Asymptotic Stability for a Class of Delay Differential Inclusions

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In this work we provide sufficient conditions for the asymptotic stability of an equilibrium for a class of differential inclusions of finite retarded type where the right-hand side is a set-valued mapping having a bounded, closed, convex graph. In particular our result includes "linear differential inclusions" in which the right-hand side is a closed, convex process. This result partially generalizes an earlier result of Leizarowitz (1985, SIAM J. Control Optim. 23, 514-522) in the non-delay case.

1. INTRODUCTION AND MOTIVATION

In this work we study the asymptotic stability of the differential inclusion

$$\dot{x}(t) \in G(x_t) \quad t \geq 0$$

where $G(\cdot): C_{\mathbb{R}}[-r, 0] \to 2^{\mathbb{R}^n} \setminus \emptyset$ is a given set-valued mapping having a closed, bounded convex graph

$$M = \{(\eta, z); z \in G(\eta)\},$$

$C_{\mathbb{R}}[-r, 0]$ denotes the space of $n$-vector valued continuous functions defined on $[-r, 0]$ for some fixed $r > 0$ endowed with the uniform topology. Here the notation $x_t \in C_{\mathbb{R}}[-r, 0]$ means $x_t(s) = x(t + s)$ for $-r \leq s \leq 0$.

The motivation to investigate this problem arises in the study of variational problems with time delay in which the objective is described by an improper integral. To be specific, consider the problem of determining the minimizers of the improper integral

$$J(x) = \int_0^{+\infty} L(x_t, x(t)) \, dt$$

over all absolutely continuous arcs $x: [-r, +\infty) \to \mathbb{R}^n$ satisfying a fixed initial condition $x(s) = \phi(s)$ for $-r \leq s \leq 0$. Here $L: C_{\mathbb{R}}[-r, 0] \times \mathbb{R}^n \to \mathbb{R}$ is

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convex, lower semicontinuous and enjoys the usual coercivity growth conditions. Dictated by specific applications of these models without delay one must assume that generally this problem is not well-defined in the traditional sense due to the possible lack of convergence of the objective functional. The study of these models for problems without delay is extensive with most of the applications arising in mathematical economics. Recently however, there has been interest in the study of these problems in engineering as well. Specifically, these models have been used to investigate thermodynamical equilibrium for materials (see e.g., Leizarowitz and Mizel [20] and Coleman, Marcus, and Mizel [14]) and also to study the tracking of a periodic signal (see Artstein and Leizarowitz [2]). For their role in mathematical economics, as well as a detailed introduction of this theory, we refer the reader to the monograph of Carlson, Haurie, and Leizarowitz [8]. For variational problems of this type these differential inclusions arise naturally in that to establish the existence of an overtaking optimal solution one must assume the following (see Carlson [7] for specific details):

1. There exists a unique constant trajectory \( \bar{x}(t) \equiv \bar{x} \) and a vector \( \bar{p} \) such

   \[
   L_0(\eta, v) = L(\eta, v) - L(\bar{x}, 0) + \langle \bar{p}, v \rangle \geq 0
   \]

   for all \( (\eta, v) \in C_1[-r, 0] \times \mathbb{R}^n \).

2. All the trajectories \( x: [-r, +\infty) \to \mathbb{R}^n \) which satisfy

   \[
   L_0(x_t, \bar{x}(t)) = 0 \quad \text{for all} \quad t \geq 0
   \]

   are uniformly attracted to \( \bar{x} \) as \( t \to +\infty \).

It is easy to see that (2) can be investigated by studying the asymptotic convergence properties a differential inclusion of the type discussed here. Indeed, in the case of interest the set-valued mapping \( \mathcal{G}(\cdot) \) is given by

\[
\mathcal{G}(\eta) = \{ v \in \mathbb{R}^n : L_0(\eta, v) = 0 \}. \tag{1.1}
\]

The observation that (2) is related to differential inclusions was first recognized in the work of Leizarowitz [19] where results analogous to [7] for the nondelay case are studied. In this paper we extend the result found in Leizarowitz [18] for ordinary differential inclusions to hereditary systems.

With these brief remarks, the plan of our paper is as follows. In Section 2 we present the basic hypotheses and preliminary lemmas needed to prove our main result which we state as Theorem 2.1. In Section 3 we give a proof of our main result. We demonstrate it's utility in Section 4 by giving
two examples to which our result may be applied and we make some concluding remarks in Section 5.

2. BASIC HYPOTHESES AND PRELIMINARY LEMMAS

To begin we let \( X \subseteq C_a[-r, 0] \) be a closed bounded convex subset and let \( G: X \to \mathbb{R}^n \setminus \emptyset \), where \( \mathbb{R}^n \) denotes the power set of \( \mathbb{R}^n \), be a set-valued mapping having a closed, bounded, convex graph

\[
M = \{ (\eta, z) \in X \times \mathbb{R}^n : z \in G(\eta) \}.
\]  

With this notation we consider the differential system

\[
\begin{align*}
\dot{x}(t) &\in G(x_t) & 0 \leq t \\
x_t &\in X & 0 \leq t.
\end{align*}
\]  

For (2.2) we have the following definitions.

**Definition 2.1.** 1. A function \( x: [-r, \infty) \to \mathbb{R}^n \) will be called a **viable solution** of (2.2) if \( x|_{[-r, 0]} \) is continuous, \( x|_{(0, T]} \) is absolutely continuous for each \( T > 0 \), and (2.2) is satisfied.

2. A viable solution \( x(\cdot) \) of (2.2) will be called elliptic if

\[
x(t) = a \cos(\pi t) + b \sin(\pi t)
\]

for some vectors \( a, b \in \mathbb{R}^n \) satisfying \( \|a\| + \|b\| \neq 0 \) and real number \( \pi \neq 0 \).

3. A vector \( z \in \mathbb{R}^n \) will be called a **stationary point** of (2.2) if \( \zeta: [-r, 0] \to \mathbb{R}^n \) defined by \( \zeta(s) = z \) for \( -r \leq s \leq 0 \) satisfies

\[
0 \in G(\zeta).
\]

**Remark 2.1.** The assumption that \( M \), the graph of \( G \), is closed and bounded implies that there exists constants \( K \) and \( \bar{K} \) such that \( \|\phi\|_\infty = \sup_{s \leq \omega \leq 0} \|\phi(s)\| \leq K \) and \( \|z\| \leq \bar{K} \) for all \( (\phi, z) \in M \). In particular we note that this implies that for any viable trajectory we have

\[
\|x_t\|_\infty \leq K \quad \text{and} \quad \|\dot{x}(t)\| \leq \bar{K}
\]

for almost all \( t \geq 0 \). Thus, by the Arzela-Ascoli theorem, the set of functions \( \{x_t\}_{t \geq 0} \) is a relatively compact subset of \( C_a[-r, 0] \).

Our goal is to establish the following theorem.
Theorem 2.1. If, in addition to the general hypotheses given above, we have

1. \( 0 \in G(\eta) \) if and only if \( \eta(s) \equiv 0 \) for \( -r \leq s \leq 0 \), and
2. The system (2.2) has no elliptic solutions,

then, every viable solution of (2.2) converges to zero as \( t \to +\infty \). Moreover, this convergence is uniform over any set of trajectories \( \Omega \) whose initial data \( (i.e., x_0 \in C_\alpha[-r, 0]) \) lies in a compact subset of \( C_\alpha[-r, 0] \). That is, for every \( \varepsilon > 0 \) there exists \( T_\varepsilon > 0 \) such that for all \( t > T_\varepsilon \) and all \( x \in \Omega \) one has

\[ \| x(t) \| < \varepsilon. \]

Remark 2.2. In Leizarowitz [18] it is shown that the two conditions given in the above theorem are both necessary and sufficient in the non-delay case. That this is not the case for the delay inclusions considered here is shown in the following example.

Example. We let \( X = \{ \eta \in C_\alpha[-r, 0]: 0 \leq \eta(s) \} \) and denote \( G : X \to 2^\mathbb{R} \setminus \{ \emptyset \} \) by the following

\[ G(\eta) = \{ z \in \mathbb{R}: -K \leq z \leq -\gamma \eta(0) + p(\eta(-r)) \}, \]

in which \( K, \gamma, \) and \( p \) are positive constants with \( 0 < p < \gamma \). It is an easy matter to see that the set-valued map \( G \) has a closed, bounded, convex graph \( M \). Additionally for any \( \eta \in X \) satisfying \( -\gamma \eta(0) + p(\eta(-r)) > 0 \) we have \( 0 \in G(\eta) \). Consequently, condition (1) of Theorem 2.1 does not hold yet it is well known that any positive solution of the differential inequality

\[ \dot{v}(t) \leq -\gamma v(t) + p \| v \|_\infty \quad \text{for} \quad t \geq 0 \]

is exponentially stable (see Driver [16, p. 389] or Halanay [17]). In fact once can show that such a solution satisfies the following exponential estimate

\[ 0 \leq v(t) \leq \| v_0 \|_\infty \ e^{-\lambda t}, \]

where \( \lambda \) is the unique positive definite solution of the equation

\[ \lambda = \gamma - pe^{\lambda}. \]

It is an easy matter to see that any viable solution for this example meets these requirements. Therefore the conclusion of Theorem 2.1 holds yet condition (1) does not hold.

To prove this theorem we require two technical results. The first of these is a closure result first proven in a finite dimensional setting independently
by Berkovitz [5], Bidaut [6], and Cesari [9]. In infinite dimension this result has been given in a variety of settings by a variety of authors and we refer the reader to the works of Cesari [10–12], Angell [1], Balder [4] for a representative sampling. The second result is a compactness result concerning the convergence of viable solutions to a viable solution.

**Theorem 2.2 (Closure theorem).** Let \( X \subset C([-r, 0]) \) be a closed convex set and let \( G: X \to 2^{\mathbb{R}^n} \setminus \emptyset \) be a set-valued mapping which is convex valued and has closed bounded graph. Further let \( \{ t^k_0 \}_{k=1}^{\infty} \) and \( \{ t^k_1 \}_{k=1}^{\infty} \) be two sequences of real numbers such that \( t^k_0 < t^k_1 \) for all \( k \in \mathbb{N} \) with \( \lim_{k \to \infty} t^k_1 = t_i \) (finite or infinite) for \( i = 0, 1 \). Let \( x^k: [t^k_0, t^k_1] \to \mathbb{R}^n, \ x_i: [t_0, t_1] \to \mathbb{R}^n, \ \psi^k: [t_0, t_1] \to \mathbb{R}^n, \) and \( \psi_i: [t_0, t_1] \to \mathbb{R}^n \) be given measurable functions satisfying

1. \( x^k \in C([-r, t^k_1]) \) and \( x^k_i \in X \) for all \( t \in [t^k_0, t^k_1] \) for all \( k = 1, 2, \ldots \).
2. \( \psi^k \in L_1^{\text{loc}}([t^k_0, t^k_1], \mathbb{R}^n) \) with \( \psi^k(t) \in G(x^k) \) a.e. on \( [t^k_0, t^k_1] \).
3. \( \| x^k_t - x_t \|_\infty \to 0 \) as \( k \to \infty \) for all \( t \in [t_0, t_1] \) and \( \psi^k \to \psi \) weakly in \( L_1^{\text{loc}}([t_0, t_1], \mathbb{R}^n) \) as \( k \to +\infty \) (Here we extend \( x^k \) by constancy and \( \psi^k \) by zero when necessary).

Then for all \( t \in [t_0, t_1] \) we have \( x_t \in X \) and \( \psi(t) \in G(x_t) \) a.e. on \( [t_0, t_1] \).

**Remark 2.3.** In most statements of closure theorems, the set valued mapping is required to satisfy the Kuratowski upper semicontinuity property known as property \((K)\). That is,

\[
G(\eta) = \bigcap_{\delta > 0} \text{cl} \left( \bigcup \{ G(\phi): \phi \in X \text{ and } \| \phi - \eta \|_\infty < \delta \} \right)
\]

holds for all \( \eta \in X \). In the above statement the fact that \( G \) has a closed graph, \( M \), insures that property \((K)\) holds (see e.g., Cesari [13]).

**Theorem 2.3 (Compactness theorem).** Suppose that \( x^j: [t^j_0 - r, t^j_1] \to \mathbb{R}^n \) is a sequence of solutions to (2.2) with \( t^j_i \to t_i, \ i = 0, 1 \), as \( j \to \infty \) (here only \( t_1 \) may be finite or infinite), and such that \( \{ x^j \}_j \) is a compact subset of \( C([-r, 0]) \). Then there exists a function \( x: [t_0 - r, t_1] \to \mathbb{R}^n \) and a subsequence \( \{ x^{j_k} \}_k \) such that

1. \( \| x^{j_k}_t - x_t \|_\infty \to 0 \) as \( j \to \infty \) for each \( t \in [t_0, t_1] \),
2. \( x^{j_k} \to x \) weakly in \( L_1^{\text{loc}}([t_0, t_1], \mathbb{R}^n) \) as \( k \to \infty \), and
3. \( x(t) \in G(x_t) \) a.e. on \( [t_0, t_1] \) and \( x_t \in X \) on \( [t_0, t_1] \).

**Proof.** It is sufficient to prove that (1) and (2) hold since (3) follows immediately from the closure theorem once (1) and (2) are known.
Without loss of generality we may assume that the sequence of functions \( \{x_k(t)\}_{k=1}^\infty \) converges uniformly to some function \( \phi: [-r, 0] \to \mathbb{R}^n \). Since \( M \) is closed and bounded we know there exists \( K \) and \( \tilde{K} \) such that
\[
\|x_k(t)\| \leq \|x^*_k\| \leq K \quad \text{and} \quad \|x_k(t)\| \leq \tilde{K}
\]
almost everywhere on \([t_0, t_1]\). These boundedness conditions imply that the sequence \( \{x_k\} \) (extended by zero to \((t_0, t_1]\) if necessary) has equi-absolutely continuous integrals on compact subsets of \([t_0, t_1]\), and thus is a relatively weakly sequentially compact sequence in \(L^1_{\text{loc}}([t_0, t_1]; \mathbb{R}^n)\). Therefore there exists a subsequence, say still \( \{x_k\} \), and a locally integrable function \( y: [t_0, t_1] \to \mathbb{R}^n \) such that
\[
\hat{x}^k \to y \quad \text{weakly in } L^1_{\text{loc}}([t_0, t_1]; \mathbb{R}^n)
\]
as \( k \to \infty \). This further implies that the sequence \( \{x_k^\circ\}_{k=1}^\infty \) (extended by constancy and continuity if necessary) is equicontinuous and equibounded on compact subsets of \([t_0, t_1]\) and so there exists a continuous function \( \hat{x}: [t_0, t_1] \to \mathbb{R}^n \) and a subsequence \( \{x_{kj}\}_{j=1}^\infty \) which converges uniformly on compacta to \( \hat{x} \). In addition we may also write,
\[
x_k^\circ(t) = x^\circ_j(t) + \int_t^{t_j} x_k(s) \, ds
\]
for any fixed \( t \in [t_0, t_1] \). Letting \( j \to \infty \) gives us
\[
\hat{x}(t) = \hat{x}(t) + \int_t^{t_1} y(s) \, ds
\]
from which it follows that \( \hat{x} \) is locally absolutely continuous with
\[
\frac{d\hat{x}}{dt} = y(t) \quad \text{a.e. on } [t_0, t_1]
\]
The desired result follows now by defining \( x: [t_0-r, t_1] \to \mathbb{R}^n \) by the formula
\[
x(t) = \begin{cases} 
\phi(t-t_0) & \text{if } t_0-r \leq t \leq t_0 \\
\hat{x}(t) & \text{if } t_0 \leq t \leq t_1
\end{cases}
\]
and observing that the sequence \( \{x_k^\circ\} \) and the function \( x \) defined above satisfy the conditions of Theorem 2.2.
3. THE PROOF OF THEOREM 2.1

It is sufficient to prove that if the conditions (1) and (2) of Theorem 2.1 both hold, then every viable solution of the system (2.2) converges to zero and that this convergence is uniform for any set of viable trajectories whose initial data lie in a compact subset of $C_n[-r, 0]$. To this end we define the sets

$$M_x = \{ z \in \mathbb{R}^n : (\phi, z) \in M \text{ for some } \phi \in C_n[-r, 0] \}$$

(3.1)

and

$$M_\phi = \{ \phi \in C_n[-r, 0] : (\phi, z) \in M \text{ for some } z \in \mathbb{R}^n \}.$$  

(3.2)

Observe that as a result of (1) of Theorem 2.1 we clearly have that $0 \in M_x$ and that the zero function (i.e., $\phi(s) \equiv 0$ for $-r \leq s \leq 0$) is in $M_\phi$. We divide our considerations into cases.

**Case 1.** The constant vector $0 \in \text{ri } M_x$ (where $\text{ri } M_x$ denotes the relative interior of $M_x$). Define the map $T: M_x \to 2^{M_\phi \setminus \emptyset}$ by the formula

$$T(z) = \{ \phi \in X : (\phi, z) \in M \},$$

and observe that

1. $T(0) = \{ 0 \}$ (otherwise there exists $\psi \neq 0$ such that $0 \in G(\psi)$) and
2. graph $T = \{ (\phi, z) : \phi \in T(z) \} = \{ (\phi, z) : (\phi, z) \in M \}$ is a convex set.

These conditions imply that $T$ is a single valued map. To see this we observe that since $0 \in \text{ri } M_x$ and $M_x$ is a convex set in $\mathbb{R}^n$ there exists a positive integer $m \in \mathbb{N}$ with $m \leq n$ and an affine map $S: \text{aff } M_x \to \mathbb{R}^m$ (aff $M_x$ denotes the affine hull of $M_x$) which is one-to-one and onto, with $S(M_x) \subset \mathbb{R}^m$ and $0 \in \text{int } S(M_x)$ (int $S(M_x)$ denotes the interior of $S(M_x)$). Thus, for $w \in M_x$ and $\lambda \in [0, 1]$ we have $\pm \lambda w \in M_x$. Now suppose that for $i = 1, 2$ we have $\phi_i \in T(w)$. Since the graph of $T$ is convex and $0 \in T(0)$ we have $(\lambda \phi_1, \lambda w) \in \text{graph } T$ for $i = 1, 2$ and $\lambda \in [0, 1]$. We also have that for each $\lambda \in [0, 1]$ there exists $\psi_j \in T(-\lambda w)$. Fixing $\lambda \in [0, 1]$ we notice that, since $T$ has a convex graph, for $i = 1, 2$

$$\frac{1}{2} \lambda \phi_i + \frac{1}{2} \psi_j \in T(\frac{1}{2} \lambda w + \frac{1}{2} (-\lambda w)) = T(0).$$

Therefore we have for each $s \in [-r, 0]$

$$\frac{1}{2} \phi_1(s) + \frac{1}{2} \psi_j(s) = 0 = \frac{1}{2} \phi_2(s) + \frac{1}{2} \psi_j(s).$$
Clearly this implies \( \phi_1(s) = \phi_2(s) \) for all \(-r \leq s \leq 0\) implying that \( T \) is a singleton.

As \( T: M_Z \to M_\phi \) is a single valued map with a convex graph we must have that \( T \) is linear and onto. This means that we can extend \( T \) to the linear span of \( M_Z \), denoted \( \text{span} M_Z \), by its linear extension, say \( \tilde{T} \). That is, if \( w = \sum_{i=1}^j \alpha_i w_i, \alpha_i \in \mathbb{R}, \ w_i \in M_Z \), then \( \tilde{T}(w) = \sum_{i=1}^j \alpha_i T(w_i) \). Observe that \( \tilde{T}: \text{span} M_Z \to \text{span} M_\phi \) is an onto map and so,

\[
\dim(\text{span} M_\phi) \leq \dim(\text{span} M_Z) \leq n.
\]

Moreover, if \( P: \text{span} M_\phi \to \mathbb{R}^n \) is defined by \( P(\phi) = \phi(0) \) we have that \( P \) is linear and onto \( P(\text{span} M_\phi) \) with

\[
\dim(P(\text{span} M_\phi)) \leq \dim(\text{span} M_\phi).
\]

With these maps we recursively define the following sequences of sets. We initialize these sequences by setting \( X_0 = \text{span} M_Z, \ Y_0 = \text{span} M_\phi, \ Z_0 = P(\ Y_0), \) and \( H_0 = M \). For \( k \geq 0 \) we define

\[
X_{k+1} = X_k \cap Z_k, \quad Y_{k+1} = \tilde{T}(X_{k+1}), \quad Z_{k+1} = P(Y_{k+1}), \quad \text{and} \quad H_{k+1} = \{(\phi, z) \in H_k; \phi \in Y_{k+1} \text{ and } z \in X_{k+1}\}.
\]

Observe that for \( k = 1, 2, \ldots \) we have that \( X_k, \ Y_k, \) and \( Z_k \) are closed, finite-dimensional subspaces of \( \mathbb{R}^n, \ C([ -r, 0], \mathbb{R}), \) and \( \mathbb{R}^n \) respectively. Now let \( x: [ -r, \infty ) \to \mathbb{R}^n \) be a viable solution of (2.2) such that

\[
(x_t, x(t)) \in H_k \quad \text{a.e on } [0, \infty)
\]

(3.3)

for some \( k \in \mathbb{N} \). This implies that \( x_t \in Y_k \) and \( x(t) \in X_k \) almost everywhere on \([0, \infty)\) so that \( x(t) = P(x_t) \in Z_k \). Further, for almost all \( t \in [0, \infty) \) we have (since \( Z_k \) is closed),

\[
\dot{x}(t) = \lim_{h \to 0} \frac{x(t+h) - x(t)}{h} \in Z_k.
\]

Thus, \( \dot{x}(t) \in X_k \cap Z_k = X_{k+1} \) for almost all \( t \geq 0 \). This gives us that \( x_t = \tilde{T}(\dot{x}(t)) \in Y_{k+1} \) holds for almost all \( t \geq 0 \). This means that if \( x: [ -r, \infty ) \to \mathbb{R}^n \) is a viable solution of (2.2) we must have that (3.3) holds for all \( k \in \mathbb{N} \). In addition, since \( X_k \) and \( Z_k \) are finite dimensional subspaces with \( \dim Z_k \leq \dim X_k \), if \( X_{k+1} \neq Z_k \), then necessarily we have \( \dim X_{k+1} < \dim X_k \). Since \( X_k \) has finite dimension it follows that there must exist an
index, say \( I \), such that \( X_I = Z_I \). This implies that the map \( F : X_I \to Z_I \) defined by

\[
F(z) = P(\hat{T}(z))
\]
is a linear bijection and therefore has a continuous inverse, say \( A = F^{-1} \). Further, we also have

\[
\dim Z_I \leq \dim Y_I \leq \dim X_I,
\]
giving us

\[
\dim Z_I = \dim Y_I = \dim X_I.
\]

To conclude case 1 we observe that if \( x : [ -r, \infty ) \to \mathbb{R}^n \) is a viable solution, the above discussion implies for almost all \( t > 0 \)

\[
F(\dot{x}(t)) = P(\hat{T}(\dot{x}(t))) = x(t),
\]
or equivalently

\[
\dot{x}(t) = A(x(t)).
\] (3.4)

We further notice that \( 0 \in \text{ri}(H_I)_Z \), where \( (H_I)_Z = \{ z \in \mathbb{R}^n : (\phi, z) \in H_I \text{ for some } \phi \in C_\varepsilon([-r, 0]) \} \). Indeed since \( 0 \in \text{ri} M_Z \) there exists \( \varepsilon > 0 \) such that \( \varepsilon B \cap \text{aff } M_Z \subset M_Z \). As \( H_I \subset M \) we observe that if \( w \in \varepsilon B \cap \text{aff } (H_I)_Z \subset M_Z \) there exists \( \phi \in M_0 \) so that \( (\phi, w) \in M \). However, \( w \in \text{aff } (H_I)_Z \) means that \( w = \sum_{i=1}^{d} \lambda_i z_i \), where \( \sum_{i=1}^{d} \lambda_i = 1 \) and \( z_i \in (H_I)_Z \subset X_I \) and so there exists \( \phi_i \in Y_I \) so that \( (\phi_i, z_i) \in H_I \) for all \( i = 1, 2, ... \). Thus, \( \phi = \sum_{i=1}^{d} \lambda_i \phi_i \in Y_I \) and \( w \in X_I \) giving us \( w \in (H_I)_Z \). Therefore, \( \varepsilon B \cap \text{aff } (H_I)_Z \subset (H_I)_Z \) so that \( 0 \in \text{ri}(H_I)_Z \) as desired.

Finally, from the above we see that each viable solution \( x(\cdot) \) is a bounded function that satisfies the linear ordinary differential Eq. (3.4) and additionally satisfies \( x(t) \in (H_I)_Z \). Our assumption on the solutions of (2.2) guarantee that this linear system has no elliptic solutions implying that the matrix \( A \) has no purely imaginary eigenvalues. Thus \( x(t) \to 0 \) as \( t \to +\infty \) since it is bounded on \( [0, +\infty) \). In addition, we observe that the initial points \( x(0) \) lie in a compact subset of \( \mathbb{R}^n \) and so the uniform convergence is immediate. Thus under the additional assumption \( 0 \in \text{ri } M_Z \) the desired result follows. We now turn to the general case.

**Case 2.** The constant vector \( 0 \notin M_Z \). We begin by letting \( F \subset M_Z \) be a face. That is, \( F \) is the largest convex subset of \( M_Z \) such that \( 0 \in \text{ri } F \) (the existence of \( F \) is discussed in Rockafellar [21]). With this set \( F \) we define the set \( H_0 \subset M \) by the formula

\[
H_0 = \{ (\phi, z) \in M : z \in F \}
\]
and define the set-valued mapping \( \mathcal{H}: X \to 2^{\mathbb{R}^n} \) by the formula
\[
\mathcal{H}(\phi) = \{ z : (\phi, z) \in H_0 \}.
\]
If \( x: [-r, +\infty) \to \mathbb{R}^n \) is such that
\[
\dot{x}(t) \in \mathcal{H}(x_t) \quad \text{a.e. on } [0, +\infty).
\]  
(3.5)
it follows from case 1 that \( x(t) \to 0 \) as \( t \to +\infty \). Thus, we are done if we show that every viable solution of (2.2) converges uniformly on compact subsets to a solution of (3.5). To do this we first observe that by the separation theorem for convex sets and the fact that \( F \) is a proper subset of \( M_Z \), there exists \( \eta \in \mathbb{R}^n \) such that
\[
\langle \eta, w_0 \rangle \geq 0
\]
for all \( w_0 \in M_Z \) with at least one \( v \in M_Z \) such \( \langle \eta, v \rangle > 0 \), and
\[
\langle \eta, w_0 \rangle = 0
\]
for every \( w_0 \in F \). Let \( \Sigma = \{ z \in M_Z : \langle \eta, z \rangle = 0 \} \) and note that \( F \subseteq \Sigma \). Now let \( x: [-r, +\infty) \to \mathbb{R}^n \) be any viable solution of (2.2) and observe that
\[
\frac{d}{dt} \langle \eta, x(t) \rangle = \langle \eta, \dot{x}(t) \rangle \geq 0 \quad \text{a.e. on } [0, +\infty)
\]
so that we have the map \( t \to \langle \eta, x(t) \rangle \) is a bounded nondecreasing function. This implies there exists \( x \in \mathbb{R}^n \) such that \( \lim_{t \to +\infty} \langle \eta, x(t) \rangle = x \). Therefore for any \( \varepsilon > 0 \), there exists \( T_\varepsilon > 0 \) such that
\[
|\langle \eta, x(t) \rangle - x| < \frac{\varepsilon}{2}
\]
whenever \( t \geq T_\varepsilon \). For \( T > T_\varepsilon \) consider
\[
\left| \frac{1}{T_0} \int_0^T \langle \eta, x(t) \rangle \, dt - x \right| \leq \frac{1}{T_0} \int_0^T \left| \langle \eta, x(t) \rangle \right| \, dt
\]
\[
+ \frac{1}{T} \int_0^T \left| \langle \eta, x(t) \rangle \right| \, dt - x \right| + \frac{T_\varepsilon \varepsilon}{T}
\]
\[
\leq \frac{1}{T_0} \int_0^T |\eta| K \, dt + \frac{T_\varepsilon |x|}{T} + \frac{1}{T} \int_{T_\varepsilon}^T \frac{T_\varepsilon}{3} \, dt
\]
\[
= \frac{|\eta| K T_\varepsilon + T_\varepsilon |x|}{T} + \frac{c(T - T_\varepsilon)}{3T}.
\]
From this we see that
\[
\lim_{T \to \infty} \frac{1}{T} \int_0^T \langle \eta, x(t) \rangle \, dt = \alpha.
\]
In addition we also have that the family of functions \( \{z^T \in C_n[-r, 0]: T > 0\} \) defined by
\[
z^T(s) = \frac{1}{T} \int_0^T x(t + s) \, dt
\]
is uniformly bounded and equicontinuous since \( M \), the graph of \( G \), is bounded. Thus we can select a sequence of times \( T_k \to \infty \) and a corresponding sequence of functions \( \{z^{T_k}\}_{k=1}^\infty \) from this family that converges uniformly to some function \( z \in C_n[-r, 0] \). Moreover, we also observe that,
\[
\left\{ \frac{1}{T_k} \int_0^{T_k} \dot{x}(t) \, dt \right\} = \left\{ \frac{x(T_k) - x(0)}{T_k} \right\} \leq \frac{2K}{T_k},
\]
giving us
\[
\lim_{T_k \to \infty} \frac{1}{T_k} \int_0^{T_k} \dot{x}(t) \, dt = 0.
\]
Finally we notice that the convexity and closedness of \( M \) gives us
\[
\left( \frac{1}{T_k} \int_0^{T_k} x(t) \, dt \right) \frac{1}{T_k} \int_0^{T_k} \dot{x}(t) \, dt \in M
\]
for each \( k \in \mathbb{N} \) giving us \( (z, 0) \in M \). This of course implies that \( z(s) \equiv 0 \). In particular, if we let \( s = 0 \) we have
\[
\alpha = \lim_{T_k \to \infty} \frac{1}{T_k} \int_0^{T_k} \langle \eta, x(t) \rangle \, dt = 0,
\]
from which it follows
\[
\lim_{t \to \infty} \langle \eta, x(t) \rangle = 0. \tag{3.6}
\]
Thus, we have
\[
\text{dist}(x(t), \Sigma) \to 0 \quad \text{as} \quad t \to \infty. \tag{3.7}
\]
We now show that (3.7) holds uniformly over any set of viable trajectories whose initial data \( x|_{[-r, 0]} \) is a compact subset of \( C_n[-r, 0] \). To see this let \( \Omega \) be such a set of viable trajectories and assume that there exists an
\( \varepsilon_0 > 0 \), a sequence of trajectories \((x^k)_{k=1}^{\infty} \in \Omega\) and a strictly increasing sequence of times \( t_k \to \infty \) such that for each \( k \) we have
\[
\langle \eta, x^k(t) \rangle < -\varepsilon_0 \quad \text{for all } \ t \in [0, t_k].
\] (3.8)

By appealing to the compactness theorem, with \( t_k^0 = 0 \) and \( t_k^1 = t_k \) for all \( k \in \mathbb{N} \) there exists a function \( x : [-r, \infty) \to \mathbb{R}^n \) and a subsequence \((x^j)_{j=1}^{\infty}\) such that
\[
(i) \quad \|x^j_t - x_t\| \to 0 \text{ as } j \to \infty \quad \forall t \in [0, +\infty),
(ii) \quad \dot{x}^j \to \dot{x} \text{ weakly in } L^1_{loc}([-r, \infty), \mathbb{R}^n), \text{ and}
(iii) \quad \dot{x}(t) \in G(x) \text{ a.e. on } [0, \infty).
\]

From (3.8) we clearly get \( \langle \eta, x(t) \rangle \leq -\varepsilon_0 < 0 \) for all \( t \geq 0 \) which is clearly a contradiction since we have (3.6).

We now construct a finite sequence, say \((K_j)_{j=0}^{\infty}\), of closed, bounded convex subsets of \( \mathbb{R}^n \) satisfying:
\[
K_0 = M_Z, \quad K_j \supseteq K_{j+1} \quad \text{for all } j = 1, 2, ..., \text{ and } K_s = F
\]
as follows:

We let \( \eta_1 = \eta \) and define
\[
K_1 = \{ z \in K_0 = M_Z : \langle \eta_1, z \rangle = 0 \} = \Sigma.
\]
If \( 0 \in \text{ri} K_1 \) we are done since then \( K_1 = F \). Otherwise \( F \subset K_1 \) and we can define \( \eta_2 \) analogously to \( \eta_1 \) with \( K_0 = M_Z \) replaced by \( K_1 \) and define
\[
K_2 = \{ z \in K_1 : \langle \eta_2, z \rangle = 0 \}.
\]

We continue this construction until we arrive at a set \( K_s \) with the property that \( 0 \notin \text{ri} K_s \). As each set \( K_j \) is a proper convex subset of \( K_{j-1} \) it follows that their respective dimensions are strictly decreasing. Consequently, this process will terminate in a finite number of steps. Further as \( F \subset K_j \) for all \( j = 1, 2, ..., s \) it follows that \( F = K_s \).

Now let \( x : [-r, \infty) \to \mathbb{R}^n \) be any solution of (2.2) and suppose that for some \( i \in \{1, 2, ..., s-1\} \) we have \( \dot{x}(t) \in K_i \) for almost all \( t \in \mathbb{R} \). This implies \( \langle \eta_{i+1}, \dot{x}(t) \rangle \geq 0 \) a.e. on \( [-r, \infty) \). For \( \tau \geq 0 \) define the family of functions \( x^\tau : [-r, \infty) \to \mathbb{R}^n \) by the formula
\[
x^\tau(t) = x(t + \tau) \quad \text{for } \tau \geq -r.
\]

Observe that for almost all \( t > -r \) and \( \tau \in \mathbb{R} \)
\[
(x^\tau_t, \dot{x}(t)) = (x_{t+\tau}, \dot{x}(t + \tau)) \in M
\]
holds (i.e., \( x'(\cdot) \) is a viable solution for each \( \tau \geq 0 \)). Therefore, by proceeding as above with \( \eta \) replaced by \( \eta_{i+1} \) we have

\[
\lim_{t \to +\infty} \langle \eta_{i+1}, x'(t) \rangle = 0
\]  

(3.9)

for each \( \tau \geq 0 \). Moreover, since \( x(\cdot) \) is viable, we have

\[
\|x'(t)\| \leq \sup_{-\tau \leq s \leq 0} \|x'(s)\| \leq K
\]

and

\[
\|\tilde{x}'(t)\| \leq \tilde{K}
\]

for all \( t \geq 0 \) and all \( \tau \geq 0 \). In particular this implies that the corresponding set of “initial data” \( \{x'(\cdot)_{[-\tau,0]}\}_{\tau \geq 0} \) is a relatively compact set. Therefore, we see that (3.9) holds uniformly with respect to \( \tau \) and so for each \( \varepsilon > 0 \) there exists \( T_{\tau} > 0 \) such that for all \( t > T_{\tau} \) and all \( \tau \geq 0 \) we have

\[
-\varepsilon \leq \langle \eta_{i+1}, x'(t) \rangle \leq 0.
\]

Thus, as \( \varepsilon \) and \( \tau \) are arbitrary we have,

\[
\langle \eta_{i+1}, x(t) \rangle = 0
\]

for all \( t \in \mathbb{R} \) giving us

\[
\langle \eta_{i+1}, \tilde{x}(t) \rangle = \frac{d}{dt} \langle \eta_{i+1}, x(t) \rangle = 0
\]

a.e. on \([ -\tau, +\infty ) \). That is, \( \tilde{x}(t) \in K_{i+1} \) a.e. on \([ -\tau, +\infty ) \). In particular we observe that this implies every solution \( x : [ -\tau, +\infty ) \to \mathbb{R}^n \) of (2.2) necessarily satisfies

\[
\tilde{x}(t) \in F \quad \text{a.e. on } \mathbb{R},
\]

which implies

\[
(x_*, \tilde{x}(t)) \in H_0 \quad \text{a.e. on } \mathbb{R}.
\]

Defining the family of functions \( x'(\cdot) \) as above (\( \tau \geq 0 \)) we conclude that \( x(t) \equiv 0 \) for all \( t \in \mathbb{R} \).

We now show that if \( \Omega \) is a set of viable solutions of (2.2) with initial data lying in a compact subset of \( C_{[-\tau,0]} \), then for every \( \varepsilon > 0 \) there exists \( T_{\tau} > 0 \) such that for all \( t \geq T_{\tau} \)

\[
\|x(t)\| < \varepsilon \quad \text{for all } x \in \Omega.
\]
To see this we suppose that there exists an \( \varepsilon_0 > 0 \), a sequence \( \{x^k\}_{k=1}^{+\infty} \) in \( \Omega \) and a monotonic sequence \( t_k \to +\infty \) as \( k \to +\infty \) such that

\[
\|x^k(t_k)\| \geq \varepsilon_0.
\]

Define the sequence \( s^k: [-t_k - r, +\infty) \to \mathbb{R}^n \), for \( k = 1, 2, \ldots \), by the formula

\[
s^k(t) = x^k(t + t_k).
\]

Observe that for all \( k = 1, 2, \ldots \), we have

1. \( \|s^k(0)\| = \|x^k(t_k)\| \geq \varepsilon_0 \),
2. \( s^k(\tau) = s^k(\tau - t_k) = x^k(\tau) \) for all \( \tau \in [-r, 0] \), and
3. \( (s^k, s^k(t)) \in M \) a.e. \( -t_k \leq t \).

Thus by appealing to Theorem 2.3 there exists a subsequence \( \{s^k\}_{j=1}^{+\infty} \) and a function \( s: (-\infty, +\infty) \to \mathbb{R}^n \) (here we take \( t_0 = -t_k \) and \( t_1 = +\infty \)) such that \( \|s^k - s\|_\infty \to 0 \) as \( j \to +\infty \) for each \( t \in \mathbb{R} \), \( s^k \to s \) weakly in \( L^1_{\text{loc}}(-\infty, +\infty) \) as \( f \to +\infty \), and \( (s, s(t)) \in M \) a.e. on \( (-\infty, +\infty) \). From the above this implies that \( s(t) \equiv 0 \) on \( \mathbb{R} \). Clearly this is a contradiction since \( s(0) = \lim_{j \to +\infty} s^k(0) \geq \varepsilon_0 \). Thus, the result is proved.

4. EXAMPLES

In this section we present two examples to which the above theorem may be applied.

**Example 1 (Linear Functional Differential Equations).** In this example we let \( \Phi: \mathbb{R}^n \to [0, +\infty) \) be a fixed strictly convex function satisfying \( \Phi(0) = 0 \) and let \( L: C([-r, 0]) \to \mathbb{R}^n \) be a continuous, linear functional. Define \( G: C([-r, 0]) \to \mathbb{R}^n \) by the formula

\[
G(\psi) = \{z \in \mathbb{R}^n: \Phi(z - L(\psi)) = 0\}.
\]

From the above definitions it is an easy matter to see that \( x(\cdot): [-r, +\infty) \to \mathbb{R}^n \) is a solution of the differential inclusion

\[
x(t) \in G(x_t) \quad \text{a.e. } t \geq 0
\]

if and only if it is a solution of the linear differential equation

\[
\dot{x} = L(x_t) \quad \text{a.e. } t \geq 0.
\]
As is well known, all the solutions of Eq. (4.1) converge to zero if and only if the characteristic equation

\[ \det \left( J - \int_{-\tau}^{0} e^{\lambda s} \, d\eta(s) \right) = 0, \]

in which \( \eta: [-\tau, 0] \to \mathbb{R}^n \) is the unique vector-valued function whose components are normalized functions of bounded variation for which

\[ L(\eta) \equiv \int_{-\tau}^{0} \eta(s) \, d\eta(s) \]

for all \( \psi \in C([-\tau, 0]) \), has no solutions with nonnegative real part. Moreover, this convergence is uniform over any bounded subset of initial data. Clearly another way of saying this is that the linear retarded functional differential equation has no elliptic solutions. Thus we see that for any closed bounded set \( X \subset C([-\tau, 0]) \) that contains 0 in its interior, all of the solutions of the inclusion \( x(t) \in G(x) \) satisfying \( x(\tau) \in X \) for \( t \geq 0 \) converges to zero whenever the Eq. (4.1) is asymptotically stable.

**Example 2 (Closed Convex Processes).** In the monograph [3] it is remarked that the natural generalization of a continuous linear operator in set-valued analysis is the closed convex process. Indeed one need only read chapter 2 in [3] to appreciate the strength of this remark. In this example we promote this idea once more by considering a differential inclusion whose right hand side is a closed convex process. Specifically we assume that \( G: C([-\tau, 0]) \to 2^{\mathbb{R}^n} \setminus \emptyset \) is a set-valued mapping whose graph is a closed convex cone. That is

1. the graph of \( G \) is closed,
2. the graph of \( G \) is convex, and
3. for every \( \lambda > 0 \) and \( \phi \in C([-\tau, 0]) \) we have \( \lambda G(\phi) = G(\lambda \phi) \).

An immediate consequence of these properties is that 0 \( \in \) \( G(0) \) so that the constant function 0 is an equilibrium for the differential inclusion. From this we see that almost all of the general hypotheses of Theorem 2.1 are satisfied. We only need require the graph of \( G \) to be bounded. This is a severe restriction in general, however we observe that if \( G \) has linear growth (i.e., there exists a constant \( k > 0 \) such that for all \( (z, \phi) \in \text{graph}(G) \) one has \( \|z\| \leq k \|\phi\|_\infty \)), then for any closed, bounded set \( X \subset C([-\tau, 0]) \) that contains 0 in its interior we have that the restriction of \( G \) to the set \( X \) has a bounded graph. Thus in this situation we can apply Theorem 2.1. We further notice if \( G \) has a closed bounded graph, then the closed convex process enjoys linear growth. Indeed, if we consider \( \mathcal{G} = \{ (z, \phi) : z \in G(\phi) \} \),
\[ \| \phi \| \leq 1 \}, \] then there is a constant k > 0 such that \( \| z \| \leq k \) for all \( z \in \mathbb{R}^n \) such that \( (z, \phi) \in \mathcal{G} \) for some \( \phi \) in the unit ball of \( C_n[0, -r, 0] \). Therefore if \( \phi \neq 0 \) then we have, by the positive homogeneity of \( G \), for any \( z \in G(\phi) \) that \( z/\| \phi \| \leq k \| \phi \| \), giving us \( \| z \| \leq k \| \phi \| \) as desired.

Therefore, in the case of closed, convex process we see that the restriction of a closed, bounded graph is more restrictive than linear growth. However if attention is confined to solutions lying in fixed bounded set these ideas are equivalent. Further, these observations allow us to establish the following result as a corollary of Theorem 2.1.

**Proposition 4.1.** Let \( G : C_n[-r, 0] \to 2^{\mathbb{R}^n} \) be a closed convex process. Then a sufficient condition for all bounded solutions of the differential inclusion

\[ \dot{x}(t) \in G(x(t)) \]

to converge to zero as \( t \to +\infty \) is that

1. \( 0 \in G(\eta) \) if and only if \( \eta(s) \equiv 0 \) for \( -r \leq s \leq 0 \)
2. The system (2.2) has no elliptic solutions.

**Proof.** Let \( X \) denote the closed unit ball in \( C_n[-r, 0] \) and observe that if \( x(\cdot) : [-r, +\infty) \to \mathbb{R}^n \) is a bounded, say \( \| x(t) \| \leq K \) for all \( t \geq -r \), solution of the differential inclusion, then \( y(\cdot) : [-r, +\infty) \to \mathbb{R}^n \) defined for \( t \geq 0 \) by

\[ y(t) = \frac{1}{K} x(t) \]

is a solution to the differential inclusion for which \( \| y(t) \| \leq 1 \) for all \( t \geq 0 \). The result now follows immediately from Theorem 2.1.

There are apparently no results of this type in the literature (other than Leizarowitz \[18\]) when \( r = 0 \). Related results for ordinary differential inclusions concerning these ideas appear in several references. These are nicely collected in the monograph of Deimling \[15\]. In particular we direct the reader's attention to Theorem 14.3(a) and the remarks 5 and 7(iv) in Section 14.5 of this work. Deimling's result gives conditions for exponential asymptotic stability the zero solution of a differential inclusion

\[ \dot{x} \in F(x(t)) \]

in which \( F : \mathbb{R}^n \to 2^{\mathbb{R}^n} \) is an upper semicontinuous set-valued mapping with compact, convex values which is positively homogeneous. For the convex processes considered above these properties are satisfied. The
specific result says that if there exists $r > 0$ such that all of the solutions of
the above inclusion defined on $[0, +\infty)$ with initial data $\|y(0)\| \leq r$ tend
to zero as $t \to +\infty$, then $y(t) \equiv 0$ is strongly asymptotically stable and
there exists two positive constants $\alpha$ and $\beta$ such that for all solutions
$y: [0, +\infty) \to \mathbb{R}^n$ of the inclusion satisfy
$$\|y(t)\| \leq \alpha \|y(0)\| e^{-\beta t}.$$ Clearly, we see that in this situation we have no elliptic solutions.

5. CONCLUSION

In this work we have established sufficient conditions for the asymptotic
stability of an equilibrium for a class of differential inclusions of retarded
type. This class includes inclusions in which the right-hand side is closed,
convex process and the result obtained is clearly related to well known
results for linear retarded differential equations. However, we are quick to
point out that our conditions are merely sufficient and not necessary. It is
conjectured that if condition (1) of Theorem 2.1 is replaced by the require-
ment that the zero function is the only stationary solution then the theorem
is not only sufficient but also necessary. The proof offered here is not
adequate since condition (1) is a crucial assumption in the first part of the
proof. It has been observed that in a finite dimensional setting the notion
of an eigenvector for a closed convex process has been defined with proper-
ties similar to those of a linear mapping. An interesting question is whether
it is possible to characterize asymptotic stability in this case by showing
that all eigenvectors have negative real part and moreover establish the
usual spectral decomposition that permits one to describe all of the solu-
tions of a linear differential equation. To the limited knowledge of this
author, these results have apparently not been studied in the literature.

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