Retractions in hyperspaces

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Abstract

Let $C(X)$ and $2^X$ denote the hyperspaces of subcontinua and of closed subsets of a metric continuum $X$, respectively. For $p \in X$, define the hyperspace $2^X_p = \{ A \in 2^X : p \in A \}$. Moreover, consider the mapping $A \mapsto A \cup \{p\}$. In this paper we find necessary and sufficient conditions under which this mapping is a deformation retraction or a strong deformation retraction from $2^X$ onto $2^X_p$.

We obtain similar results for the case in which the domain of the mapping is $C(X)$. Finally, we characterize local connectedness of a continuum in terms of the behavior of these mappings.

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1. Introduction

Throughout this paper a continuum means a compact, connected metric space. If $X$ is a continuum, $2^X$ will denote the hyperspace of closed subsets of $X$, which is assumed to be equipped with the Hausdorff metric (see Definitions 1.5, 1.6 and 2.1 in [2]). Also, $C(X)$ will denote the hyperspace of subcontinua of $X$, and $C_2(X) = \{ A \in 2^X : A \text{ has at most } 2 \text{ components} \}$. Further, for $p \in X$ define the hyperspaces $2^X_p = \{ A \in 2^X : p \in A \}$, $C(p,X) = \{ A \in C(X) : p \in A \}$ and $C_2(p,X) = \{ A \in C_2(X) : p \in A \}$.

In this article we investigate retractions in hyperspaces. More precisely, for a continuum $X$ and $p \in X$ we consider the mapping $\psi_p : 2^X \to 2^X_p$ given by $\psi_p(A) = A \cup \{p\}$. The main purpose of the paper is to give necessary and sufficient conditions under which $\psi_p$ is a deformation retraction and a strong deformation retraction.

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We consider also a mapping \( \phi_p : C(X) \to C_2(p, X) \) given by \( \phi_p(A) = A \cup \{p\} \) and we study similar properties. In Section 4 we characterize locally connected continua in terms of the behavior of these mappings. Finally, in Section 5 we establish a relation between the mapping \( \phi_p |_{F_1(X)} \) and some topological properties of the continuum \( X \), namely unicoherence and arcwise connectedness.

2. Preliminaries

By a mapping we mean a continuous function. We denote by \( I \) the unit interval, by \( \mathbb{N} \) the set of all positive integers and by \( \mathbb{R} \) the set of all real numbers. Let \( C \) denote the set of all complex numbers (equipped with the natural topology), and let \( S^1 = \{z \in C : |z| = 1\} \).

Further, for a continuum \( X \), and \( A \subseteq X \), we denote by \( \text{cl}(A) \) and \( \text{int}(A) \) the closure and the interior of \( A \) with respect to \( X \). Also, \( \text{diam}(X) \) will denote the diameter of a continuum \( X \), and \( X \approx Y \) means that \( X \) is homeomorphic to \( Y \). However, if \( f, g \) are mappings, \( f \approx g \) means that \( f \) and \( g \) are homotopic.

Finally, if the continuum \( X \) has a metric \( d \), \( x \in X \) and \( A \) is a closed subset of \( X \), let \( d(x, A) = \inf\{d(x, a) : a \in A\} \). Moreover, \( N(\varepsilon, A) \) denotes the set \( \{x \in X : d(x, A) < \varepsilon\} \). Also, if \( U \) is an open subset of \( X \), let \( \langle U \rangle \) denote a Vietoris set, namely: \( \langle U \rangle = \{A \in 2^X : A \subseteq U\} \). It is known that \( \langle U \rangle \) is an open subset of \( 2^X \) (see [2, Theorem 3.1]).

Throughout this paper, the following hyperspaces will be considered. For a continuum \( X \) and \( n \in \mathbb{N} \), define the hyperspaces \( F_n = \{A \subseteq X : 1 \leq |A| \leq n\} \) and \( F(X) = \bigcup\{F_n(X) : n \in \mathbb{N}\} \) (see [2, p. 6]). Let \( (X, T) \) be a topological space, and let \( \{A_i\}_{i=1}^{\infty} \) a sequence of subsets of \( X \). We define the limit inferior of \( \{A_i\}_{i=1}^{\infty} \) and the limit superior of \( \{A_i\}_{i=1}^{\infty} \) as follows:

1. \( \liminf A_i = \{x \in X : \text{for any } U \in T \text{ such that } x \in U, U \cap A_i \neq \emptyset \text{ for all but finitely many } i\} \).
2. \( \limsup A_i = \{x \in X : \text{for any } U \in T \text{ such that } x \in U, U \cap A_i \neq \emptyset \text{ for infinitely many } i\} \).

For more information on this subject, we refer the reader to [2, p. 20–24].

Finally, let \( A, B \in C(X) \). An order arc from \( A \) to \( B \) is a mapping \( \alpha : I \to C(X) \) such that \( \alpha(0) = A, \alpha(1) = B, \) and \( \alpha(r) \subseteq \alpha(s) \) whenever \( r < s \) (see [4, 1.2–1.8]).

3. General tools

Let \( X \) be a topological space and \( A \subseteq X \). Recall that a mapping \( r : X \to A \) is a retraction provided that \( r|_A \) is the identity map on \( A \). Further, \( r \) is a deformation retraction whenever \( r \) is a retraction which is homotopic to the identity map on \( X \). Finally, \( r \) is a strong deformation retraction provided that there exists a homotopy \( F : X \times I \to X \) such that \( F(x, 0) = x, F(x, 1) = r(x) \) and \( F(a,t) = a \) for every \( a \in A \) and \( t \in I \). In these cases, \( A \) is called a retract, a deformation retract or a strong deformation retract, respectively.
The idea of considering the maps in Definitions 3.1 and 3.2 below came from investigating the following questions: Is $2^X_p$ a retract, a deformation retract or a strong deformation retract of $2^X$? Under which conditions can we obtain these properties? In order to answer these questions, we introduce the following mapping.

**Definition 3.1.** Let $X$ be a continuum and let $p \in X$. We define the mapping $\psi_p : 2^X \rightarrow 2^X_p$ by $\psi_p(A) = A \cup \{p\}$ (note that $\psi_p$ is a retraction).

We would also like to analyze a similar situation in $C(X)$, and then compare the results with those of $2^X$. We introduce another mapping.

**Definition 3.2.** Let $X$ be a continuum and let $p \in X$. We define the mapping $\phi_p : C(X) \rightarrow C_2(p, X)$ by $\phi_p(A) = A \cup \{p\}$.

In this case we cannot ask whether $\phi_p$ is a retraction, because $C_2(p, X) \not\subseteq C(X)$. However, this mapping turned out to have interesting properties, so we establish the following convention. Since $\phi_p|_{C(p, X)}$ is the identity map in $C(p, X)$, we shall say that $\phi_p$ is a contraction in $C_2(X)$ if there exists a homotopy $G : C(X) \times I \rightarrow C_2(X)$ between $\phi_p$ and the identity map in $C(X)$. Finally, we will say that $\phi_p$ is a strong deformation retraction in $C_2(X)$ if the homotopy $G$ is such that $G(A, t) = A$, whenever $A \in C(p, X)$ and $t \in I$.

Most of the results will be shown only for the mapping $\phi_p$. We will also mention similar results for $\psi_p$; in these cases we shall not include the proof, for it can be done in a similar way to that for $\phi_p$.

We shall need the following auxiliary results.

**Lemma 3.3** [4, Lemma 1.48]. The union function $u : 2^{2^X} \rightarrow 2^X$ given by $u(A) = \bigcup A = \bigcup \{A : A \in A\}$ is a mapping.

**Lemma 3.4** [4, Lemma 1.49]. Let $X$ be a continuum and $A$ be a subcontinuum of $2^X$. If $A \cap C(X) \neq \emptyset$, then $\bigcup A$ is a subcontinuum of $X$.

Let $X$ and $Y$ be topological spaces. Recall that $X$ is contractible in $Y$ provided that every mapping $f : X \rightarrow Y$ is homotopic to a constant map. Moreover, $X$ is contractible if it is contractible in itself. The mapping $f$ is a contraction.

The proof of the following lemma is similar to the proof of [3, 3.3], as well as to the proof of (4) implies (1) of [2, Theorem 20.1].

**Lemma 3.5.** Let $Y$ be a subcontinuum of a continuum $X$. If $C(Y)$ is contractible in $C_2(X)$, then there exists a mapping $G : C(Y) \times I \rightarrow C(X)$ such that for each $A \in C(Y)$, we have $G(A, 0) = A$, $G(A, 1) = X$ and $G(A, s) \subseteq G(A, t)$ whenever $s \leq t$.

The proof of the following result is similar to the proof of Lemma 3.5.
Lemma 3.6. Let $Y$ be a subcontinuum of a continuum $X$. If $2^Y$ is contractible in $2^X$, then there exists a mapping $G : 2^Y \times I \to 2^X$ such that for each $A \in 2^Y$, we have $G(A, 0) = A$, $G(A, 1) = X$ and $G(A, s) \subset G(A, t)$ whenever $s \leq t$.

4. Characterizations

In this section we will present conditions under which $\phi_p$ is a (strong) deformation retraction in $C_2(X)$, and $\psi_p$ is a (strong) deformation retraction.

Theorem 4.1. Let $X$ be a continuum. Then $\phi_p$ is a deformation retraction in $C_2(X)$ for each $p \in X$ if and only if $C(X)$ is contractible.

Proof. Let $p \in X$ and suppose that $\phi_p$ is a deformation retraction in $C_2(X)$. Then, there exists a homotopy $\Phi : C(X) \times I \to C_2(X)$ such that $\Phi(A, 0) = A$ and $\Phi(A, 1) = \phi_p(A)$, for each $A \in C(X)$. Take an order arc $\alpha$, from $\{p\}$ to $X$, and consider a function $\hat{H} : C(X) \times I \to C(X)$ given by

$$\hat{H}(A, t) = \begin{cases} \bigcup \{\Phi(A, 2s) : s \in [0, t]\}, & \text{if } t \in [0, \frac{1}{2}], \\ \bigcup \{\Phi(\{A\} \times t)\} \cup \alpha(2t - 1), & \text{if } t \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

Since $\Phi(\{A\} \times \{0, r\}) \in C(2^X)$ for each $A \in C(X)$ and $r \in I$, applying Lemma 3.3 we get that

$$\hat{H}|_{C(X) \times [0, \frac{1}{2}]}(A, t) \in 2^X \quad \text{and} \quad \hat{H}|_{C(X) \times [0, \frac{1}{2}]} \text{ is continuous.} \quad (1)$$

Furthermore, since $\Phi(A, 0) = A \in C(X)$, Lemma 3.4 yields

$$\hat{H}(A, t) \in C(X), \quad (2)$$

for each $(A, t) \in C(X) \times [0, \frac{1}{2}]$. Now, note that

$$p \in \Phi(A, 1) \cap \alpha(0) \subseteq \bigcup \{\Phi(\{A\} \times t)\} \cap \alpha(2t - 1), \quad (3)$$

for each $(A, t) \in C(X) \times [\frac{1}{2}, 1]$. Thus, $\hat{H}(A, t) \in C(X)$, whenever $(A, t) \in C(X) \times [\frac{1}{2}, 1]$. Hence, according to (1), (2), (3) and Lemma 3.3, we get that $\hat{H}$ is well defined and continuous. On the other hand, clearly $\hat{H}(A, 0) = \Phi(A, 0) = A$ and $X = \alpha(1) \subseteq \hat{H}(A, 1)$ for each $A \in C(X)$; therefore, $\hat{H}$ is a contraction in $C(X)$.

Conversely, suppose that $C(X)$ is contractible and let $p \in X$. Then there exists a homotopy $G : C(X) \times I \to C(X)$ such that $G(A, 0) = A$ and $G(A, 1) = B$ for some $B \in C(X)$ and for each $A \in C(X)$. We may suppose that $B = \{p\}$. Consider now the function $g : C(X) \times I \to C_2(X)$ given by

$$g(A, t) = \begin{cases} G(A, 2t), & \text{if } t \in [0, \frac{1}{2}], \\ \{p\} \cup G(A, 2 - 2t), & \text{if } t \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

It is easy to see that $g$ is well defined and continuous. Moreover, $g(A, 0) = G(A, 0) = A$ and $g(A, 1) = \{p\} \cup G(A, 0) = A \cup \{p\} = \phi_p(A)$, for each $A \in C(X)$. Therefore, $\phi_p$ is a deformation retraction in $C_2(X)$. \(\square\)
The proof of the following result is similar to the proof of Theorem 4.1.

**Theorem 4.2.** Let $X$ be a continuum. Then $\psi_p$ is a deformation retraction for each $p \in X$ if and only if $2^X$ is contractible.

From now on, we shall use the following notation. Let $W$ be a subset of a continuum $X$. Then $C(W)$ will denote the subcontinua of $X$ which are contained in $W$. Similarly, $C_2(W) = \{ A \in C_2(X) : A \subset W \}$. Also, $H$ will denote the Hausdorff distance in $2^X$.

**Lemma 4.3.** Let $X$ be a continuum and $p \in X$. If $\{ V_n \}_{n=1}^\infty$ is a basis of nested neighborhoods of $p$, such that $C(V_{n+1})$ is contractible in $C_2(V_n)$ for each $n \in \mathbb{N}$, then there exists a basis of closed nested neighborhoods $\{ K_n \}_{n=1}^\infty$ of $p$, such that $C(K_{n+1})$ is contractible in $C_2(K_n)$ for each $n \in \mathbb{N}$.

**Proof.** Since $\{ V_n \}_{n=1}^\infty$ is a basis of nested neighborhoods of $p$, we may suppose that $\text{cl}(V_{n+1}) \subset V_n$, for every $n \in \mathbb{N}$. Fix $n \in \mathbb{N}$. Then $C(\text{cl}(V_{n+2})) \subset C(V_{n+1})$, which is contractible in $C_2(V_n)$. Thus, $C(\text{cl}(V_{n+2}))$ is contractible in $C_2(\text{cl}(V_n))$. \qed

**Remark.** In most of the results of this section, we will consider an element $p$ of a continuum $X$ and a basis of closed, nested neighborhoods $\{ K_n \}_{n=1}^\infty \subset C(X)$, of $p$, such that $C(K_{n+1})$ is contractible in $C_2(K_n)$ for each $n \in \mathbb{N}$. In fact, we do not need to ask the neighborhoods to be closed, as Lemma 4.3 shows. If we do, it is just for convenience.

**Theorem 4.4.** Let $X$ be a continuum and let $p \in X$. If $\psi_p$ is a strong deformation retraction in $C_2(X)$, then $C(X)$ is contractible and there exists a basis of closed, nested neighborhoods $\{ K_n \}_{n=1}^\infty \subset C(X)$, of $p$, such that $C(K_{n+1})$ is contractible in $C_2(K_n)$ for each $n \in \mathbb{N}$.

**Proof.** Let $\Phi : C(X) \times I \rightarrow C_2(X)$ be a homotopy such that $\Phi(A,0) = A$, $\Phi(A,1) = \psi_p(A)$ and $\Phi([p],t) = [p]$ for each $t \in I$. Using Theorem 4.1, we get directly that $C(X)$ is contractible.

We will construct inductively the basis of closed, nested neighborhoods $K_n$. Define $K_2 = K_1 = X$. Then $C(K_2)$ is contractible in $C(K_1) \subset C_2(K_1)$. Assume now that $K_n$ is already constructed. We shall construct a compact, connected neighborhood $K_{n+1}$ of $p$, in such a way that $K_{n+1} \subset N(\frac{1}{n}, p) \cap K_n$ and $C(K_{n+1})$ is contractible in $C_2(K_n)$.

Let $\varepsilon > 0$ be such that $\varepsilon \in (0, \frac{1}{n})$ and $N(\varepsilon, p) \subset \text{int}(K_n)$. Since $\Phi$ is uniformly continuous we can take $\delta \in (0, \varepsilon)$ such that, for each $A \subset C(X)$ and $t, t' \in I$, we have that if $H(A,\{p\}) \leq \delta$, then

$$H(\Phi(A,t),\{p\}) = H(\Phi(A,t),\Phi([p],t')) < \varepsilon.$$  \hspace{1cm} (4)

Similarly, we may consider $\delta_1 \in (0, \delta)$ such that if $H(A,\{p\}) \leq \delta_1$, then $H(\Phi(A,t),\{p\}) = H(\Phi(A,t),\Phi([p],t')) < \delta$.

Define $K_{n+1} = \bigcup \{ \Phi([a] \times I) : a \in \text{cl}(N(\delta_1, p)) \}$. We will prove in a series of steps that $K_{n+1}$ satisfies the required conditions.

**Step 1.** $K_{n+1}$ is a neighborhood of $p$.
Let \( a \in N(\delta_1, p) \). Since \( \Phi([a], 0) = \{a\} \), then \( a \in \bigcup \Phi([a] \times I) \subset K_{n+1} \). Thus, \( N(\delta_1, p) \subset K_{n+1} \) and, therefore, \( K_{n+1} \) is a neighborhood of \( p \).

Step 2. \( K_{n+1} \subset C(X) \).

Let \( a \in \operatorname{cl}(N(\delta_1, p)) \). Since \( \{a\} \in \Phi([a] \times I) \), it follows from Lemma 3.4 that \( \bigcup \Phi([a] \times I) \subset C(X) \). Moreover, \( a, p \in \bigcup \Phi([a] \times I) \). Thus, \( K_{n+1} \subset C(X) \).

Step 3. \( K_{n+1} \subset N(\frac{1}{n}, p) \cap K_n \).

Let \( y \in K_{n+1} \), then \( y \in \Phi([a], t) \) for some \((a, t) \in \operatorname{cl}(N(\delta_1, p)) \times I \). Thus, \( H(\Phi([a], t), \{p\}) < \delta \). Hence, \( y \in N(\delta, p) \) and

\[
K_{n+1} \subset N(\delta, p) \subset N(\varepsilon, p) \subset N\left(\frac{1}{n}, p\right) \cap K_n.
\]  

(5)

Step 4. \( C(K_{n+1}) \) is contractible in \( C_2(K_n) \).

Take an order arc \( \alpha \), from \( \{p\} \) to \( K_{n+1} \) and consider the function \( G : C(K_{n+1}) \times I \to C_2(K_n) \) given by

\[
G(A, t) = \begin{cases} 
\Phi(A, 2t), & \text{if } t \in \left[0, \frac{1}{2}\right], \\
A \cup \alpha(2t - 1), & \text{if } t \in \left[\frac{1}{2}, 1\right].
\end{cases}
\]

For each \( A \in C(K_{n+1}) \), note that \( \Phi(A, 2t) \in C_2(X) \), whenever \( t \in \left[0, \frac{1}{2}\right] \) and that \( A \cup \alpha(2t - 1) \in C_2(X) \) for each \( t \in \left[\frac{1}{2}, 1\right] \). Thus, to see that \( G \) is well defined, it is enough to prove that \( G(A, t) \subset K_n \) for every \((A, t) \in C(K_{n+1}) \times I \). Let \((A, t) \in C(K_{n+1}) \times I \). If \( t \in \left[\frac{1}{2}, 1\right] \), by (5) we have that \( G(A, t) = A \cup \alpha(2t - 1) \subset K_{n+1} \subset K_n \). On the other hand, if \( t \in \left[0, \frac{1}{2}\right] \), then \( G(A, t) = \Phi(A, 2t) \). By (5) we know that \( A \subset K_{n+1} \subset N(\delta, p) \), hence, according to (4), we get that \( G(A, t) = \Phi(A, 2t) \subset N(\varepsilon, p) \subset K_n \). Therefore, \( G \) is well defined. Furthermore, using Lemma 3.3 it is easy to see that \( G \) is continuous. Finally, \( G(A, 0) = \Phi(A, 0) = A \) and \( G(A, 1) = K_{n+1} \), for each \( A \in C(X) \). Thus, \( C(K_{n+1}) \) is contractible in \( C_2(K_n) \). \( \Box \)

The proof of the following result is similar to the proof of Theorem 4.4.

**Theorem 4.5.** Let \( X \) be a continuum and let \( p \in X \). If \( \psi_p \) is a strong deformation retraction, then \( 2^X \) is contractible and there exists a basis of closed, nested neighborhoods \( \{K_{n} \}_{n=1}^{\infty} \subset C(X) \), of \( p \), such that \( 2^{K_{n+1}} \) is contractible in \( 2^{K_n} \) for each \( n \in \mathbb{N} \).

**Lemma 4.6.** Let \( X \) be a continuum, let \( p \in X \) and let \( \{K_i\}_{i=1}^{\infty} \subset C(X) \) be a basis of neighborhoods of \( p \) such that \( K_{i+1} \subseteq \text{int}(K_i) \) for every \( i \in \mathbb{N} \) and \( K_1 = X \). For each \( A \in 2^X \), with \( A \neq \{p\} \), define

\[
w(A) = \begin{cases} 
\max\{i \in \mathbb{N} : A \subset K_i\}, & \text{if } A \subset K_2, \\
2, & \text{if } A \setminus K_2 \neq \emptyset.
\end{cases}
\]

If \( A \in 2^X \), \( A \neq \{p\} \) and \( \{A_n\}_{n=1}^{\infty} \subset 2^X \) is a sequence which converges to \( A \), then there exists \( N \in \mathbb{N} \) such that if \( n > N \), then \( w(A_n) \in [w(A), w(A) - 1] \).

**Proof.** If \( A \setminus K_2 \neq \emptyset \), it is easy to see that there exists \( N \in \mathbb{N} \) such that \( A_n \setminus K_2 \neq \emptyset \) for every \( n > N \). Therefore, \( w(A) = 2 = w(A_n) \) whenever \( n > N \). Suppose now that \( A \subset K_2 \).
In this case we know that \( A \subset K_{w(A)} \subset \text{int}(K_{w(A)-1}) \), in other words, \( A \in (\text{int}(K_{w(A)})) \). Hence, there exists \( n \in \mathbb{N} \) such that \( A_n \in (\text{int}(K_{w(A)-1})) \) whenever \( n > N \). Therefore, \( w(A_n) \geq w(A) - 1 \) if \( n > N \). Now, if \( w(A_n) \geq w(A) + 1 \) for a subsequence \( \{A_{n_k}\}_{k=1}^{\infty} \) of \( \{A_n\}_{n=1}^{\infty} \), then we have \( A = \lim A_{n_k} \subset K_{w(A)+1} \), a contradiction. \( \square \)

**Proposition 4.7.** Let \( X \) be a continuum such that \( C(X) \) is contractible and let \( p \in X \). If there exists a basis of closed, nested neighborhoods \( \{K_i\}_{i=1}^{\infty} \subset C(X) \) of \( p \), in such a way that \( C(K_i) \) is contractible in \( C_2(K_i) \) for every \( i \in \mathbb{N} \), then there exists a mapping \( G : C(X) \times I \rightarrow C(X) \) such that \( G(A, 0) = A, \ p \in G(A, 1) \) for each \( A \in C(X) \) and \( G((p), t) = \{p\} \), for each \( t \in I \).

**Proof.** We may suppose that \( K_1 = X \), for \( C(K_2) \) is contractible in \( C(X) \), and that \( K_{i+1} \subseteq \text{int}(K_i) \) for each \( i \in \mathbb{N} \). We will develop the proof of the proposition in four steps.

**Step 1. Definitions.**

Since \( C(X) \) is contractible, by Lemma 3.5, there exists a mapping \( G_1 : C(X) \times I \rightarrow C(X) \) such that \( G_1(A, 0) = A, \ G_1(A, 1) = X = K_1 \) and \( G_1(A, s) \subset C_1(A, t) \), whenever \( 0 \leq s \leq t \leq 1 \) and \( A \in C(X) \). Further, for each \( i \in \mathbb{N} \setminus \{1\} \) we can take a mapping \( G_i : C(K_{i+1}) \times I \rightarrow C(K_i) \) such that \( G_i(A, 0) = A, \ G_i(A, 1) = K_i \) and \( G_i(A, s) \subset G_i(A, t) \), whenever \( 0 \leq s \leq t \leq 1 \) and \( A \in C(K_{i+1}) \). Take also a sequence \( \{x_n\}_{n=1}^{\infty} \subset I \), which converges to \( 1 \), in such a way that \( 0 = x_1 < x_n < x_{n+1} < 1 \) for every \( n > 1 \). Moreover, for each \( n \in \mathbb{N} \) we will need a mapping \( f_n : [x_n, x_{n+1}] \rightarrow I \) such that \( f_n(x_n) = 0 \) and \( f_n(x_{n+1}) = 1 \). We also define \( A_n = \{A \in C(X) : A \setminus \text{int}(K_{n+2}) \neq \emptyset \} \) and \( m_n = \inf\{H(A, C(K_{n+2}) : A \in A_n\} \). It is easy to see that \( A_n = C(X) \setminus \langle \text{int}(K_{n+2}) \rangle \), thus, \( A_n \) is a closed subset of \( C(X) \). In particular, \( A_n \) is compact. Hence, since \( A \) takes values in \( A_n \), \( H \) is continuous and \( C(K_{n+2}) \) is compact too, we obtain that \( m_n = \min\{H(A, C(K_{n+2}) : A \in A_n\} \). Now, if \( H(A, C(K_{n+3})) = 0 \) for some \( A \in C(X) \), then we have that \( A \subset C(K_{n+3}) \subset \text{int}(K_{n+2}) \). Therefore, \( m_n > 0 \) for each \( n \in \mathbb{N} \).

**Step 2. The mappings \( \hat{G}_i \). Construction and properties.**

For \( i \in \mathbb{N} \setminus \{1\} \), in this step we will define a family of mappings \( \hat{G}_i : C(K_{i+1}) \times [x_1, x_{i+1}] \rightarrow C(X) \), and also a particular mapping \( \hat{G}_1 : C(K_1) \times [x_1, x_2] \rightarrow C(X) \). We shall present the basic properties of these functions, which will later be very useful. Define \( \hat{G}_1 : C(K_1) \times [x_1, x_2] \rightarrow C(X) \) by \( \hat{G}_1(A, t) = G_1(A, \min\{1, \frac{f_1(t)}{m_1} \cdot H(A, C(K_1))\}) \). For \( i \geq 2 \) define:

\[
\hat{G}_i(A, t) = \bigcup_{j=1}^{i-1} G_j \left( A, \min\left\{ 1, \frac{1}{m_j} \cdot H(A, C(K_j+3)) \right\} \right) \\
\cup G_i \left( A, \min\left\{ 1, \frac{f_i(t)}{m_i} \cdot H(A, C(K_i+3)) \right\} \right).
\]

**Property 1.** It is easy to see that \( \hat{G}_i \) is well defined for every \( i \in \mathbb{N} \). The continuity follows from Lemma 3.3. Hence, \( \hat{G}_i \) is a mapping for every \( i \in \mathbb{N} \).
Property 2. Let $A \subset K_{i+1}$ and $A \setminus \text{int}(K_{i+2}) \neq \emptyset$. Since $A \in \mathcal{A}_i$, we get that $\frac{1}{m_i} \cdot H(A, C(K_{i+3})) \geq 1$, therefore,

$$G_i\left(A, \min \left\{1, \frac{1}{m_i} \cdot H(A, C(K_{i+3}))\right\}\right) = G_i(A, 1) = K_i.$$

Property 3. Using a straightforward computation, one can easily prove that $\tilde{G}_i(A, x_{i+1}) = \tilde{G}_{i+1}(A, x_{i+1})$ for every $i \in \mathbb{N}$ and every $A \in C(K_{i+2})$.

Property 4. Let $i \geq 4$. If $A \in C(K_i)$, $t \in [x_k, x_{k+1}]$ and $k \leq i - 3$, then $A \in C(K_i) \subset C(K_{k+3}) \subset C(K_j)$ for each $j \leq k$. Hence,

$$\tilde{G}_k(A, t) = \bigcup_{j=1}^{k-1} G_j\left(A, \min \left\{1, \frac{1}{m_j} \cdot H(A, C(K_{j+3}))\right\}\right)$$

$$+ G_k\left(A, \min \left\{1, \frac{f_k(t)}{m_k} \cdot H(A, C(K_k))\right\}\right)$$

$$= \bigcup_{j=1}^{k} G_j\left(A, \min \left\{1, f(t) \cdot H(A, C(K_i))\right\}\right) = A.$$

Property 5. Let $i \in \mathbb{N}$, $A \in C(K_{i+2})$ and $t \in [x_i, x_{i+1}]$. Clearly $\tilde{G}_1(A, t) \subset K_1 = X$ whenever $t \in [x_1, x_2]$ and $A \in C(K_3)$. On the other hand, if $i \geq 2$ and $A \in C(K_{i+2})$, by Property 4 we get

$$\tilde{G}_i(A, t) = \bigcup_{j=1}^{i-1} G_j\left(A, \min \left\{1, \frac{1}{m_j} \cdot H(A, C(K_{j+3}))\right\}\right)$$

$$+ G_i\left(A, \min \left\{1, \frac{f_i(t)}{m_i} \cdot H(A, C(K_{i+3}))\right\}\right)$$

$$= A \cup G_i\left(A, \min \left\{1, f_i(x) \cdot H(A, C(K_{i+1}))\right\}\right) \subset A \cup K_i \subset K_i.$$

Property 6. If $A \setminus K_2 \neq \emptyset$, then $\tilde{G}_1(A, x_2) = G_1(A, 1) = X$. Further, if $i \in \mathbb{N} \setminus \{1\}$ is such that $A \in C(K_{i+1})$ and that $A \setminus \text{int}(K_{i+2}) \neq \emptyset$, then clearly $\frac{f_i(x_{i+1})}{m_i} \cdot H(A, C(K_{i+3})) = \frac{1}{m_i} \cdot H(A, C(K_{i+3})) \geq 1$. Therefore,

$$\tilde{G}_i(A, x_{i+1}) = \bigcup_{j=1}^{i-1} G_j\left(A, \min \left\{1, \frac{1}{m_j} \cdot H(A, C(K_{j+3}))\right\}\right)$$

$$+ G_i\left(A, \min \left\{1, \frac{f_i(x_{i+1})}{m_i} \cdot H(A, C(K_{i+3}))\right\}\right)$$

$$= \tilde{G}_{i-1}(A, x_i) \cup G_i(A, 1) = \tilde{G}_{i-1}(A, x_i) \cup K_i.$$

For each $A \in C(X)$, $A \neq \{ p \}$, we may consider a number

$$w(A) = \begin{cases} \max\{ i \in \mathbb{N}: A \subseteq K_i\}, & \text{if } A \subseteq K_2, \\ 2, & \text{if } A \setminus K_2 \neq \emptyset. \end{cases}$$

Define a function $G': (C(X) \setminus \{ \{ p \} \}) \times I \to C(X)$ given by

$$G'(A,t) = \begin{cases} \hat{G}_i(A,t), & \text{if } x_i \leq t \leq x_{i+1} \leq x_{w(A)}, \\ \hat{G}_{w(A)-1}(A,x_{w(A)}), & \text{if } t \geq x_{w(A)}. \end{cases}$$

According to Property 3, the function $G'$ is well defined. In order to prove continuity of $G'$, let $(\{A_n, t_n\})_{n=1}^\infty \subseteq (C(X) \setminus \{ \{ p \} \}) \times I$ be a sequence which converges to $(A, t_0) \in (C(X) \setminus \{ \{ p \} \}) \times I$.

Case 1. $t_0, t_n \leq x_{w(A)}$ for each $n \in \mathbb{N}$.

If $t_0 \in (x_i, x_{i+1})$ for some $i \leq w(A) - 1$, then $t_n \in (x_i, x_{i+1})$ for large enough $n$. Hence, using Property 1, if $n$ is large enough we have $G'(A_n, t_n) = \hat{G}_i(A_n, t_n) \to \hat{G}_i(A, t_0) = G'(A, t_0)$.

Case 2. $t_0 \geq x_{w(A)}$.

By Lemma 4.6, we may assume that either $w(A_n) = w(A)$ or $w(A_n) = w(A) - 1$. In this case we may also assume that $t_n \geq x_{w(A)} \geq x_{w(A_n)}$ for each $n \in \mathbb{N}$. It suffices to analyze two subcases.

Subcase 1. $w(A_n) = w(A)$ for each $n \in \mathbb{N}$.

In this subcase it follows that $G'(A_n, t_n) = \hat{G}_{w(A_n)-1}(A_n, x_{w(A_n)}) \to \hat{G}_{w(A)-1}(A, x_{w(A)}) = G'(A, t_0)$.

Subcase 2. $w(A_n) = w(A) - 1$ for every $n \in \mathbb{N}$.

If $w(A) = 2$, by Lemma 4.6 we know that $w(A_n) = 2$ and we may apply the previous subcase. Thus, we may assume that $w(A) \geq 3$.

Since $A_n \to A$, in this subcase $A \not\subseteq \text{int}(K_{w(A)})$. Then

$$G'(A, t_0) = \hat{G}_{w(A)-1}(A, x_{w(A)})$$

$$= \bigcup_{j=1}^{w(A)-2} G_j(A, \min\left\{ 1, \frac{1}{m_j} \cdot H(A, C(K_{j+3})) \right\})$$

$$\cup \hat{G}_{w(A)-1}(A, \min\left\{ 1, \frac{1}{m_{w(A)-1}} \cdot H(A, C(K_{w(A)+2})) \right\})$$

$$= \bigcup_{j=1}^{w(A)-3} G_j(A, \min\left\{ 1, \frac{1}{m_j} \cdot H(A, C(K_{j+3})) \right\})$$

$$\cup K_{w(A)-2} \cup K_{w(A)-1}$$

$$= \hat{G}_{w(A)-3}(A, x_{w(A)-2}) \cup K_{w(A)-2}. \quad (6)$$
On the other hand, in this case we know that $A_n \subseteq K_{w(A)} - 1$ and $A \setminus \text{int}(K_{w(A)}) \neq \emptyset$. Taking $i = w(A) - 2$, Property 6 and (6) yield

$$G'(A_n, t_n) = \widehat{G}_{w(A)-1}(A_n, x_{w(A_n)}) = \widehat{G}_{w(A)-2}(A_n, x_{w(A)-1})$$
$$= \widehat{G}_{w(A)-3}(A_n, x_{w(A)-2}) \cup K_{w(A)-2}$$
$$\rightarrow \widehat{G}_{w(A)-3}(A, x_{w(A)-2}) \cup K_{w(A)-2} = G'(A, t_0).$$

Therefore, in any case we get that $G'$ is continuous.

**Step 4. The mapping $G$.**

We finally define $G : C(X) \times I \to C(X)$ given by

$$G(A, t) = \begin{cases} G'(A, t), & \text{if } A \neq \{p\}, \\ \{p\}, & \text{if } A = \{p\}. \end{cases}$$

We proceed to present the properties of this function that we are interested on.

(i) $G$ is a mapping.

Clearly $G$ is well defined. Since $G'$ is continuous, it is enough to take a sequence $(A_n, t_n)_{n=1}^{\infty} \subseteq (C(X) \setminus \{\{p\}\}) \times I$ which converges to $(\{p\}, t_0)$, and prove that $G(A_n, t_n) \to \{p\}$. Notice that $w(A_n) \to \infty$.

**Case 1.** $t_0 < 1$.

Let $k \in \mathbb{N}$ be such that $x_k \leq t_0 \leq x_{k+1}$ and define $x_0 = x_1 = 0$ and $G_0 = \widehat{G}_1$.

In this case, we may choose a large enough $n$, in such a way that $t_n \in [x_{k-1}, x_k] \cup [x_k, x_{k+1}] \cup [x_{k+1}, x_{k+2})$, $t_n \leq x_{w(A)-1}$ and $k \leq w(A_n) - 4$. One can also verify that $G(A_n, t_n) = G'(A_n, t_n) \in \{G_{r-1}(A_n, t_n), \widehat{G}_k(A_n, t_n), \widehat{G}_{k+1}(A_n, t_n)\}$. Now, from Property 4 we deduce that $G_r(A_n, t_n) = A_n$ whenever $r \in \{k - 1, k, k + 1\}$. Hence, $G(A_n, t_n) = A_n \to \{p\}$.

**Case 2.** $t_0 = 1$.

Let $\varepsilon > 0$ and $N \in \mathbb{N}$ be such that if $n \geq N$, then diam$(K_n) < \varepsilon$. Take $n > N$ such that $x_{N+1} < t_n \leq 1$ and $N + 1 < w(A_n) - 1$. Then, for $n > N$ we have that $G(A_n, t_n) = G'(A_n, t_n) = \widehat{G}_r(A_n, t_n)$, for some elements $r \in \{N + 1, w(A_n) - 1\}$, and $z_n \in \{t_n, x_{w(A_n)}\}$.

Note that $A_n \subseteq K_{r+1}$. Hence, by Property 5, we have that $\widehat{G}_{r-1}(A_n, x_r) \subseteq K_{r-1}$. Therefore,

$$\widehat{G}_r(A_n, z_n) = \bigcup_{j=1}^{r-1} G_j \left( A_n, \min \left\{ 1, \frac{1}{m_j} \cdot H(A_n, C(K_{j+3})) \right\} \right)$$
$$\cup G_r \left( A_n, \min \left\{ 1, \frac{f_r(z_n)}{m_r} \cdot H(A_n, C(K_{r+3})) \right\} \right)$$
$$= \widehat{G}_{r-1}(A_n, x_r)$$
$$\cup G_r \left( A_n, \min \left\{ 1, \frac{f_r(z_n)}{m_r} \cdot H(A_n, C(K_{r+3})) \right\} \right)$$
$$\subseteq K_{r-1} \cup K_r = K_{r-1} \subseteq K_N.$$

Hence, diam$(G(A_n, t_n)) < \varepsilon$ for every $n > N$. Thus, diam$(G(A_n, t_n)) \to 0$. Therefore, $G(A_n, t_n) \to \{p\}$ and we may conclude that $G$ is a mapping.
Let $\text{Theorem 4.9.}$ exists a basis of closed, nested neighborhoods \{\cnu\}

\text{Proposition 4.8.}$ $G(A, x_2) = X$. Therefore, $p \in G(A, 1)$.

Suppose now that $A \in C(K_2)$ and $A \neq \{p\}$. Since $A \subset K_w(A)$ and $A \setminus K_w(A) + 1 \neq \emptyset$, by Property 6 we get that $p \in K_w(A) - 1 \cup \hat{G}_w(A) - 2(A, x_w(A) - 1) = \hat{G}_w(A) - 1(A, x_w(A)) = G'(A, 1) = G(A, 1)$.

The proof is complete. □

The proof of the following result is similar to the proof of Proposition 4.7.

\text{Proposition 4.8.}$ Let $X$ be a continuum such that $2^X$ is contractible and let $p \in X$. If there exists a basis of closed, nested neighborhoods \{\cnu\} of $p$, such that $2^{K_i+1}$ is contractible in $2^K$, for each $i \in \mathbb{N}$, then there exists a mapping $G : 2^X \times I \to 2^X$ such that $G(A, 0) = A$, $p \in G(A, 1)$ for every $A \in 2^X$ and $G(\{p\}, t) = \{p\}$, for every $t \in I$.

The converses of Theorems 4.4 and 4.5 are also true, as we now show.

\text{Theorem 4.9.}$ Let $X$ be a continuum such that $C(X)$ is contractible and let $p \in X$. If there exists a basis of closed, nested neighborhoods \{\cnu\} of $p$, such that $C(K_i+1)$ is contractible in $C_2(K_i)$ for each $i \in \mathbb{N}$, then $\phi_p$ is a strong deformation retraction in $C_2(X)$.

\text{Proof.}$ By Proposition 4.7, there exists a mapping $G : C(X) \times I \to C(X)$ such that $G(A, 0) = A$, $p \in G(A, 1)$ for every $A \in C(X)$ and $G(\{p\}, t) = \{p\}$, for every $t \in I$.

Define a function $\tilde{G} : C(X) \times I \to C(X)$ by

\[
\tilde{G}(A, t) = \begin{cases} 
\bigcup \{G(\{a\}, t \cdot \frac{d(p, A)}{d(p, a)}): a \in A\}, & \text{if } p \notin A, \\
A, & \text{if } p \in A.
\end{cases}
\]

We shall develop the rest of the proof in a series of steps.

\text{Step 1.}$ $\tilde{G}$ is well defined.

Let $A \in C(X) \setminus C(p, x) \text{ and } a \in A$. Then $0 \leq t \cdot \frac{d(p, A)}{d(p, a)} \leq 1$, whenever $t \in I$.

Hence, $G(\{a\}, t \cdot \frac{d(p, A)}{d(p, a)}) \in \hat{C}(X)$. Now, consider the function $\eta_A(\hat{X} \setminus \{p\}) \times I \to \hat{C}(X)$ given by $\eta_A(a, t) = G(\{a\}, t \cdot \frac{d(p, A)}{d(p, a)})$ and note that it is continuous. Therefore, $\{G(\{a\}, t \cdot \frac{d(p, A)}{d(p, a)}): a \in A\} = \eta_A(A \times \{t\}) \subset \hat{C}(C(X))$ for each $(A, t) \in \hat{C}(X) \times I$. Hence, applying Lemma 3.4 we get that $\bigcup \{G(\{a\}, t \cdot \frac{d(p, A)}{d(p, a)}): a \in A\} \subset \hat{C}(X)$. Thus, $\tilde{G}$ is well defined.

\text{Step 2.}$ $\tilde{G}$ is continuous.

Let $([A_n, b_n])_{n=1}^\infty \subset \hat{C}(X) \times I$ be a sequence converging to some $(A, t) \in \hat{C}(X) \times I$.

\text{Case 1.}$ $p \notin A$.

In this case $d(p, a) > 0$ for every $a \in A$. The continuity of $\tilde{G}$ follows from Lemma 3.3.

\text{Case 2.}$ $p \in A$.

We may assume that $p \notin A_n$ for each $n \in \mathbb{N}$.
Let \( x \in \lim sup \hat{G}(A_n, t_n) \). Then there exist a subsequence \( \{A_{n_k}\}_{k=1}^\infty \) of the original sequence and \( x_{n_k} \in \hat{G}(A_{n_k}, t_{n_k}) \) for each \( k \), such that \( x_{n_k} \to x \). Hence, \( x_{n_k} \in G(A_{n_k}, t_{n_k}) \). Since \( A \) is compact, we may suppose that \( a_{n_k} \to b \) for some \( b \in A \). Now, if \( x = p \) we get directly that \( x \in A \). In case that \( x \neq p \), we may suppose that \( d(p, a_{n_k}) \) does not converge to 0. However \( d(p, A_{n_k}) \to 0 \), thus, \( x_{n_k} \in G((a_{n_k}), t_{n_k}) \to G((b), 0) \subset A \). Therefore, \( x \in A \) and \( \lim sup \hat{G}(A_n, t_n) \subset A \).

Now take \( a \in A \). We shall see that \( a \in \lim inf \hat{G}(A_n, t_n) \).

Subcase 1. \( a \neq p \).

Let \( \{a_n\}_{n=1}^\infty \) be a sequence which converges to \( a \), and such that \( a_n \in A_n \) for each \( n \in \mathbb{N} \).
In this subcase \( d(p, a_n) \to d(p, a) \neq 0 \), hence, \( G((a_n), t_n) \cdot \frac{d(p, A_n)}{d(p, a_n)} \to G((a), t) \cdot \frac{d(p, A)}{d(p, a)} = G((a), 0) = \{a\} \). Thus, \( a \in \lim inf \hat{G}(A_n, t_n) \).

Subcase 2. \( a = p \).

Since every \( A_n \) is compact, for each \( n \in \mathbb{N} \) take \( b_n \in A_n \) such that \( d(p, A_n) = d(p, b_n) \).
Then it is easy to see that \( b_n \to p \) and that \( G((b_n), t_n) \cdot \frac{d(p, A_n)}{d(p, b_n)} \to G((p), t) = \{p\} \). Hence, \( a = p \in \lim inf \hat{G}(A_n, t_n) \). Therefore, \( A \subset \lim inf \hat{G}(A_n, t_n) \).

We may now conclude that \( \hat{G}(A_n, t_n) \to A \). Thus, \( \hat{G} \) is continuous.

Step 3. Note that \( \hat{G}(A, t) = A \) for each \( A \in C(p, X) \) and \( t \in I \).

Step 4. If \( A \in C(X) \setminus C(p, X) \), then \( \hat{G}(A, 0) = \bigcup \{G([a], 0) : a \in A\} = \bigcup \{[a] : a \in A\} = A \).

Step 5. Let \( A \in C(X) \setminus C(p, X) \) and take \( q \in A \) such that \( d(p, A) = d(p, q) \). Then \( p \in G((q), 1) = G((q), \frac{d(p, A)}{d(p, q)}) \subset \hat{G}(A, 1) \).

Step 6. \( \phi_p \) is a strong deformation retraction in \( C_2(X) \).

Let \( g : C(X) \times I \to C_2(X) \) be given by

\[
g(A, t) = \begin{cases} 
\hat{G}(A, 2t), & \text{if } t \in [0, \frac{1}{2}], \\
\hat{G}(A, 2(1-t)) \cup [p], & \text{if } t \in [\frac{1}{2}, 1].
\end{cases}
\]

According to step 5, it is easy to see that \( g \) is a mapping. Moreover, as a consequence of steps 3 and 4, we obtain that \( g \) is a homotopy between the identity map and \( \phi_p \), such that \( g(A, t) = A \) for each \( t \) whenever \( A \in C(p, X) \). Thus, \( \phi_p \) is a strong deformation retraction in \( C_2(X) \).

The proof is complete. \( \square \)

The proof of the following result is similar to the proof of Theorem 4.9.

**Theorem 4.10.** Let \( X \) be a continuum such that \( 2^X \) is contractible and let \( p \in X \). If there exists a basis of closed, nested neighborhoods \( \{K_i\}_{i=1}^\infty \subset C(X) \) of \( p \), such that \( 2^{K_{i+1}} \) is contractible in \( 2^{K_i} \) for each \( i \in \mathbb{N} \), then \( \psi_p \) is a strong deformation retraction.

As a corollary of Theorems 4.4 and 4.9 we obtain the following characterization.

**Corollary 4.11.** Let \( X \) be a continuum and let \( p \in X \). Then \( \phi_p \) is a strong deformation retraction in \( C_2(X) \) if and only if \( C(X) \) is contractible and there exists a basis of closed,
nested neighborhoods $\{K_i\}_{i=1}^{\infty} \subset C(X)$ of $p$, such that $C(K_{i+1})$ is contractible in $C_2(K_i)$ for each $i \in \mathbb{N}$.

Similarly, Theorems 4.5 and 4.10 yield the following result.

**Corollary 4.12.** Let $X$ be a continuum and let $p \in X$. Then $\psi_p$ is a strong deformation retraction if and only if $2^X$ is contractible and there exists a basis of closed, nested neighborhoods $\{K_i\}_{i=1}^{\infty} \subset C(X)$ of $p$, such that $2^{K_{i+1}}$ is contractible in $2^{K_i}$ for each $i \in \mathbb{N}$.

Another nice characterization is the following.

**Corollary 4.13.** For a continuum $X$, the following three conditions are equivalent:

1. $X$ is locally connected,
2. for every $p \in X$, $\psi_p$ is a strong deformation retraction, and
3. for every $p \in X$, $\phi_p$ is a strong deformation retraction in $C_2(X)$.

**Proof.** Suppose that $\psi_p$ is a strong deformation retraction. Then, by Corollary 4.12, we get that $X$ is connected in kleinen at $p$, for every $p \in X$. Thus, $X$ is locally connected. Similarly, assuming that $\phi_p$ is a strong deformation retraction in $C_2(X)$ we obtain that $X$ is locally connected.

Further, suppose that $X$ is locally connected and let $p \in X$. We may take a basis of nested, compact and connected neighborhoods $\{K_i\}_{i=1}^{\infty}$ of $p$, which are themselves locally connected. Now, by [4, Theorem 1.93], the hyperspaces $C(X)$ and $C(K_i)$ are contractible for each $i \in \mathbb{N}$, in particular, $C(K_{i+1})$ is contractible in $C_2(K_i)$ for each $i \in \mathbb{N}$. Hence, applying Corollary 4.11 we get that $\phi_p$ is a strong deformation retraction in $C_2(X)$. In a similar way, as a consequence of [4, Theorem 1.93] and Corollary 4.12, we obtain that $\psi_p$ is a strong deformation retraction. \[\square\]

5. A particular case

A question which arises naturally is what happens with the mappings $\phi_p$ and $\psi_p$ in the hyperspaces $F_n(X)$. We will only present a particular case which, nevertheless, turned out to have interesting properties.

Recall that a continuum $X$ is unicoherent provided that $A \cap B$ is connected for every $A, B \in C(X)$ such that $A \cup B = X$.

The following theorem presents a relation between the topological structure of a continuum $X$ and the behavior of the mapping $\phi_p|_{F_2(X)}$.

**Theorem 5.1.** Let $X$ be a continuum and let $p \in X$. If $\phi_p|_{F_1(X)}$ is a deformation retraction in $F_2(X)$ (i.e., $\phi_p|_{F_1(X)}$ is homotopic to the inclusion map from $F_1(X)$ into $F_2(X)$), then $X$ is unicoherent and arcwise connected.

**Proof.** (i) $X$ is unicoherent.
Let $h : F_1(X) \to F_2(X)$ denote the inclusion map of $F_1(X)$ into $F_2(X)$. Since $\phi_p|_{F_1(X)}$ is a deformation retraction in $F_2(X)$, we have $\phi_p|_{F_1(X)} \approx h$.

Suppose that $X$ is not unicoherent. Then, by [5, 12.64], there exists a mapping $f : X \to S^1$ which is not homotopic to a constant map. Consider the following functions:

(a) $\lambda : F_2(X) \to F_2(S^1)$ given by $\lambda([x, y]) = [f(x), f(y)]$, 
(b) $g : X \to F_1(X)$ given by $g(x) = [x]$, and 
(c) $\pi : F_2(S^1) \to S^1$ given by $\pi([z, w]) = zw$.

It is easy to see that all of them are continuous and that $\pi \circ \lambda \circ h \circ g \approx \pi \circ \lambda \circ \phi_p|_{F_1(X)} \circ g$. Now, $(\pi \circ \lambda \circ h \circ g)(x) = (\pi \circ \lambda \circ h)(x) = \pi((f(x))) = f^2(x)$.

Note also that $(\pi \circ \lambda \circ \phi_p|_{F_1(X)} \circ g)(x) = (\pi \circ \lambda \circ \phi_p|_{F_1(X)})(x) = \pi((f(x), f(p))) = f(x)f(p)$. Thus, $f^2 \approx f \cdot f(p)$. However, this yields $f \approx f(p)$, which contradicts the choice of $f$. Therefore, $X$ is unicoherent.

(ii) $X$ is arcwise connected.

Let $x, y \in X$. Since $\phi_p|_{F_1(X)}$ is a deformation retraction in $F_2(X)$, there exists a mapping $F : F_1(X) \times I \to F_2(X)$ such that $F([x], 0) = [x]$ and $F([x], 1) = [x, p]$. Note that, for each $z \in X$, the mapping $\alpha_z : I \to F_2(X)$ given by $\alpha_z(t) = F([z], t)$ is a path in $F_2(X)$, which begins at $[z]$ and finishes at $[z, p]$. In particular, $\alpha_x(I)$ and $\alpha_y(I)$ are locally connected subcontinua of $F_2(X)$. Hence, by [1, Lemma 2.2] and Lemma 3.4 it follows that $\bigcup \alpha_x(I)$ and $\bigcup \alpha_y(I)$ are locally connected subcontinua of $X$. Since both contain $p$, the set $u(\alpha_x(I)) \cup u(\alpha_y(I))$ is a locally connected subcontinuum of $X$ which contains $x$ and $y$.

The conclusion follows from [5, Theorem 8.23].

One could ask whether the converse of the last theorem holds. This is not the case, as we now show.

Example 5.2. Let $X$ be the continuum presented in [2, 19.12] (notice that $C(X)$ is not contractible). Clearly, $X$ is unicoherent and arcwise connected. Now, if we suppose that $\phi_p|_{F_1(X)}$ is a deformation retraction in $F_2(X)$, then, proceeding in a similar way to Theorem 4.1, we obtain that $C(X)$ is contractible, which is absurd. Hence, the converse of Theorem 5.1 is not true.

Lemma 5.3. If $X$ is a contractible continuum, then $\phi_p|_{F_1(X)}$ is a deformation retraction in $F_2(X)$.

Proof. Since $X$ is contractible, we may take a mapping $g : X \times I \to X$ such that $g(x, 0) = x$ and $g(x, 1) = p$ for every $x \in X$. Define $H : F_1(X) \times I \to F_2(X)$ by $H([x], t) = [g(x, t), x]$. Then it is easy to see that $H$ is a mapping such that $H([x], 0) = [x]$ and $H([x], 1) = [x, p] = \phi_p([x])$. Therefore, $\phi_p|_{F_1(X)}$ is a deformation retraction in $F_2(X)$. 

We could also ask whether the contractibility of a continuum $X$ is equivalent to the condition that $\phi_p|_{F_1(X)}$ is a deformation retraction in $F_2(X)$. In fact, this is an open
question, and it is related to the following question: If $F_2(X)$ is contractible, is $X$ contractible? (see [2, 78.20]).

**Questions.**

1. Give necessary and sufficient conditions under which $2^X_p$ is a (strong) deformation retract of $2^X$.
2. Give necessary and sufficient conditions under which $C(p, X)$ is a (strong) deformation retract of $C(X)$.
3. For $n \in \mathbb{N}$ define $F_n(p, X) = \{A \in F_n(X): p \in A\}$. Give necessary and sufficient conditions under which $F_n(p, X)$ is a (strong) deformation retract of $F_n(X)$.
4. Suppose that $\phi_p|_{F_1(X)}$ is a deformation retraction in $F_2(X)$. Does that imply that $X$ is contractible?

**References**