whether $N$ acts nonunipotently on both, neither or one of $V_i$ and $V/V_{i-1}$.) □

Note that Lemmas 3.3 and 3.4 imply the characteristic zero cases of both Theorem 1.1 and Corollary 1.2. We now consider the second approach.

**Lemma 3.5.** Let $G$ be a (Zariski) closed subgroup of $GL(n, F)$, $U$ a maximal unipotent subgroup of $G$ and $N$ a closed subgroup of $G$ containing every diagonalizable element of $G$. Then $G = UN$.

**Proof.** As in the proof of Lemma 2.6 we merely need to note that the maximal unipotent (and here necessarily connected) subgroups of $G$ are conjugate in $G$. This follows from 8.2 of [2]. □

**Proposition 3.6.** Let $U$ be a unipotent normal subgroup of right Engel elements of the subgroup $G$ of $FGL(V)$. Then:

(a) $G/C_G(U)$ is a Fitting group,

(b) for every finite subset $X$ of $G$ there is a normal subgroup $K$ of $G$ such that $K \supseteq X$ and $U \cap K$ lies in some finitely indexed term of the upper central series of $K$. □

Part (b) here is a weak version of Lemma 3.3. To prove Proposition 3.6 repeat the proof of Proposition 2.9, using Lemma 3.5 in place of Lemma 2.6.

**4. The proof of Corollary 1.3**

In this section $F$ is a field of characteristic $p \geq 0$, $D$ is a locally finite-dimensional division $F$-algebra, $V$ is a left vector space over $D$ and $G$ is a subgroup of $FGL(V)$.

**Lemma 4.1.** Let $J$ be a normal subgroup of $G$. Then $N = \langle L(J), R(J) \rangle$ is
generated by soluble normal subgroups of $G$. In particular,
\[
L(G) = \eta(G) = \sigma(G) = \bar{\eta}(G),
\]
\[
R(G) = \rho(G) = \bar{\rho}(G),
\]
and the four Engel sets are normal subgroups of $G$.

**Proof.** The group $G$ is locally linear, so by the linear case $N$ is locally nilpotent [8, 8.15, 8.2]. Pick $x$ in $N$ and set $M = \langle x^G \rangle$. Then $u(M)$ is nilpotent and $M/u(M)$ is isomorphic to a subdirect product of (in fact irreducible) locally nilpotent skew linear groups over $D$ of bounded degree, see 3.4 of [11]. Consequently $M$ is soluble by [7, 3.3.8]. For the final part take $J = G$ and apply 4.1 of [10]. \(\square\)

Define the normal subgroup $G^+$ of $G$ as in [10, Section 4]. Always $G/G^+$ is locally finite.

**Lemma 4.2.** Let $N \triangleleft J$ be normal subgroups of $G$ such that $NeJ$ (i.e. such that $R(J) \supseteq N$) and $u(N) = \langle 1 \rangle$.

(a) $[N, J]$ and $N/(N \cap \zeta_1(J))$ are locally finite $p'$-groups. Also $N^+ \subseteq \zeta_1(J)$.

(b) Suppose $V = \bigoplus_{\omega \in \Omega} V_{\omega}$ is a system of imprimitivity for $J$ in $V$, where $\Omega$ is infinite and $J$ acts transitively and almost primitively on $\Omega$. Then $N = \langle 1 \rangle$.

(c) Assume $u(G) = \langle 1 \rangle$. For every finite subset $X$ of $G$ there is a normal subgroup $K$ of $G$ with $K \supseteq X$, $R(J \cap K) = \zeta_{\omega_2}(J \cap K)$ and $R(J \cap K) = \zeta_{\omega}(J \cap K)$.

(d) For every finite subset $X$ of $G$ there exists a normal subgroup $K$ of $G$ with $K \supseteq X$ and $N \cap K \subseteq \zeta_{\omega_2}(J \cap K)$. If in fact $N \cap J$ (i.e. if $R(J) \supseteq N$) then $N \cap K \subseteq \zeta_{\omega_2}(J \cap K)$.

(e) Assume $u(G) = \langle 1 \rangle$. For every finite subset $X$ of $G$ there is a normal subgroup $K$ of $G$ with $K \supseteq X$ and

\[
L(K) = \eta_1(K) = \zeta_{\omega_2}(\eta_1(K)), \quad R(K) = \zeta_{\omega_2}(K),
\]
\[
\bar{L}(K) = \bar{\eta_1}(K) = \zeta_{\omega_2}(\bar{\eta_1}(K)), \quad \bar{R}(K) = \zeta_{\omega_2}(K).
\]

**Proof.** Repeat the proof of [10, 4.3] with the following substitutions:

(a) In place of [10, 4.2 and 2.3] use Lemmas 4.1 and 2.8.

(b) For [10, 4.2] use Lemma 4.1 and in place of the linear case (8.15 of [8]) use [7, 3.5.1].

(c) In place of the linear case use [7, 3.5.2, 3.2.11 and 3.4.13]. To deduce that $H$ is soluble use Lemma 4.1 in place of [10, 4.2] and [5, Proposition 1] (also labeled [5] in [10]). For Proposition 3(i) and (ii) of [5] use [11, 3.1 and 3.2].

(d) No changes required.

(e) In place of the linear case use [7, 3.5.2, 3.2.11, 3.4.13 and 3.5.3(b)]. Note that if $I$ is a subdirect product of nilpotent groups, then $L = \zeta_{\omega}(L)$. \(\square\)
Lemma 4.3. Assume $F$ is perfect. Let $N$ be a normal subgroup of right Engel elements of $G$ and let $X$ be a finite subset of $G$. Then there exists a normal subgroup $K$ of $G$ with $K \supseteq X$ and $N \cap K \leq \xi_{\omega}(K)$. If in fact $N$ consists of bounded right Engel elements of $G$ then we can choose such a $K$ with $N \cap K \leq \xi_{\omega}(K)$.

Proof. Repeat the proof of [10, 5.1], using Corollary 1.2 and Lemma 4.2(d) in place of [10, 3.3(b) and 4.3(d)]. □

Lemma 4.4. Assume $F$ is perfect. Set $K = \iota(G)$ and $\tilde{G} = K_u G = K_{\omega} G$. Then $\xi(\tilde{G}) = K_u \times K_{\omega}$ and $\xi_i(G) = G \cap \xi_i(\tilde{G})$ for all $i \leq \omega$.

Proof. Copy the proof of [7, 3.1.8]. □

Lemma 4.5. (a) If $G$ is irreducible and $\dim_D V$ is infinite, then $\xi_i(G) = \langle 1 \rangle$.

(b) $G/\iota(G)$ has central height at most $\omega 2$.

(c) Assume $F$ is perfect. Then the central height of $G$ is bounded by the maximum of $\omega 2$ and the $G$-central height of $\xi(G) = \xi(G)/\iota(G) \cap \iota(G)$.

Proof. Repeat the proof of [10, 5.2], using [7, 3.4.13], the skew linear case, for part (b) and using Lemma 4.4 for part (c). □

4.6. Proof of Corollary 1.3. Copy Sections 5.3, 5.4 and 5.5 of [10], replacing the results 2.3, 4.2, 4.3(a), 4.3(d), 5.1 and 5.2(b) of [10] by, respectively, Lemmas 2.8, 4.1, 4.2(a), 4.2(d), 4.3 and 4.5(b). Also for [5, Theorem B(i)] (also labeled [5 in [10]) use [11, 2.1(a), 2.4 and 2.1(d)(iii)] and for [5, Theorem B(vi)] use [11, 2.3 and 2.1(d)(iii)]. □

References