Positive Solutions of Asymptotically Linear Elliptic Eigenvalue Problems

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0. INTRODUCTION

In this paper we are concerned with the existence of positive solutions of the nonlinear elliptic eigenvalue problem

\[ Lu = \lambda f(u) \quad \text{in} \quad \Omega, \]
\[ u = 0 \quad \text{on} \quad \partial \Omega. \]  

(0.1)

Here \( \Omega \) denotes a bounded domain in \( \mathbb{R}^N \) \( (N \geq 1) \) with smooth boundary \( \partial \Omega \), and \( \mathcal{L} \),

\[ \mathcal{L}u = -\sum_{i,k=1}^{N} a_{ik} \frac{\partial^2 u}{\partial x_i \partial x_k} + \sum_{i=1}^{N} a_i \frac{\partial u}{\partial x_i} + au, \]

with real, smooth coefficients \( a_{ik}, a_i, a, a_{ik} = a_{ki} \), and \( a \geq 0 \) in \( \Omega \), is a strongly uniformly elliptic linear differential expression. Further \( f: \mathbb{R}^+ \rightarrow \mathbb{R} \) is a continuously differentiable function which is asymptotically linear in the sense that there exist \( m_x > 0 \), a function \( g \) and a constant \( C \) such that

\[ f(s) = m_x s + g(s), \quad |g(s)| \leq C, \quad \forall s \in \mathbb{R}^+. \]

(0.2)

We suppose \( f(0) \geq 0 \).

The existence of positive solutions of elliptic eigenvalue problems of the form (0.1) has been investigated extensively in recent years; we refer to the survey article of Amann [1] and the references to work by Keener and Keller, Laetsch, Dancer, and others therein. In these papers it is assumed that \( f \) maps \( \mathbb{R}^+ \) into \( \mathbb{R}^+ \) and, in general, the main tool in the proofs is the theory of positive operators in ordered Banach spaces. Clearly, this theory is no longer available if \( f \) is allowed to assume negative values also. We will be interested mainly in this case.

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here. Using a different approach based on topological degree and global bifurcation techniques, we obtain rather complete results which cover both the case $f(s) > 0$ for all $s > 0$, and the new one.

The paper is organized as follows. Section 1 contains the statement of the results; Section 2 deals with the abstract framework; in Section 3 we discuss the bifurcation from infinity, particular interest being posed in the study of the asymptotic behavior of the bifurcating branch, showing how it closely depends on a classical condition given by Keener and Keller [7]; in Section 4 the case $f(0) = 0$ is studied (bifurcation from the trivial solution); Section 5 deals with the case $f(0) > 0$, and Section 6, finally, concerns the intertwining of the branches bifurcating from infinity and from the trivial solution, showing the essential role played by the fact whether $f$ admits nonnegative values on $(0, +\infty)$ or not.

A problem similar to ours is discussed in the recent paper [5] by Brown and Budin; there the authors use the theory of sub- and supersolutions, but are able to give applications to ordinary differential equations only.

A preliminary announcement of our results has been given in [6].

1. Statement of the Results

We shall work essentially in the space $E := C(\bar{\Omega})$, the norm of which we denote by $\| \cdot \|$. By a positive solution of Problem (0.1) we mean a pair $(\lambda, u)$, where $\lambda > 0$ and $u$ is a (classical) solution of (0.1) with $u > 0$ (i.e., $u \geq 0$, in $\Omega$ and $u \neq 0$). Let $\Sigma \subset \mathbb{R}^+ \times E$ be the closure of the set of positive solutions of (0.1). It is our aim to investigate the existence and global behavior of components of $\Sigma$. According to the behavior of the function $f$ near 0, we distinguish between three cases. In Theorems A-C below we collect the results for each case separately.

It is well known that the linear eigenvalue problem

$$ Lu = \lambda u \quad \text{in} \quad \Omega, $$
$$ u = 0 \quad \text{on} \quad \partial \Omega, $$

has a principal (i.e., least) eigenvalue $\lambda_1 > 0$. Further, let $f'_+(0)$ denote the right-sided derivative of $f$ at 0. We first discuss the case $f(0) = 0$.

**Theorem A.** Suppose $f(0) = 0$ and $f'_+(0) > 0$.

(i) Set $\lambda_\infty := (m_\infty)^{-1} \lambda_1$. Then $\lambda_\infty$ is a bifurcation point from infinity for positive solutions, and it is the only one. More precisely, there exists a component $\Sigma_\infty$ of positive solutions which meets $\lambda_\infty, \infty$. If

$$ \lim_{t \to +\infty} \inf_{s \in [-t, 0]} g(s) > 0 \quad (1.1) $$


respectively, then \( \Sigma_\alpha \) bifurcates to the left (right) of \( \lambda_\infty \).

(ii) Set \( \lambda_0 := (f'_+(0))^{-1} \lambda_1 \). Then \( \lambda_0 \) is a bifurcation point from the trivial solution, and it is the only one for positive solutions. From \((\lambda_0 , 0)\) there emanates an unbounded component \( \Sigma_0 \) of positive solutions.

(iii) If \( f(s) > 0 \) for all \( s > 0 \), there is a number \( \lambda^* > 0 \) such that problem (0.1) admits no positive solution with \( \lambda > \lambda^* \). In this case \( \Sigma_0 = \Sigma_\infty \).

(iv) If \( f(s_0) \leq 0 \) for some \( s_0 > 0 \), there exists no positive solution \((\lambda, u)\) with \( \| u \| = s_0 \). Hence the two components \( \Sigma_0 \) and \( \Sigma_\infty \) are disjoint, and Problem (0.1) admits at least two positive solutions for all \( \lambda > \max(\lambda_\infty , \lambda_0) \).

**Theorem B.** Suppose still \( f(0) = 0 \), but \( f'_+(0) \leq 0 \). Then

(i) Assertion (i) of Theorem A holds.

(ii) There is no bifurcation of positive solutions from the line of trivial solutions \( \mathbb{R}^+ \times \{0\} \).

In case \( f(0) > 0 \), our results are given by

**Theorem C.** Suppose \( f(0) > 0 \). Then

(i) Assertion (i) of Theorem A holds.

(ii) There exists an unbounded component \( \Sigma_0 \) of positive solutions meeting \((0,0)\). If \( f(s) > 0 \) for all \( s > 0 \), then \( \Sigma_0 = \Sigma_\infty \).

(iii) If \( f(s_0) \leq 0 \) for some \( s_0 > 0 \), there exists no positive solution \((\lambda, u)\) with \( \| u \| = s_0 \). Hence the two components \( \Sigma_0 \) and \( \Sigma_\infty \) are disjoint, and problem (0.1) has at least two positive solutions for \( \lambda > \lambda_\infty \).

Our main tools in the proof of Theorems A–C are topological degree arguments and a variant of the global bifurcation theorem of Rabinowitz [8]. We extend the function \( f \) to a continuous function \( \tilde{f} \) defined on \( \mathbb{R} \) in such a way that \( \tilde{f}(s) > 0 \) for all \( s < 0 \) (note that if \( f(0) = 0 \) and \( f'_+(0) > 0 \), then \( \tilde{f} \) can not be differentiable at 0). For \( \lambda \geq 0 \) we then look at (arbitrary) solutions \( u \) of the eigenvalue problem

\[
\mathcal{L} u = \lambda f(u) \quad \text{in} \quad \Omega, \\
\quad u = 0 \quad \text{on} \quad \partial \Omega. \tag{1.2}
\]

By the maximum principle, such solutions are nonnegative and hence solutions of our given problem (0.1). Therefore, the closure of the set of nontrivial solutions \((\lambda, u)\) of (1.2) in \( \mathbb{R}^+ \times E \) is exactly \( \Sigma \).
All our results hold, without modification, also for homogeneous Neumann or regular oblique derivative boundary conditions.

Remarks. (a) It will be clear from the proofs below that

\[ \| u \| \geq \max \{ s : f(s) \leq 0 \} \quad \text{for all } (\lambda, u) \in \Sigma. \]

(b) In order to render prominent the simplicity of our method, we have restricted attention to functions \( f \) not depending explicitly on \( x \in \Omega \). It is an easy exercise to extend the present results to functions \( f \) defined on \( \bar{\Omega} \times \mathbb{R}^+ \) and satisfying suitable variants of our hypotheses, provided one assumes that the function \( m_\infty \) is then positive on \( \bar{\Omega} \) and, in case \( f(\cdot, 0) = 0 \), distinguishes between the cases \( \partial_{2,-} f(\cdot, 0) > 0 \) and \( \partial_{2,+} f(\cdot, 0) \leq 0 \) on \( \bar{\Omega} \) (where \( \partial_{2,\cdot} f \) denotes the right-sided partial derivative of \( f \) with respect to the second variable). The case where \( m_\infty \) or \( \partial_{2,-} f(\cdot, 0) \) changes sign in \( \Omega \) poses new difficulties and will be treated in a later paper.

(c) Concerning the precise number of positive solutions of problem (0.1), we refer to the paper [4] by one of the authors where upper bounds are given without any assumption on the positiveness of \( f \). These results, together with the present existence theorems, allow us to improve results of Amann [2] and Amann and Laetsch [3].

2. Notations and Preliminary Results

Let \( H := L^2(\Omega) \), with inner product \( \langle \cdot, \cdot \rangle \) and norm \( \| \cdot \|_{L^2} \), and set \( E := C(\bar{\Omega}) \subset H \). Further let \( L_0 \) be the linear operator induced by \( \mathcal{L} \) in \( H \), with domain \( \mathcal{D}(L_0) = H^1_0(\Omega) \cap H^2(\Omega) \). Then \( L_0 \) is a closed operator with compact resolvent, and \( 0 \in \rho(L_0) \). It is known that \( L_0 \) has a principal eigenvalue \( \lambda_1 > 0 \); the corresponding eigenspace is one dimensional and spanned by a function \( \varphi \in E \) which can be chosen to be positive in \( \Omega \). We normalize \( \varphi \) by \( \| \varphi \|_{L^2} = 1 \). The adjoint operator \( L_0^* \) has also the eigenvalue \( \lambda_1 \), and by the Krein–Rutman theorem, the corresponding eigenspace is again one dimensional and spanned by the positive eigenfunction \( \psi \in E \). Let \( \psi \) be normalized by \( \| \psi \|_{L^2} = 1 \). The space \( H \) admits the topological direct decomposition

\[ H = H_1 + \mathbb{R} \varphi, \quad (2.1) \]

where \( H_1 \) is the orthogonal complement (in \( H \)) of \( \mathbb{R} \varphi \). Note that \( H_1 \) and \( \mathbb{R} \varphi \) are invariant under \( L_0 \). Since \( \varphi \in E \) and \( E \subset H \), the space \( E \) is also decomposed in

\[ E = E_1 + \mathbb{R} \varphi, \quad (2.2) \]

where \( E_1 = H_1 \cap E \).
By the regularity theory for linear elliptic boundary value problems, the restriction \( L_0^{-1} \mid E \) maps \( E \) into itself and is \textit{compact} (as an operator in \( E \)). Let \( L \) be the restriction of \( L_0 \) to \( E \), defined by

\[
D(L) = \{ u \in E : u \in D(L_0), L_0 u \in E \}, \\
L u = L_0 u, \quad u \in D(L).
\]

Then \( L \) is a closed operator in \( E \) with compact inverse.

Let \( f \) be an extension of the function \( f \) to \( \mathbb{R} \) as described in Section 1, and let \( F \) denote the Nemytskii operator associated with \( \tilde{f} : F(u)(x) = \tilde{f}(u(x)) \) for any function \( u \) defined on \( \overline{Q} \). Then \( F \) is a bounded and continuous operator of \( E \) into itself. Since \( f(s) > 0 \) for \( s < 0 \), the following result is an immediate consequence of the maximum principle.

**Lemma 2.1.** Let the function \( u \in D(L) \) be such that \( Lu \geq \lambda F(u) \) in \( Q \), \( \lambda \geq 0 \). Then \( u \geq 0 \) in \( Q \).

Problem (0.1), with \( \lambda \geq 0 \), is now equivalent to the functional equation

\[
u = \lambda L^{-1} F(u)
\]

in the Banach space \( E \). In the following we shall apply the Leray–Schauder degree theory, mainly to the mapping \( \Phi_\lambda : E \to E \),

\[
\Phi_\lambda(u) = u - \lambda L^{-1} F(u).
\]

For \( R > 0 \), let \( B_R = \{ u \in E : \| u \| < R \} \), let \( \deg(\Phi_\lambda, B_R, 0) \) denote the degree of \( \Phi_\lambda \) on \( B_R \) with respect to 0, and let \( i(\Phi_\lambda, u_0, 0) \) be the index of the solution \( u_0 \) of the equation \( \Phi_\lambda(u) = 0 \) (provided they are defined).

### 3. Bifurcation from Infinity

It is assumed that (0.2) holds. In order to investigate the bifurcation from infinity, we follow the standard pattern (e.g., \([1, 9])\) and perform the change of variable \( z := \| u \|^2 u \) (\( u \neq 0 \)). We thus consider the mapping \( \Psi_\lambda : E \to E \) defined by

\[
\Psi_\lambda(z) = z - \lambda \| z \|^2 L^{-1} F \left( \frac{z}{\| z \|^2} \right), \quad z \neq 0, \\
= 0, \quad z = 0.
\]

Then \( \lambda \) is a bifurcation point from the trivial solution for \( \Psi_\lambda(z) = 0 \) if and only if \( \lambda \) is a bifurcation point from infinity for \( \Phi_\lambda(u) = 0 \).
LEMMA 3.1. Let $\Lambda \subset \mathbb{R}^+$ be a compact interval with $\lambda_\infty \notin \Lambda$. Then there exists a number $R > 0$ such that $\forall u \in E$ with $\| u \| \geq R$, $\forall \lambda \in \Lambda$,

$$\Phi_\lambda(u) = 0.$$ 

Proof. Suppose to the contrary that there exist $u_n \in E$ with $\| u_n \| \to \infty$ ($n \to \infty$) and $\lambda_n \in \Lambda$ such that $\Phi_{\lambda_n}(u_n) = 0$. We may assume $\lambda_n \to \bar{\lambda} \in \Lambda$. By Lemma 2.1, $u_n \geq 0$ in $\Omega$. Set $v_n := \| u_n \|^{-1} u_n$. Then

$$v_n = \lambda_n L^{-1} \frac{F(u_n)}{\| u_n \|}.$$ 

Since $\| u_n \|^{-1} F(u_n)$ is bounded in $E$, $(v_n)_{n \in \mathbb{N}}$ is a relatively compact set in $E$ by the compactness of $L^{-1}$. Suppose $v_n \rightharpoonup v$ in $E$. Then $\| v \| = 1$ and $v \geq 0$ in $\Omega$. Further

$$L v_n = \lambda_n m_{\infty} v_n + \lambda_n \| u_n \|^{-1} G(u_n) \to \bar{\lambda} m_{\infty} v$$

in $E$ ($G$ denotes the Nemytskii operator associated with $g$). Since $L$ is closed in $E$, it follows that $\bar{v} \in D(L)$ and $L \bar{v} = \bar{\lambda} m_{\infty} \bar{v}$. Hence $\bar{v}$ is a positive eigenfunction of $L$ to the eigenvalue $\bar{\lambda} m_{\infty}$. But the only eigenvalue with a positive eigenfunction is $\lambda_\infty$. Thus $\bar{\lambda} = \lambda_\infty \in \Lambda$, a contradiction. \quad \Box

COROLLARY 3.2. For $\lambda \in (0, \lambda_\infty)$, $i(\Psi_\lambda, 0 \ 0) = 1$.

Proof. Lemma 3.1, applied to the interval $\Lambda = [0, \lambda]$, guarantees the existence of $R > 0$ such that

$$u - t \lambda L^{-1} F(u) \neq 0$$

$\forall u \in E$ with $\| u \| \geq R$, $\forall t \in [0, 1]$. Thus, performing the transformation $z = \| u \|^{-2} u$ ($u = 0$), we get

$$z - t \lambda \| z \|^{2} L^{-1} F \left( \frac{z}{\| z \|^{2}} \right) \neq 0$$

$\forall z \in E$ with $0 < \| z \| \leq R^{-1}$, $\forall t \in [0, 1]$. Hence for any $\varepsilon \in (0, R^{-1}]$,

$$\text{deg}(\Psi_\lambda, B_\varepsilon, 0) = \text{deg}(I, B_\varepsilon, 0) = 1,$$

which implies the assertion. \quad \Box

On the other hand we have

LEMMA 3.3. Suppose $\lambda > \lambda_\infty$. Then there exists $R > 0$ with the property that $\forall u \in E$ with $\| u \| \geq R$, $\forall t \geq 0$,

$$\Phi_\lambda(u) \neq \tau p.$$
Proof. Let us assume that for some sequence \((u_n)_{n \in \mathbb{N}}\) in \(E\) with \(\|u_n\| \to \infty\) and numbers \(\tau_n \geq 0\), \(\Phi_n(u_n) = \tau_n \varphi\). Then
\[
Lu_n = \lambda F(u_n) + \tau_n \lambda_1 \varphi,
\]
and since \(\tau_n \lambda_1 \varphi \geq 0\) in \(\Omega\), it follows by Lemma 2.1 that \(u_n \geq 0\) in \(\Omega\). Let \(u_n = w_n + s_n \varphi\) be decomposed according to the decomposition (2.2). Then
\[
s_n = (u_n, \psi)(\varphi, \psi)^{-1} > 0, \quad \forall n \in \mathbb{N}.
\]
We first prove that \(s_n \to 0\) as \(n \to \infty\). Suppose that the sequence \((s_n)_{n \in \mathbb{N}}\) is bounded. Then \(\|w_n\| \to \infty\). Set \(v_n := \|w_n\|^{-1} w_n\). Let \(P : E \to E\) be the (continuous) projection of \(E\) onto \(E_1\) parallel to \(\mathbb{R} \varphi\). Applying \(P\) to Eq. (3.1) we obtain
\[
Lv_n = \lambda \|w_n\|^{-1} PF(u_n) = \lambda m_n v_n + \lambda \|w_n\|^{-1} PG(u_n).
\]
Since \(G(u_n)\) is bounded in \(E\), we infer as in the proof of Lemma 3.1 that (for a subsequence) \(v_n \to \bar{v}\) in \(E\), \(\|\bar{v}\| = 1\), and \((\bar{v}, \psi) = 0\). Thus \(\bar{v}\) has to change sign in \(\Omega\). On the other hand, \(u_n \geq 0\) in \(\Omega\) implies that \(v_n \geq -s_n \|w_n\|^{-1} \varphi\) and in the limit \(v \geq 0\), a contradiction.

Taking the inner product of (3.1) with \(\psi\), we get
\[
s_n \lambda_1 (\varphi, \psi) = (Lu_n, \psi)
\]
\[
= \lambda m_n (u_n, \psi) + \lambda (G(u_n), \psi) + \tau_n \lambda_1 (\varphi, \psi)
\]
\[
\geq \lambda m_n s_n (\varphi, \psi) + \lambda (G(u_n), \psi).
\]
Hence
\[
\lambda_1 \geq \lambda m_n + (s_n (\varphi, \psi))^{-1} \lambda (G(u_n), \psi) \to \lambda m_n,
\]
\((n \to \infty)\), a contradiction to the assumption \(\lambda > \lambda_x\).

Corollary 3.4. For \(\lambda > \lambda_w\), \(i(\Psi_\lambda, 0, 0) = 0\).

Proof. By Lemma 3.3, there exists \(R > 0\) such that \(\Phi_\lambda(u) \neq t \|u\|^{\alpha} \varphi\), \(\forall u \in E\) with \(\|u\| \geq R\), \(\forall t \in [0, 1]\). Then
\[
\Psi_\lambda(z) \neq t \varphi \quad \forall z \in E : 0 < \|z\| \leq R^{-1}, \quad \forall t \in [0, 1].
\]
We conclude that
\[
\text{deg}(\Psi_\lambda, B_\varepsilon, 0) = \text{deg}(\Psi_\lambda - \varphi, B_\varepsilon, 0) = 0
\]
for all \(\varepsilon \in (0, R^{-1}]\). The assertion follows.

We are now ready to prove

Proposition 3.5. \(\lambda_x\) is a bifurcation point from infinity for positive solutions, and the only one. There exists an unbounded component \(\Sigma_\infty\) of positive solutions which meets \((\lambda_w, \infty)\).
Proof. That \( \lambda_\infty \) is a bifurcation point from the trivial solution for the equation \( \Psi_A(z) = 0 \), and that there bifurcates an unbounded continuum of (positive) solutions from \( (\lambda_\infty, 0) \), follows from a simple modification of the global bifurcation theorem of Rabinowitz [8, Theorem 1.3]: The assumptions about the differentiability at \( z = 0 \) of the mappings involved and the oddness of the multiplicity of the eigenvalue \( \lambda_z \) of the "linearized" problem at \( z = 0 \) are replaced here by the assertions of Corollaries 3.2 and 3.4, which have to be compared with Relation (1.11) in [8]. By the transformation \( u = \| z \|^{-2} z \) we then obtain the continuum \( \Sigma_\infty \).

The uniqueness of \( \lambda_\infty \) as a bifurcation point from infinity for positive solutions of \( \Phi_A(u) = 0 \) follows with the same arguments as in the proof of Lemma 3.1.

We now turn to the more detailed study of the nature of the bifurcation at \( (\lambda_\infty, \infty) \). For that purpose we suppose that \( g \) satisfies either (1.1) or (1.1').

**Lemma 3.6.** (i) Assume (1.1) holds. Then the assertion of Lemma 3.1 holds with \( A = [\lambda_\infty, \beta] \), where \( \beta > \lambda_\infty \).

(ii) Assume (1.1'). Then we can take \( A = [0, \lambda_\infty] \) in Lemma 3.1.

The important point in this sharpening of Lemma 3.1 is the fact that \( \lambda_z \) is now included in the set \( A \).

**Proof.** We prove statement (i); (ii) follows similarly. By Lemma 3.1 the assertion holds for any interval \( \Lambda = [\lambda_\infty + \epsilon, \beta] \), \( \epsilon > 0 \). Suppose now there exist sequences \( (u_n) \in E \) and \( (\lambda_n) \in R^+ \) with \( \| u_n \| \to \infty \), \( \lambda_n \downarrow \lambda_\infty \), such that \( \Phi_{\lambda_n}(u_n) = 0 \) \( \forall n \). As in the proof of Lemma 3.1, we have \( u_n \to 0 \) and, setting \( v_n = \| u_n \|^{-1} u_n \), we conclude that \( v_n \to v \) in \( E \), \( \| v \| = 1 \) and \( v \geq 0 \) in \( \Omega \). Further \( L \bar{v} = \lambda_z m_z \bar{v} \), which implies that \( \bar{v} = \alpha \varphi \) for some \( \alpha > 0 \). Hence \( \bar{v} > 0 \) in \( \Omega \) and \( u_n(x) = \| u_n \| v_n(x) \to +\infty \), \( \forall x \in \Omega \). In the decomposition \( u_n = v_n + s_n \varphi \) according to (2.2), we have \( s_n = (u_n, \varphi)(\varphi, \varphi)^{-1} > 0 \). Moreover

\[
s_n \lambda_1(\varphi, \psi) = (Lu_n, \psi) = \lambda_n m_z s_n(\varphi, \psi) + \lambda_n(G(u_n), \psi).
\]

As \( \lambda_n m_z > \lambda_1 \), we conclude that \( \int_{\Omega} g(u_n) \psi \, dx < 0 \) \( \forall n \in \mathbb{N} \). By the Fatou lemma, it follows that

\[
0 < \int_{\Omega} c\psi \, dx \leq \lim \inf \int_{\Omega} g(u_n) \psi \, dx \leq 0
\]

(\( c := \lim \inf_{n \to +\infty} g(s) > 0 \)), a contradiction. ■

Assertion (i) of Theorems A–C follows now directly.
4. BIFURCATION FROM THE TRIVIAL SOLUTION

We suppose now \( f(0) = 0 \) and investigate first the case \( f'(0) > 0 \). The assumptions imply that there is a constant \( \epsilon : \, |s^{-1}f(s)| \leq \epsilon \forall s > 0 \). Set \( \lambda_0 := (f'(0))^{-1} \lambda_1 \).

**Lemma 4.1.** Let \( A \subseteq \mathbb{R}^+ \) be a compact interval such that \( \lambda_0 \notin A \). Then there exists \( \delta > 0 \) with the property that \( \Phi_\lambda(u) \neq \tau \) \( \forall u \in E \) with \( 0 < \|u\| \leq \delta \), \( \forall \lambda \in A \).

**Proof.** We suppose, to the contrary, that there exist sequences \( (\lambda_n)_{n \in \mathbb{N}} \) in \( A \) and \( (u_n)_{n \in \mathbb{N}} \) in \( E \): \( \lambda_n \to \lambda \in A \), \( u_n \to 0 \) in \( E \), such that \( \Phi_\lambda(u_n) = \tau \) \( \forall n \). By Lemma 2.1, \( u_n \geq 0 \) in \( \Omega \).

Set \( v_n := \|u_n\|^{-1} u_n \). Then \( L v_n = \lambda_n \|u_n\|^{-1} F(u_n) \). Since \( \|u_n\|^{-1} F(u_n) \) is bounded in \( E \), we infer that \( (v_n)_{n \in \mathbb{N}} \) is relatively compact in \( E \); hence (for a subsequence) \( v_n \to \bar{v} \) in \( E \) with \( \bar{v} \geq 0 \) in \( \Omega \), \( \|\bar{v}\| = 1 \). Further \( \Phi_\lambda(u_n) \to \Phi_\lambda(0) \) \( \text{in } E \).

We conclude that \( L \bar{v} = \lambda \bar{v} \) and, since \( \|\bar{v}\| = 1 \), \( \bar{v} \geq 0 \) in \( \Omega \), that \( \lambda \bar{v} = \lambda_1 \), i.e., \( \lambda = \lambda_0 \). But by assumption, \( \lambda_0 \notin A \).

**Corollary 4.2.** For \( \lambda \in [0, \lambda_0] \), \( i(\Phi_\lambda, 0, 0) = 1 \).

We now turn to \( \lambda > \lambda_0 \).

**Lemma 4.3.** Let \( \lambda > \lambda_0 \). Then there exists \( \delta > 0 \) such that \( \forall u \in E \) with \( 0 < \|u\| \leq \delta \), \( \forall \tau \geq 0 \), \( \Phi_\lambda(u) \neq \tau \).

**Proof.** We assume again to the contrary that there exist \( \tau_n \geq 0 \) and a sequence \( u_n \to 0 \) in \( E \) such that \( \Phi_\lambda(u_n) = \tau_n \) \( \forall n \). As \( L u_n = \lambda F(u_n) + \tau_n \lambda_1 \phi \) and \( \tau_n \lambda_1 \phi \geq 0 \) in \( \Omega \), we conclude by Lemma 2.1 that \( u_n \geq 0 \) in \( \Omega \). Employing the decomposition \( u_n = w_n + s_n \phi \) according to (2.2), with \( s_n > 0 \), we obtain

\[
\lambda_1 \phi \geq \lambda F(u_n) + \tau_n \lambda_1 \phi + s_n \phi.
\]

Choose \( \sigma > 0 \) such that \( \sigma < f'(0) - \lambda^{-1} \lambda_1 \). For all sufficiently large \( n \),

\[
F(u_n) \geq (f'(0) - \sigma) u_n \quad \text{in} \quad \Omega.
\]

Thus

\[
s_n \lambda_1 \phi \geq \lambda (f'(0) - \sigma) s_n \phi + \tau_n \lambda_1 \phi
\]

\[
> \lambda_1 s_n \phi,
\]

a contradiction. \( \square \)
Corollary 4.4. For $\lambda > \lambda_0$, $i(\Phi_\lambda, 0, 0) = 0$.

Proof. Let $0 < \epsilon \leq \delta$, where $\delta$ is the number asserted in Lemma 4.3. As $\Phi_\lambda$ is bounded on $B_\epsilon$, there exists $a > 0$ such that $\Phi_\lambda(u) \neq au \quad \forall u \in B_\epsilon$. By Lemma 4.3, $\Phi_\lambda(u) \neq \tau \quad \forall u \in \partial B_\epsilon \quad \forall \tau \in [0, 1]$. Hence

$$\deg(\Phi_\lambda, B_\epsilon, 0) = \deg(\Phi_\lambda - au, B_\epsilon, 0) = 0.$$

In the same way as before Proposition 3.5, we now obtain

Proposition 4.5. $\lambda_0$ is a bifurcation point from the trivial solution, and it is the only one for positive solutions. There exists an unbounded component $\Sigma_0$ of positive solutions emanating from $(\lambda_0, 0)$.

This is exactly Assertion (ii) of Theorem A.

We conclude this section by considering the case $f(0) = 0$, $f'(0) \leq 0$. Assertion (ii) of Theorem B follows readily by the arguments used in the proof of Lemma 4.1:

Lemma 4.6. In case $f(0) = 0$, $f'(0) \leq 0$, there is no bifurcation of positive solutions from the trivial solution.

5. Continua without Bifurcation

We briefly look at the situation where $f(0) > 0$.

Lemma 5.1. There exists an unbounded component $\Sigma_0$ of positive solutions of problem $(0.1)$ meeting $(0, 0)$.

Proof. By Theorem 3.2 of Rabinowitz [8] there exists an unbounded continuum $\Sigma_0$ of solutions $(\lambda, u)$ of the equation $\Phi_\lambda(u) = 0$ in $\mathbb{R}^+ \times E$, meeting $(0, 0)$. Note that $(0, u)$ is a solution only for $u = 0$, while $(\lambda, 0)$ is a solution only for $\lambda = 0$. Thus $(\lambda, u) \in \Sigma_0 \setminus \{(0, 0)\}$ is a positive solution, by Lemma 2.1.

The first part of Theorem C(ii) follows from Lemma 5.1.

6. Global Behavior of the Components of Positive Solutions

We start with

Lemma 6.1. Suppose $f(s) \geq \alpha s \quad \forall s \geq 0$, with $\alpha > 0$. Then there exists a number $\lambda^* > 0$ such that there is no positive solution $(\lambda, u)$ of $\Phi_\lambda(u) = 0$ with $\lambda > \lambda^*$. 
Proof. Let \((\lambda, u)\) be a positive solution of \(\Phi_\lambda(u) = 0\). Then \(Lu = \lambda F(u) \geq \lambda u\) and hence

\[ \lambda_1(u, \psi) = (Lu, \psi) \geq \lambda_\alpha(u, \psi). \]

Since \((u, \psi) \succ 0\), it follows that \(\lambda \leq \alpha^{-1} \lambda_\Psi =: \lambda^*\).

Note that (i) if \(f(0) = 0, f_s(0) > 0\), and \(f(s) > 0 \ \forall s > 0\), or (ii) if \(f(0) > 0\) and \(f(s) > 0 \ \forall s > 0\), the hypothesis of Lemma 6.1 is satisfied (of course (0.2) is assumed).

The assertion that \(\Sigma_0 = \Sigma_\infty\) in both Theorem A(iii) and Theorem C(ii) now follows easily. For, in both cases \(\Sigma_0 \) and \(\Sigma_\infty\) are contained in the strip \([0, \lambda^*] \times E\), and \(\lambda_\infty\) is the only bifurcation point from infinity. In Theorem A(iii), the unbounded component \(\Sigma_0\) has to meet \((\lambda_\infty, \infty)\), since \((\lambda_0, 0)\) is the only bifurcation point from the trivial solution for positive solutions. A similar argument applies in Theorem C(ii).

Finally, we suppose \(f(s_0) \leq 0\) for some \(s_0 > 0\).

**Lemma 6.2.** If \(f(s_0) \leq 0\) for some \(s_0 > 0\), then \(\Phi_\lambda(u) \neq 0 \ \forall u \in E\) with \(\|u\| = s_0, \ \forall \lambda \geq 0\).

**Proof.** Suppose \(\Phi_\lambda(u) = 0\) for some \(\lambda \geq 0\) and \(\|u\| = s_0\). By Lemma 2.1, \(u \geq 0\) in \(\Omega\) and thus \(0 \leq u(x) \leq s_0\), \(\forall x \in \Omega\). There exists \(m \geq 0\) such that \(f(s) + ms\) is monotone increasing in \(s\) for \(s \in [0, s_0]\). Then

\[ (L + \lambda m)u = \lambda(F(u) + mu) \]

and, since \(Ls_0 = 0 \geq F(s_0)\),

\[ (L + \lambda m)s_0 \geq \lambda(F(s_0) + ms_0). \]

Subtracting, we get

\[ (L + \lambda m)(s_0 - u) \geq 0 \text{ in } \Omega, \quad (s_0 - u) > 0 \text{ on } \partial\Omega. \]

The maximum principle implies that \(s_0 - u > 0\) in \(\bar{\Omega}\) and hence \(\|u\| < s_0\), a contradiction.

**References**


